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HOPF ALGEBRA OF PROJECTION FUNCTIONS

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ABSTRACT. We study Hopf algebras over projection functions of the complex vector C^X appropriate for computing inversion formulas from discrete mathematics. Using calculus of projection functions introduced in this way, we derived various inversion formulas, including Gould's inversion formula and its generalizations.

1. Introduction

The main idea of the paper is explained by the following example. Let C be the set of complex numbers, Z the set of integers, N the set of nonnegative integers and $\mathbf{C}^Z$ the complex vector space. We remind that a function $\pi_n\colon C^Z \to C$, $n \in Z$, is a projection if $\pi_n(f) = f(n)$, $f \in C^Z$. Let A be the subspace of the vector space of linear functionals of $\mathbf{C}^Z$ where A is generated by projections π_n, $n \in Z$. We introduce an associative and commutative algebra $\mathcal{A} = (A, \cdot)$ over A defining multiplication of projections by $\pi_m \cdot \pi_n = \pi_{m+n}$. Obviously, the power $(\pi_1)^n$ is equal to π_n. Hence, if π denotes π_1, then $\pi^n = \pi_n$. By projection calculus we shall mean calculation in the algebra $\mathcal{A}$. It appears that the algebra $\mathcal{A}$ is very appropriate for computing various inversion formulas from discrete mathematics.

Now we proceed to our example. Let $\mathcal{F} = \{f \in C^Z \colon \bigwedge_{n<0} f(n) = 0\}$. Obviously $\mathcal{F}$ is a subspace of $\mathbf{C}^Z$ and we may identify $\mathcal{F}$ and $\mathbf{C}^N$. Also, $f \in \mathcal{F}$ if and only if for all $n < 0$, $\pi^n(f) = \pi_n(f) = 0$. We shall prove by projection calculus the following well known inversion formula

$$g_n = \sum_k \binom{n}{k} f_{n-2k} \Leftrightarrow f_n = \sum_{2k \leqslant n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} g_{n-2k}, \quad f, g \in \mathcal{F}. \tag{1.1}$$

For this purpose let us introduce the functional $\theta = \pi + \pi^{-1}$. Then

$$\theta^n = \sum_k \binom{n}{k} \pi^{n-2k}. \tag{1.2}$$

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If $f \in \mathcal{F}$ and $g \in F$ is defined by $g_n = g(n) = \theta^n(f)$, $n \in N$, then by (1.2) we have $g_n = \sum_k \binom{n}{k} f_{n-2k}$. For the proof of equivalence (1.1), we express π^n by a polynomial of θ using Tchebychev polynomials. Let $T_n(x)$ be a Tchebychev polynomial of the first kind and $C_n(x) = \frac{1}{2}T_n(2x)$. Then $C_n(x)$ is also called Tchebychev polynomial of the first kind and it is well known $C_n(x)$ satisfies the following identities (see for example [**1, 13**]):

$$C_n(x + x^{-1}) = x^n + x^{-n}, \quad n \geqslant 0, \tag{1.3}$$

$$C_n(x) = \sum_{2k \leqslant n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \quad n > 0. \tag{1.4}$$

Hence, we have

$$\pi^n = C_n(\theta) - \pi^{-n}, \quad n \in Z. \tag{1.5}$$

Therefore, if $f \in \mathcal{F}$ and $n > 0$ then, as $\pi^{-n}(f) = 0$, we have $f_n = \pi^n(f) = C_n(\theta)(f) - \pi^n(f) = C_n(\theta)(f)$. Hence

$$f_n = \sum_{2k \leqslant n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \theta^{n-2k}(f) = \sum_{2k \leqslant n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} g_{n-2k}.$$

Thus we proved direction ($\Rightarrow$) of equivalence (1.1). The other direction follows from the following observation. The equalities $g_m = \sum_k \binom{m}{k} f_{m-2k}$, $m = 0, 1, \dots, n$, can be written as $P \cdot F = G$, where P is a regular triangular matrix, $G = [g_0, g_1, \dots, g_n]$ and $F = [f_0, f_1, \dots, f_n]$. Then the righthand side of the equivalence (1.1) is written as $F = Q \cdot G$ where $Q = P^{-1}$ and entries of Q are exactly coefficients appearing in expansion of f_n by g_{n-2k}. As from $F = Q \cdot G$ follows $P \cdot F = G$, it also follows the direction ($\Leftarrow$) in (1.1).

2. Hopf algebra of projection functions

We show that the algebra $\mathcal{A}$ discussed in the previous section and similar algebras naturally bear the structure of Hopf algebra. Even if the next definitions and analysis can be applied to an arbitrary field $\mathbf{K}$, we shall assume $\mathbf{K} = \mathbf{C}$. Let $I \subseteq C$, $I \neq \emptyset$ and A the subspace of the vector space of linear functionals of the complex space $\mathbf{C}^I$, generated by projection functions π_i, $i \in I$. Suppose that I is a subgroup of the additive group of $\mathbf{C}$ or of the multiplicative part C^* of $\mathbf{C}$. Assuming the usual notation for Hopf algebras and related notions (see for example [**4**] or [**18**]), it is easy to see that in both of the following two cases we obtain a Hopf algebra.

