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SEGAL'S MULTISIMPLICIAL SPACES

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Abstract. Some sufficient conditions on a simplicial space $X\colon \Delta^{op} \to \mathrm{Top}$ guaranteeing that $X_1 \simeq \Omega|X|$ were given by Segal. We give a generalization of this result for multisimplicial spaces. This generalization is appropriate for the reduced bar construction, providing an n-fold delooping of the classifying space of a category.

1. Introduction

This note makes no great claim to originality. It provides a complete inductive argument for a generalization of [**17**, Proposition 1.5], which was spelled out, not in a precise manner, in [**2**, paragraph preceding Theorem 2.1]. The authors of [**2**] did not provide the proof of this generalization. Some related, but quite different, results are given in [**4**] and [**3**].

The main result of [**5**] reaches its full potential role in constructing a model for an n-fold delooping of the classifying space of a category only with the help of such a generalization of [**17**, Proposition 1.5]. The aim of this note is to fill in a gap in the literature concerning these matters.

Segal, [**17**, Proposition 1.5], gave conditions on a simplicial space $X\colon \Delta^{\mathrm{op}} \to \mathrm{Top}$ guaranteeing that $X_1 \simeq \Omega|X|$. His intention was to cover a more general class of simplicial spaces than we need for our purposes, therefore he worked with nonstandard geometric realizations of simplicial spaces. We generalize his result, in one direction, by passing from simplicial spaces to multisimplicial spaces, but staying in a class appropriate for the standard geometric realization. Our result is formulated to be directly applicable to the reduced bar construction of [**5**], providing an n-fold delooping of the classifying space of a category.

We work in the category (here denoted by Top) of compactly generated Hausdorff spaces. (This category is denoted by $\mathcal{K}e$ in [**7**] and by **CGHaus** in [**9**].) The objects of Top are called *spaces* and the arrows are called *maps*. Products are given

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the compactly generated topology. We adopt the following notation throughout: $\simeq$ for homotopy of maps or same homotopy type of spaces, $\approx$ for homeomorphism of spaces.

The category Δ (denoted by Δ^+ in [**9**]) is the standard topologist's *simplicial* category defined as in [**9**, VII.5]. We identify this category with the subcategory of Top whose objects are the standard ordered simplices (one for each dimension), i.e., with the standard cosimplicial space $\Delta \to \mathrm{Top}$.

The objects of Δ are the nonempty ordinals $1, 2, 3, \ldots$, which are rewritten as $[0], [1], [2]$, etc. The coface arrows from $[n-1]$ to $[n]$ are denoted by δ_i^n, for $0 \leqslant i \leqslant n$, and the codegeneracy arrows from $[n]$ to $[n-1]$ are denoted by σ_i^n, for $0 \leqslant i \leqslant n-1$.

For the opposite category Δ^{op}, we denote the arrow $(\delta_i^n)^{\mathrm{op}} \colon [n] \to [n-1]$ by d_i^n and $(\sigma_i^n)^{\mathrm{op}} \colon [n-1] \to [n]$ by s_i^n. For f an arrow of Δ^{op} (or $(\Delta^{\mathrm{op}})^n$), we abbreviate $X(f)$ by f whenever the (multi)simplicial object X is determined by the context.

We consider all the monoidal structures to be strict, which is supported by the strictification given by [**9**, XI.3, Theorem 1]. Some proofs prepared for non-specialists are given in the appendix.

2. Multisimplicial spaces and their realization

A *multisimplicial space* is an object of the category $\mathrm{Top}^{(\Delta^{\mathrm{op}})^n}$, i.e., a functor from $(\Delta^{\mathrm{op}})^n$ to Top. When $n = 0$, this is just a space and when $n = 1$, this is a *simplicial space*. As usual, for a multisimplicial space $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$, we abbreviate $X([k_1], \ldots, [k_n])$ by $X_{k_1 \ldots k_n}$.

We say that $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$ is a *multisimplicial set* when every $X_{k_1 \ldots k_n}$ is discrete. A *multisimplicial map* is an arrow of $\mathrm{Top}^{(\Delta^{\mathrm{op}})^n}$, i.e., a natural transformation between multisimplicial spaces. When $n = 1$, this is a *simplicial map*. Throughout this paper we use the standard geometric realization of (multi)simplicial spaces.

Definition 2.1. The *realization* functor $|\ | : \mathrm{Top}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$ of simplicial spaces is defined so that for a simplicial space X, we have

$$|X| = \Big(\coprod_n X_n \times \Delta^n\Big) \Big/ \sim,$$

where the equivalence relation $\sim$ is generated by

$$(d_i^n x, t) \sim (x, \delta_i^n t) \quad \text{and} \quad (s_i^n x, t) \sim (x, \sigma_i^n t).$$