Additive case, I is a subgroup of $(C, +, 0)$. Hopf algebra $H_I = (A, \nabla, \mathbf{1}, \triangle, \varepsilon)$ over the complex field $\mathbf{C}$ is defined as follows: $\nabla(\pi_i \otimes \pi_j) = \pi_{i+j}$, in multiplicative notation $\pi_i \cdot \pi_j = \pi_{i+j}$, $\mathbf{1}(z) = \pi_0$, $z \in C$, $\triangle(\pi_i) = \pi_i \otimes \pi_i$ and $\varepsilon(\pi_i) = 0$, $i, j \in I$. The map $a\colon \pi_i \mapsto \pi_{-i}$, $i \in I$, is the antipod.

Multiplicative case, I is a subgroup of $(C^*, \cdot, 1)$. Hopf algebra $H_I = (A, \nabla, \mathbf{1}, \triangle, \varepsilon)$ over the complex field $\mathbf{C}$ is defined taking: $\nabla(\pi_i \otimes \pi_j) = \pi_{ij}$, in multiplicative notation $\pi_i \cdot \pi_j = \pi_{ij}$, $\mathbf{1}(z) = \pi_1$, $z \in C$, $\triangle(\pi_i) = \pi_i \otimes \pi_i$ and $\varepsilon(\pi_i) = 1$, $i, j \in I$. The map $a\colon \pi_i \mapsto \pi_{i^{-1}}$, $i \in I$, is the antipod.

Obviously, in both cases H_I is commutative and in fact H_I is a Hopf subalgebra of the dual Hopf algebra of the group Hopf algebra $\mathbf{C}[I]$. If $I = (Z, +, 0)$ we obtain algebra $\mathcal{A}$ presented in the previous section.

Even if the additive and the multiplicative cases are similarly defined, they may produce examples of quite different nature. In an additive case if we take $\pi = \pi_1$, we may write π^i instead of π_i and if I is the set of real numbers, then the ring $\boldsymbol{A}_I = (A, +, \cdot, \mathbf{0}, \mathbf{1})$ is an integral domain and it is isomorphic to the ring of polynomials over C in the variable π, see [**5**] and [**14**]. If I is the additive group of integers, then $\boldsymbol{A}_I = \mathbb{Z}[\pi, \pi^{-1}]$ is the ring of Laurent polynomials in the indeterminate π. On the other hand, in the multiplicative case, if $I = \langle\varepsilon\rangle$, $\varepsilon^n = 1$, then $\boldsymbol{A}_I$ has divisors of zero. For example, if ε is a primitive root of $x^3 - 1$ and $a = \mathbf{1} + \pi_\varepsilon + \pi_{\varepsilon^2}$, $b = \mathbf{1} + \varepsilon^2\pi_\varepsilon + \varepsilon\pi_{\varepsilon^2}$, then $ab = \mathbf{0}$.

Let $S \subseteq I$ and $S' = I \setminus S$. Then we can identify the space $\mathbf{C}^S$ with the subspace $\mathcal{F}_S = \{f \in C^Z \colon \bigwedge_{x\in S'} f(x) = 0\}$ of $\mathbf{C}^I$. In the example from the previous section, obviously S is the set of nonnegative integers. For deriving inversion formulas for functions $f\colon S \to C$, we shall use their replicas in $\mathcal{F}_S$. This derivation is related but not the same as that one in the umbral calculus, see for example [**3**]. We also note that Tchebychev polynomials and their variants have been already the subject of investigation in context of Hopf algebras and from the purely algebraic point of view, see for example [**2**] and [**6**].

3. Linear functional $\theta = \pi + \pi^{-r}$

Let us suppose notation and definitions as previously introduced. Here we shall consider linear functionals of $\mathbf{C}^Z$ of the form

$$\theta = \pi + \pi^{-r}, \quad r \text{ is a nonnegative integer,}$$

in the ring $\boldsymbol{A}_I = \mathbb{Z}[\pi, \pi^{-1}]$. Then for $m = r + 1$

$$\theta^n = \sum_k \binom{n}{k} \pi^{n-mk}, \quad n = 1, 2, \ldots. \tag{3.1}$$

We shall prove that π^n can be expressed as stated in the following theorem.

Theorem 3.1. *There are polynomials $P_n(x)$ and $Q_n(x)$ with integer coefficients such that*

$$\pi^n = S_n(\theta) - Q_n(\pi^{-1}), \quad Q_n(0) = 0. \tag{3.2}$$

Our main aim is to find explicitly polynomials $S_n(x)$ and $Q_n(x)$. For this purpose, we shall need some properties of symmetric functions related to the polynomial $p(x) = x^m + ax^{m-1} + b$, $a, b \in C$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of $p(x)$ and $s_n(a, b) = \lambda_1^n + \lambda_2^n + \cdots + \lambda_m^n$ the n^{th} power sum of the roots. Using Girard-Waring formula for symmetric functions, Gould (see [**9**] or [**12**]) derived the formula

$$s_n(a, b) = \sum_{0\leqslant k\leqslant n/m} (-1)^{n-rk} \frac{n}{n-rk} \binom{n-rk}{k} a^{n-mk} b^k, \quad m = r + 1. \tag{3.3}$$

Using this formula, the following proposition is easily deduced.