Definition 2.2. For $p \geqslant 0$, the functor $\_^{(p)} : \mathrm{Top}^{(\Delta^{\mathrm{op}})^{n+p}} \to \mathrm{Top}^{(\Delta^{\mathrm{op}})^n}$ of *partial realization* is defined inductively as follows

$$\_^{(0)} \text{ is the identity functor, and } \_^{(p+1)} \text{ is the composition}$$

$$\mathrm{Top}^{(\Delta^{\mathrm{op}})^{n+p+1}} \xrightarrow{\cong} \big(\mathrm{Top}^{\Delta^{\mathrm{op}}}\big)^{(\Delta^{\mathrm{op}})^{n+p}} \xrightarrow{|\ |^{(\Delta^{\mathrm{op}})^{n+p}}} \mathrm{Top}^{(\Delta^{\mathrm{op}})^{n+p}} \xrightarrow{\_^{(p)}} \mathrm{Top}^{(\Delta^{\mathrm{op}})^n},$$

where the first isomorphism maps X to Y such that $(Y_{k_1 \ldots k_{n+p}})_l = X_{k_1 \ldots k_{n+p} l}$.

For a multisimplicial space $X : (\Delta^{\mathrm{op}})^p \to \mathrm{Top}$, we denote $X^{(p)}$ by $|X|$. Hence, for $X : (\Delta^{\mathrm{op}})^{n+p} \to \mathrm{Top}$, we have that $(X^{(p)})_{k_1\dots k_n} = |X_{k_1\dots k_n \underbrace{\_\dots\_}_{p}}|$.

Definition 2.3. If $X = Y^{(p)}$, for Y a multisimplicial set, then we say that X is a *partially realized* multisimplicial set (PRmss).

Definition 2.4. For $n \geqslant 0$ and $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$, let the simplicial space $\operatorname{diag} X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ be such that

$$(\operatorname{diag} X)_k = X_{k\dots k}.$$

In particular, when $n = 0$ and X is just a topological space, we have that $(\operatorname{diag} X)_k$ is X and all the face and degeneracy maps of $\operatorname{diag} X$ are $\mathbf{1}_X$.

The following lemma is a corollary of [**15**, Lemma, p. 94].

Lemma 2.5. *For $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$, we have that $|X| \approx |\operatorname{diag} X|$.*

As a consequence of Lemma 2.5 and [**10**, Theorem 14.1] we have the following lemma.

Lemma 2.6. *If X is a PRmss, then $|X|$ is a CW-complex.*

The following remark easily follows.

Remark 2.7. (a) If $X : (\Delta^{\mathrm{op}})^{n+p} \to \mathrm{Top}$ is a PRmss, then $X^{(p)}$ is a PRmss.

(b) If $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$ is a PRmss, then for every $k_1, \dots, k_n$, the space $X_{k_1\dots k_n}$ is a CW-complex.

(c) If $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$ is a PRmss, then for every $k \geqslant 0$, $X_{k\_\dots\_}$ is a PRmss.

(d) If $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$, for $n > 1$, is a PRmss, then $Y : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Top}$, defined so that $Y_{mk} = X_{mk\dots k}$, is a PRmss.

Definition 2.8. A simplicial space $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ is *good* when for every $0 \leqslant i \leqslant n-1$, the map $s_i^n : X_{n-1} \to X_n$ is a closed cofibration. It is *proper* (Reedy cofibrant) when for every $n \geqslant 1$, the inclusion $sX_n \hookrightarrow X_n$, where $sX_n = \bigcup_i s_i^n(X_{n-1})$, is a closed cofibration.

Proposition 2.9. *Every PRmss $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ is good.*

Proof. Since $d_i^n \circ s_i^n = \mathbf{1}_{X_{n-1}}$, we may consider X_{n-1} to be a retract of X_n. By Remark 2.7(b), X_n is a CW-complex and by [**6**, Corollary III.2] (see also [**8**, Corollary 2.4(a)]) a locally equiconnected space. By [**8**, Lemma 3.1] and since every X_k is Hausdorff, s_i^n is a closed cofibration. ⊣

As a corollary of [**18**, Proposition 22] (see also references listed in [**18**, Section 6, p. 19]) we have the following result.

Lemma 2.10. *Every good simplicial space is proper.*

The following result is from [**12**, Appendix, Theorem A4(ii)].

LEMMA 2.11. *Let $f: X \to Y$ be a simplicial map of proper simplicial spaces. If each $f_k: X_k \to Y_k$ is a homotopy equivalence, then $|f|: |X| \to |Y|$ is a homotopy equivalence.*

DEFINITION 2.12. The *product* $X \times Y$ of simplicial spaces X and Y is defined componentwise, i.e., $(X \times Y)_k = X_k \times Y_k$, and for every arrow $f: k \to l$ of Δ^{op} and every $x \in X_k$ and $y \in Y_k$, we have that $f(x, y) = (fx, fy)$.

Since the product of two CW-complexes in Top is a CW-complex, by reasoning as in Proposition 2.9, we have the following.

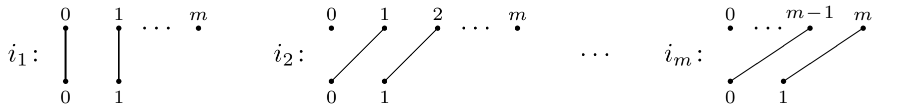
REMARK 2.13. If $X, Y : \Delta^{\mathrm{op}} \to \mathrm{Top}$ are PRmss, then $X \times Y$ is good.

The following lemma is a corollary of [**11**, Lemma 11.11].

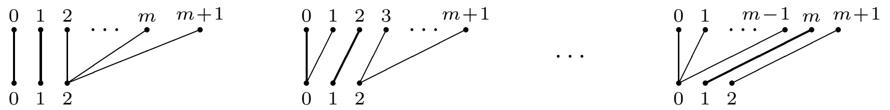
LEMMA 2.14. *If the space X_0 of a simplicial space $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ is path-connected, then $|X|$ is path-connected.*

3. Segal's multisimplicial spaces

For $m \geqslant 1$, consider the arrows $i_1, \ldots, i_m : [m] \to [1]$ of Δ^{op} given by the following diagrams.



The following images of these arrows under the functor $\mathcal{J} : \Delta^{\mathrm{op}} \to \Delta$ of [**14**, Section 6] may help the reader to see that $i_1, \ldots, i_m$ correspond to m projections. (Note that 0 and 2 in the bottom line of the images serve to project away all but one element of the top line.)



For maps $f_i: A \to B_i$, $1 \leqslant i \leqslant m$, we denote by $\langle f_1, \ldots, f_m \rangle : A \to B_1 \times \cdots \times B_m$ the map obtained by the Cartesian structure of Top. In particular, for the above-mentioned $i_1, \ldots, i_m$ and for a simplicial space $X: \Delta^{\mathrm{op}} \to \mathrm{Top}$ we have the map

$$p_m = \langle i_1, \ldots, i_m \rangle : X_m \to (X_1)^m.$$

(According to our convention from the introduction, $X(i_k)$ is abbreviated by i_k.) If $m = 0$, then $(X_1)^0 = \{*\}$ (a terminal object of Top) and let p_0 denote the unique arrow from X_0 to $(X_1)^0$. The following lemma is claimed in [**17**].

LEMMA 3.1. *If $X: \Delta^{\mathrm{op}} \to \mathrm{Top}$ is a simplicial space such that for every $m \geqslant 0$, the map p_m is a homotopy equivalence, then X_1 is a homotopy associative* H*-space whose multiplication m is given by the composition*

$$(X_1)^2 \xrightarrow{p_2^{-1}} X_2 \xrightarrow{d_1^2} X_1,$$

where p_2^{-1} is an arbitrary homotopy inverse to p_2, and whose unit $*$ is $s_0^1(x_0)$, for an arbitrary $x_0 \in X_0$.

Definition 3.2. We say that a PRmss $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ is *Segal's simplicial space* when for every $m \geqslant 0$, the map $p_m : X_m \to (X_1)^m$ is a homotopy equivalence.

Lemma 3.3. *Let* $Y : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Top}$ *be a PRmss. If for every* $k \geqslant 0$*, the simplicial space* $Y_{\_k}$ *is Segal's, then* $Y^{(1)}$ *is Segal's simplicial space.*

Definition 3.4. We say that a PRmss $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$, where $n \geqslant 1$, is *Segal's multisimplicial space*, when for every $l \in \{0, \dots, n-1\}$ and every $k \geqslant 0$, the simplicial space $X_{\underbrace{1\dots1}_{l}\_k\dots k}$ is Segal's.

Note that we do not require $X_{k_1\dots k_l\_k_{l+1}\dots k_{n-1}}$ to be Segal's for arbitrary $k_1, \dots, k_{n-1}$ (see the parenthetical remark in Section 5.)

Remark 3.5. If $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$ is Segal's multisimplicial space, then for every $l \in \{0, \dots, n-1\}$, $X_{1\dots1}$ is homotopy associative H-space with respect to the structure obtained from Lemma 3.1 applied to $X_{\underbrace{1\dots1}_{l}\_1\dots1} : \Delta^{\mathrm{op}} \to \mathrm{Top}$.

Our goal is to generalize the following proposition, which stems from [**17**, Proposition 1.5(b)]. (In the proof of that result, contractibility of $|PA|$ comes from the fact that $|PA| \simeq A_0$.)

Proposition 3.6. *Let* $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$ *be Segal's simplicial space. If* X_1 *with respect to the* H*-space structure obtained by Lemma 3.1 is grouplike, then* $X_1 \simeq \Omega|X|$.

Our generalization is the following.

Proposition 3.7. *Let* $X : (\Delta^{\mathrm{op}})^n \to \mathrm{Top}$ *be Segal's multisimplicial space. If* $X_{1\dots1}$*, with respect to the* H*-space structure obtained by Remark 3.5 when* $l = n-1$*, is grouplike, then* $X_{1\dots1} \simeq \Omega^n|X|$.

Proof. We proceed by induction on $n \geqslant 1$. If $n = 1$, the result follows from Proposition 3.6.

If $n > 1$, then we may apply the induction hypothesis to $X_{1\_\dots\_}$. Hence,

$$X_{1\dots1} \simeq \Omega^{n-1}|X_{1\_\dots\_}|.$$

By Lemma 2.5, we have that $|X_{1\_\dots\_}| \approx |\operatorname{diag} X_{1\_\dots\_}|$. By the assumption and Remark 2.7(d), the multisimplicial space $Y : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Top}$, defined so that $Y_{mk} = X_{mk\dots k}$, satisfies the conditions of Lemma 3.3. Let Z be the simplicial space $Y^{(1)} : \Delta^{\mathrm{op}} \to \mathrm{Top}$, i.e.,

$$Z_m = |Y_{m\_}| = |\operatorname{diag} X_{m\_\dots\_}|.$$

By Lemma 3.3, Z is Segal's simplicial space. By Remark 2.7(b), Z_1 is a CW-complex. Since the space Y_{10} (i.e., $X_{10\dots0}$) is by the assumption homotopic to $(X_{110\dots0})^0$, it is contractible, and hence, path-connected. By Lemma 2.14, we have that Z_1, which is equal to $|Y_{1\_}|$, is path-connected. Note also that $|Z| = |Y| \approx |\operatorname{diag} X| \approx |X|$.

By Lemma 3.1, Z_1 is a homotopy associative H-space, and since it is a path-connected CW-complex, by [**1**, Proposition 8.4.4], it is grouplike. Applying Proposition 3.6 to Z, we obtain

$$|X_{1\_\dots\_}| \approx |\operatorname{diag} X_{1\_\dots\_}| = Z_1 \simeq \Omega|Z| \approx \Omega|X|.$$

Finally, we have

$$X_{1\dots 1} \simeq \Omega^{n-1}|X_{1\_\dots\_}| \simeq \Omega^n|X|. \qquad \dashv$$

4. Segal's lax functors

Thomason, [**20**], was the first who noticed that the reduced bar construction based on a symmetric monoidal category produces a lax, instead of an ordinary, functor. The idea to use Street's rectification in that case, also belongs to him.

We use the notions of *lax functor*, *left* and *right lax transformation* as defined in [**19**]. The following theorem is taken over from [**19**, Theorem 2].

Theorem 4.1. *For every lax functor $W : \mathcal{A} \to \mathrm{Cat}$ there exists a genuine functor $V : \mathcal{A} \to \mathrm{Cat}$, a left lax transformation $E : V \to W$ and a right lax transformation $J : W \to V$ such that J is the left adjoint to E and $W = EVJ$.*

We call V a *rectification* of W. It is easy to see that if $W : \mathcal{A} \times \mathcal{B} \to \mathrm{Cat}$ is a lax functor and V is its rectification, then for every object A of $\mathcal{A}$, $W_{A\_}$ is a lax functor and $V_{A\_}$ is its rectification. As for simplicial spaces, for a (lax) functor $W : \Delta^{\mathrm{op}} \to \mathrm{Cat}$, we denote the unique arrow $W_0 \to (W_1)^0$ by p_0, and when $m \geqslant 1$, we have $p_m = \langle i_1, \dots, i_m\rangle : W_m \to (W_1)^m$.

Definition 4.2. We say that a lax functor $W : \Delta^{\mathrm{op}} \to \mathrm{Cat}$ is *Segal's*, when for every $m \geqslant 0$, $p_m : W_m \to (W_1)^m$ is the identity. We say that a lax functor $W : (\Delta^{\mathrm{op}})^n \to \mathrm{Cat}$ is *Segal's*, when for every $l \in \{0, \dots, n-1\}$ and every $k \geqslant 0$, the lax functor $W_{\underbrace{1\dots 1}_{l}\_k\dots k} : \Delta^{\mathrm{op}} \to \mathrm{Cat}$ is Segal's.

We denote by $B : \mathrm{Cat} \to \mathrm{Top}$ the *classifying space* functor, i.e., the composition $|\ | \circ N$, where $N : \mathrm{Cat} \to \mathrm{Top}^{\Delta^{\mathrm{op}}}$ is the *nerve* functor.

Proposition 4.3. *If $W : \Delta^{\mathrm{op}} \to \mathrm{Cat}$ is Segal's lax functor and V is its rectification, then $B \circ V$ is Segal's simplicial space.*

By Definitions 3.4 and 4.2, the following generalization of Proposition 4.3 is easily obtained.

Corollary 4.4. *If $W : (\Delta^{\mathrm{op}})^n \to \mathrm{Cat}$ is Segal's lax functor and V is its rectification, then $B \circ V$ is Segal's multisimplicial space.*

For Corollary 4.4, we conclude that $B \circ V$ is a PRmss as in the proof of Proposition 4.3 given in the appendix.

5. An application

Let $\mathcal{M}$ be an n-fold strict monoidal category and let $\overline{W}\mathcal{M}\colon (\Delta^{\rm op})^n \to \mathrm{Cat}$ be the n-fold reduced bar construction defined as in [**5**]. The main result of that paper says that $\overline{W}\mathcal{M}$ is a lax functor and it is easy to verify that it is Segal's. (Note that $\overline{W}\mathcal{M}_{k_1\ldots k_l\_\ldots k_{n-1}}$ is not Segal's when $k_j > 1$, for some $1 \leqslant j \leqslant l$.)

For V being a rectification of $\overline{W}\mathcal{M}$, we have the following.

Theorem 5.1. *If $BV_{1\ldots 1}$, with respect to the H-space structure obtained by Remark 3.5 when $l = n-1$, is grouplike, then $B\mathcal{M} \simeq \Omega^n|B\circ V|$.*

Proof. By Corollary 4.4, we have that $B\circ V$ is Segal's multisimplicial space. Hence, by Proposition 3.7, $BV_{1\ldots 1} \simeq \Omega^n|B\circ V|$. Since V is a rectification of $\overline{W}\mathcal{M}$, by relying on Remark A1 of the appendix, we conclude that $BV_{1\ldots 1} \simeq B\overline{W}\mathcal{M}_{1\ldots 1}$. From the fact that $\overline{W}\mathcal{M}_{1\ldots 1} = \mathcal{M}$, we obtain that $B\mathcal{M} \simeq \Omega^n|B\circ V|$. ⊣

This means that up to group completion (see [**17**] and [**13**]), for every n-fold strict monoidal category $\mathcal{M}$, the classifying space $B\mathcal{M}$ is an n-fold loop space. When $\mathcal{M}$ contains a terminal or initial object, we have that $B\mathcal{M}$, and hence $BV_{1\ldots 1}$, is path-connected. In that case, by [**1**, Proposition 8.4.4], $BV_{1\ldots 1}$ is grouplike, and $|B\circ V|$ is an n-fold delooping of $B\mathcal{M}$.

6. Appendix

Proof of Lemma 3.1. First, we prove that $\langle X_1, m, *\rangle$ is an H-space. Let $j_1\colon X_1 \to X_1\times X_1$ be such that $j_1(x) = (x,*)$, and analogously, let $j_2\colon X_1\to X_1\times X_1$ be such that $j_2(x) = (*,x)$. By the assumption, X_0 is contractible. Hence, d_0^1 is homotopic to the constant map to x_0 and therefore $s_0^1\circ d_0^1$ is homotopic to the constant map to $*$. We conclude that

$$j_1 \simeq \langle \mathbf{1}_{X_1}, s_0^1\circ d_0^1\rangle = \langle d_2^2\circ s_1^2, d_0^2\circ s_1^2\rangle = \langle d_2^2, d_0^2\rangle\circ s_1^2 = p_2\circ s_1^2,$$

i.e., $p_2^{-1}\circ j_1 \simeq s_1^2$. Hence,

$$m\circ j_1 = d_1^2\circ p_2^{-1}\circ j_1 \simeq d_1^2\circ s_1^2 = \mathbf{1}_{X_1}.$$

Analogously, $m\circ j_2 \simeq \mathbf{1}_{X_1}$ and we have that $\langle X_1, m, *\rangle$ is an H-space.

Next, we prove that m is associative up to homotopy, i.e., that

$$m\circ(m\times\mathbf{1}) \simeq m\circ(\mathbf{1}\times m).$$

Consider $p_3\colon X_3\to (X_1)^3$ for which we have

$$\begin{aligned} p_3 &= \langle\langle i_1, i_2\rangle, i_3\rangle = \langle\langle d_2^2\circ d_3^3, d_0^2\circ d_3^3\rangle, i_3\rangle = \langle p_2\circ d_3^3, i_3\rangle \\ &= (p_2\times\mathbf{1})\circ\langle d_3^3, i_3\rangle, \quad \text{and analogously} \\ p_3 &= (\mathbf{1}\times p_2)\circ\langle i_1, d_0^3\rangle. \end{aligned}$$

Since p_2 and p_3 are homotopy equivalences, we have that $\langle d_3^3, i_3\rangle$ and $\langle i_1, d_0^3\rangle$ are homotopy equivalences, too. Moreover,

$$\langle d_3^3, i_3\rangle^{-1} \simeq p_3^{-1}\circ(p_2\times\mathbf{1}), \tag{1}$$

$$\langle i_1, d_0^3\rangle^{-1} \simeq p_3^{-1}\circ(\mathbf{1}\times p_2). \tag{2}$$

Also, we show that

$$d_1^2 \times \mathbf{1} \simeq p_2 \circ d_1^3 \circ p_3^{-1} \circ (p_2 \times \mathbf{1}), \tag{3}$$

$$\mathbf{1} \times d_1^2 \simeq p_2 \circ d_2^3 \circ p_3^{-1} \circ (\mathbf{1} \times p_2). \tag{4}$$

We have

$$(d_1^2 \times \mathbf{1}) \circ \langle d_3^3, i_3 \rangle = \langle d_1^2 \circ d_3^3, i_3 \rangle = \langle d_2^2 \circ d_1^3, d_0^2 \circ d_1^3 \rangle = \langle d_2^2, d_0^2 \rangle \circ d_1^3 = p_2 \circ d_1^3,$$

which together with (1) delivers (3). Also,

$$(\mathbf{1} \times d_1^2) \circ \langle i_1, d_0^3 \rangle = \langle i_1, d_1^2 \circ d_0^3 \rangle = \langle d_2^2 \circ d_2^3, d_0^2 \circ d_2^3 \rangle = \langle d_2^2, d_0^2 \rangle \circ d_2^3 = p_2 \circ d_2^3,$$

which together with (2) delivers (4). Finally, we have

$$\begin{aligned} m \circ (m \times \mathbf{1}) &= d_1^2 \circ p_2^{-1} \circ (d_1^2 \times \mathbf{1}) \circ (p_2^{-1}) \simeq d_1^2 \circ d_1^3 \circ p_3^{-1}, \quad \text{by (3)} \\ &= d_1^2 \circ d_2^3 \circ p_3^{-1} \simeq d_1^2 \circ p_2^{-1} \circ (\mathbf{1} \times d_1^2) \circ (\mathbf{1} \times p_2^{-1}), \quad \text{by (4)} \\ &= m \circ (\mathbf{1} \times m). \end{aligned}$$

⊣

Proof of Lemma 3.3. Let $Z \colon \Delta^{\mathrm{op}} \to \mathrm{Top}$ be $Y^{(1)}$. By Remark 2.7(a), it is a PRmss. We have to show that for every $m \geqslant 0$, the map $p_m \colon Z_m \to (Z_1)^m$ is a homotopy equivalence.

Let $m = 0$ and let T be the trivial simplicial space with $T_k = \{*\}$. Consider the simplicial space $Y_{0\_} \colon \Delta^{\mathrm{op}} \to \mathrm{Top}$, which is a PRmss by Remark 2.7(c). By Proposition 2.9 and Lemma 2.10, both T and $Y_{0\_}$ are proper. The following simplicial map is obtained by the assumptions (the diagrams are commutative since $\{*\}$ is terminal).

$$\begin{array}{lcccccc} Y_{0\_}: & \cdots & Y_{02} & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & Y_{01} & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & Y_{00} \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ T: & \cdots & \{*\} & \rightleftarrows & \{*\} & \rightleftarrows & \{*\} \end{array}$$

By Lemma 2.11, we have that $|Y_{0\_}| \simeq |T| = \{*\}$ via the unique map. Since $Z_0 = |Y_{0\_}|$ and $(Z_1)^0 = \{*\}$, we are done.

Let $m > 0$. Consider the simplicial spaces $Y_{m\_}$ and $(Y_{1\_})^m$, which are proper by Remark 2.7(c), Proposition 2.9, Lemma 2.10 and Remark 2.13. The following simplicial map is obtained by the assumptions (it is straightforward to verify that the diagrams are commutative).

$$\begin{array}{lcccccc} Y_{m\_}: & \cdots & Y_{m2} & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & Y_{m1} & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & Y_{m0} \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ (Y_{1\_})^m: & \cdots & (Y_{12})^m & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & (Y_{11})^m & \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} & (Y_{10})^m \end{array}$$

By Lemma 2.11, we have that

$$|\langle Y(i_1,\_),\dots,Y(i_m,\_)\rangle|\colon |Y_{m\_}| \to |(Y_{1\_})^m|$$

is a homotopy equivalence. Also, for Top, the realization functor $|\ |$ preserves products (see [**10**, Theorem 14.3], [**7**, III.3, Theorem] and [**11**, Corollary 11.6]). Namely, for $\pi_k\colon (Y_{1\_})^m \to Y_{1\_}$, $1 \leqslant k \leqslant m$ being the kth projection,

$$\langle|\pi_1|,\dots,|\pi_m|\rangle\colon |(Y_{1\_})^m| \to |Y_{1\_}|^m$$

is a homeomorphism ($|\ |$ is strong monoidal; see [**16**, Example 6.2.2]). Hence,

$$\langle|\pi_1|,\dots,|\pi_m|\rangle \circ |\langle Y(i_1,\_),\dots,Y(i_m,\_)\rangle|\colon |Y_{m\_}| \to |Y_{1\_}|^m$$

is a homotopy equivalence.

The following easy computation, in which $\langle Y(i_1,\_),\dots,Y(i_m,\_)\rangle$ is abbreviated by α,

$$\begin{aligned}\langle|\pi_1|,\dots,|\pi_m|\rangle \circ |\alpha| &= \langle|\pi_1|\circ|\alpha|,\dots,|\pi_m|\circ|\alpha|\rangle = \langle|\pi_1\circ\alpha|,\dots,|\pi_m\circ\alpha|\rangle\\ &= \langle|Y(i_1,\_)|,\dots,|Y(i_m,\_)|\rangle = \langle Z(i_1),\dots,Z(i_m)\rangle\end{aligned}$$

shows that the map $p_m = \langle Z(i_1),\dots,Z(i_m)\rangle$ is a homotopy equivalence between $Z_m = |Y_{m\_}|$, and $(Z_1)^m = |Y_{1\_}|^m$. ⊣

Some preliminary remarks for Proposition 4.3. Let $\mathbf{2}$ be the category with two objects (0 and 1) and one nonidentity arrow $h\colon 0 \to 1$. Let $I_0, I_1\colon \mathcal{C} \to \mathcal{C}\times\mathbf{2}$ be the functors such that for every object C of $\mathcal{C}$, we have that $I_0(C) = (C,0)$ and $I_1(C) = (C,1)$. Let $F,G\colon \mathcal{C} \to \mathcal{D}$ be two functors. There is a bijection between the set of natural transformations $\alpha\colon F \dot{\to} G$, and the set of functors $A\colon \mathcal{C}\times\mathbf{2} \to \mathcal{D}$ such that $A\circ I_0 = F$ and $A \circ I_1 = G$. This bijection maps $\alpha\colon F \dot{\to} G$ to $A\colon \mathcal{C}\times\mathbf{2} \to \mathcal{D}$ such that

$$A(C,0) = FC, \quad A(C,1) = GC, \quad A(f,\mathbf{1}_0) = Ff, \quad A(f,\mathbf{1}_1) = Gf,$$

and for $f\colon C \to C'$,

$$A(f,h) = Gf \circ \alpha_C = \alpha_{C'} \circ Ff.$$

Its inverse maps $A\colon \mathcal{C}\times\mathbf{2} \to \mathcal{D}$ to $\alpha\colon F \dot{\to} G$ such that $\alpha_C = A(\mathbf{1}_C, h)$.

The nerve functor N preserves products on the nose, hence, the classifying space functor $B = |\ |\circ N$ preserves products too. Therefore, the spaces $B\mathcal{C}\times I$ (i.e., $B\mathcal{C}\times B\mathbf{2}$) and $B(\mathcal{C}\times\mathbf{2})$ are homeomorphic and we have the following.

Remark A1. Every natural transformation $\alpha\colon F \dot{\to} G$ gives rise to the homotopy

$$B\mathcal{C}\times I \xrightarrow{\approx} B(\mathcal{C}\times\mathbf{2}) \xrightarrow{BA} B\mathcal{D}$$

between the maps BF and BG.

Proof of Proposition 4.3. By the isomorphism mentioned in Definition 2.2, we have that $N\circ V$ corresponds to a multisimplicial set $X\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}} \to \mathrm{Top}$ and $B\circ V$ is $X^{(1)}$. Hence, it is a PRmss.

We have to show that for every $m \geqslant 0$, $p_m\colon BV_m \to (BV_1)^m$ is a homotopy equivalence, where we denote again by p_0 the unique map from BV_0 to $(BV_1)^0$ and by p_m the map $\langle BV(i_1),\dots,BV(i_m)\rangle$.

When $m = 0$, we show that $BJ_0 \colon BW_0 \to BV_0$ is a homotopy inverse to p_0. Since W_0 and $(V_1)^0$ are the same trivial category and $BW_0 = (BV_1)^0 = \{*\}$, it is easy to conclude that $p_0 \circ BJ_0 \simeq \mathbf{1}_{(BV_1)^0}$, and that $p_0 = BE_0$. The latter, by the adjunction $J_0 \dashv E_0$ and Remark A1, delivers $BJ_0 \circ p_0 \simeq \mathbf{1}_{BV_0}$.

When $m \geqslant 1$, we have for every $1 \leqslant j \leqslant m$, the following natural transformations.

$$\begin{tikzcd} & W_m \arrow[dr, "i_j"] & \\ V_m \arrow[ur, "E_m"] \arrow[dr, "i_j"'] \arrow[rr, phantom, "\Downarrow E_{i_j}"] & & W_1 \\ & V_1 \arrow[ur, "E_1"'] & \end{tikzcd} \qquad \begin{tikzcd} & V_m \arrow[dr, "i_j"] & \\ W_m \arrow[ur, "J_m"] \arrow[dr, "i_j"'] \arrow[rr, phantom, "\Uparrow J_{i_j}"] & & V_1 \\ & W_1 \arrow[ur, "J_1"'] & \end{tikzcd}$$

By using the monoidal structure of Cat given by 2-products and the fact that $\langle i_1, \ldots, i_m\rangle : W_m \to (W_1)^m$ is the identity, we obtain the following two natural transformations.

$$\begin{tikzcd} V_m \arrow[rr, "E_m"] \arrow[dr, "{\langle i_1, \ldots, i_m\rangle}"'] \arrow[rr, phantom, shift right=5ex, "{\langle E_{i_1}, \ldots, E_{i_m}\rangle \Downarrow}"] & & W_m \\ & (V_1)^m \arrow[ur, "(E_1)^m"'] & \end{tikzcd} \qquad \begin{tikzcd} & V_m \arrow[dr, "{\langle i_1, \ldots, i_m\rangle}"] & \\ W_m \arrow[ur, "J_m"] \arrow[rr, "(J_1)^m"'] \arrow[rr, phantom, shift left=5ex, "{\Uparrow \langle J_{i_1}, \ldots, J_{i_m}\rangle}"] & & (V_1)^m \end{tikzcd}$$

For $\pi_k \colon \mathcal{C}^m \to \mathcal{C}$, $1 \leqslant k \leqslant m$ being the kth projection,

$$\langle B\pi_1, \ldots, B\pi_m\rangle \colon B\mathcal{C}^m \to (B\mathcal{C})^m$$

is a homeomorphism whose inverse we denote by $q_m(\mathcal{C})$. It is easy to verify that for $F, F_1, \ldots, F_m : \mathcal{C} \to \mathcal{D}$ we have $B\langle F_1, \ldots, F_m\rangle = q_m(\mathcal{D})\langle BF_1, \ldots, BF_m\rangle$ and $BF^m \circ q_m(\mathcal{C}) = q_m(\mathcal{D}) \circ (BF)^m$.

By Remark A1, the transformations mentioned above give rise to

$$\begin{aligned} (\dagger) \qquad BE_m &\simeq B(E_1)^m \circ B\langle V(i_1), \ldots, V(i_m)\rangle \\ &= q_m(W_1) \circ (BE_1)^m \circ \langle BV(i_1), \ldots, BV(i_m)\rangle \\ &= q_m(W_1) \circ (BE_1)^m \circ p_m, \\ (\dagger\dagger) \qquad B(J_1)^m &\simeq B\langle V(i_1), \ldots, V(i_m)\rangle \circ BJ_m \\ &= q_m(V_1) \circ \langle BV(i_1), \ldots, BV(i_m)\rangle \circ BJ_m \\ &= q_m(V_1) \circ p_m \circ BJ_m. \end{aligned}$$

The following calculation shows that

$$BJ_m \circ q_m(W_1) \circ (BE_1)^m \colon (BV_1)^m \to BV_m$$

is a homotopy inverse to p_m.

$$
\begin{aligned}
\mathbf{1}_{BV_m} &\simeq BJ_m \circ BE_m, \quad \text{by } J_m \dashv E_m, \text{ Remark A1} \\
&\simeq BJ_m \circ q_m(W_1) \circ (BE_1)^m \circ p_m, \quad \text{by } (\dagger) \\
\mathbf{1}_{(BV_1)^m} &\simeq q_m^{-1}(V_1) \circ B(J_1)^m \circ B(E_1)^m \circ q_m(V_1), \quad \text{by } J_1 \dashv E_1, \text{ Remark A1} \\
&\simeq p_m \circ BJ_m \circ q_m(W_1) \circ (BE_1)^m, \quad \text{by } (\dagger\dagger). \qquad \dashv
\end{aligned}
$$

References

1. M. Arkowitz, *Introduction to Homotopy Theory*, Springer, Berlin, 2011.
2. C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, *Iterated monoidal categories*, Adv. Math. **176** (2003), 277–349.
3. A. K. Bousfield, *The simplicial homotopy theory of iterated loop spaces*, unpublished (1992).
4. P. V. Z. Cobb, *P_n-spaces and n-fold loop spaces*, Bull. Am. Math. Soc. **80** (1974), 910–914.
5. S. Lj. Čukić, Z. Petrić, *The n-fold reduced bar construction* (the old title was "n-fold monoidal categories"), preprint (2013) (arXiv:1309.6209).
6. E. Dyer, S. Eilenberg, *An adjunction theorem for locally equiconnected spaces*, Pac. J. Math. **41** (1972), 669–685.
7. P. Gabriel, M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergeb. Math. Grenzgeb. **35**, Springer, Berlin, 1967.
8. L. G. Lewis, *When is the Natural Map $X \to \Omega\Sigma X$ a Cofibration?*, Trans. Am. Math. Soc. **273** (1982), 147–155.
9. S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin, 1971 (expanded second edition, 1998).
10. J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, Chicago, 1967.
11. ———, *The Geometry of Iterated Loop Spaces*, Lect. Notes Math. **271**, Springer, Berlin, 1972.
12. ———, *E_∞-spaces, group completions and permutative categories*, in: G. Segal (ed.), *New Developments in Topology*, Lond. Math. Soc. Lect. Note Ser. **11**, Cambridge University Press, Cambridge, 1974, 153–231.
13. D. McDuff, G. Segal, *Homology fibrations and the "group-completion" theorem*, Invent. Math. **31** (1976), 279–284.
14. Z. Petrić, T. Trimble, *Symmetric bimonoidal intermuting categories and $\omega \times \omega$ reduced bar constructions*, Appl. Categ. Struct. **22** (2014), 467–499 (arXiv:0906.2954).
15. D. Quillen, *Higher algebraic K-theory: I*, in: H. Bass (ed.), *Higher K-Theories*, Lect. Notes Math. **341**, Springer, Berlin, 1973, 85–147.
16. E. Riehl, *Categorical Homotopy Theory*, Cambridge University Press, New York, 2014.
17. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
18. D. Stevenson, D. Roberts, *Simplicial principal bundles in parametrized spaces*, preprint (2012) (arXiv:1203.2460).
19. R. Street, *Two constructions on lax functors*, Cah. Topol. Géom. Différ. Catég. **13** (1972), 217–264.
20. R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Camb. Philos. Soc. **85**, 91 (1979), 91–109.

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