Proposition 3.1. *Let $m = r + 1$ and $u_n = s_n(-a, b)$, where $s_n(a, b)$ is defined by* (3.3)*. Then for all positive integers n*

$$u_n = \sum_{0\leqslant k\leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} a^{n-mk}b^k. \tag{3.4}$$

If $1 \leqslant n \leqslant r$, then $u_n = a^n$ and also $u_m = a^m - mb$. The sequence u_n with these initial conditions is the unique solution of the difference equation

$$v_{n+1} - av_n + bv_{n-r} = 0. \tag{3.5}$$

Proof. Identity (3.4) immediately follows from (3.3). For the second part of the proposition, we can write (3.4) as

$$u_n = a^n + \sum_{1\leqslant k\leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} a^{n-mk}b^k. \tag{3.6}$$

If $n \leqslant r$, then the sum in (3.6) is empty, hence $u_n = a^n$. Similarly,

$$u_m = a^m - mb + \sum_{2\leqslant k\leqslant n/m} (-1)^k \frac{m}{m-rk}\binom{m-rk}{k} a^{m-mk}b^k = a^m - mb.$$

The sequence u_n satisfies (3.5) since u_n is a linear combination of the roots of the characteristic equation of (3.5). The order of this recurrence is m and values for $u_1, u_2, \dots, u_m$ are determined, hence the uniqueness follows. □

Now we proceed to the proof of identity (3.2). For this purpose, we shall deliver recurrence relations for the polynomials $S_n(\theta)$ and $R_n(\pi)$.

Lemma 3.1. *Let r be a nonnegative integer, $m = r + 1$, $\theta = \pi + \pi^{-r}$ and $1 \leqslant l \leqslant r$. Then for the polynomials in* (3.2) *we can take:*

(a) *$S_l(\theta) = \theta^l$ and $S_m = \theta^m - m$.*
(b) *$Q_l(t) = t^{rl}((1+t^{-m})^l - t^{-ml})$, $Q_m(t) = t^{rm}((1+t^{-m})^m - t^{-m^2} - mt^{m-m^2})$.*

Proof. By identity (3.1), after short calculation we have

$$\pi^l = \theta^l - \sum_{k\geqslant 1}\binom{l}{k}\pi^{l-mk} = \theta^l - \pi^{-rl}((1+\pi^m)^l - \pi^{ml}).$$

Hence, $S_l(\theta) = \theta^l$ and $Q_l(\pi^{-1}) = \pi^{-rl}((1+\pi^m)^l - \pi^{ml})$ for $1 \leqslant l \leqslant r$. Taking $t = \pi^{-1}$, we have $Q_l(t) = t^{rl}((1+t^{-m})^l - t^{-ml})$.

According to identity (3.1) we also have

$$\pi^m = \theta^m - m - \sum_{k\geqslant 2}\binom{m}{k}\pi^{m-mk},$$

hence

$$\pi^m = (\theta^m - m) - \pi^{-rm}((1+\pi^m)^m - \pi^{m^2} - m\pi^{m^2-m}).$$

Therefore $S_m = \theta^n - m$ and $Q_m(\pi^{-1}) = \pi^{-rm}((1+\pi^m)^m - \pi^{m^2} - m\pi^{m^2-m})$. Taking $t = \pi^{-1}$, we have $Q_m(t) = t^{rm}((1+t^{-m})^m - t^{-m^2} - mt^{-m^2+m})$. □

Now we shall deliver the recursive relations for polynomials $S_n(t)$ and $Q_n(t)$ appearing in (3.2). In the following we shall take $m = r+1$.

Assuming identity (3.2), we have $\theta\pi^n = \theta S_n(\theta) - \theta R_n(\pi^{-1})$, hence

$$\begin{aligned}(\pi+\pi^{-r})\pi^n &= \theta S_n(\theta) - (\pi+\pi^{-r})Q_n(\pi^{-1}),\\ \pi^{n+1} &= \theta S_n(\theta) - \pi^{n-r} - (\pi+\pi^{-r})R_n(\pi^{-1}).\end{aligned} \tag{3.7}$$

Assuming recurrence (3.2) for $n-r$, i.e., $\pi^{n-r} = S_{n-r}(\theta) - Q_{n-r}(\pi^{-1})$, we obtain

$$\pi^{n+1} = \theta S_n(\theta) - S_{n-r}(\theta) - ((\pi+\pi^{-r})Q_n(\pi^{-1}) - Q_{n-r}(\pi^{-1})). \tag{3.8}$$

With regard to (3.2) we have $\pi^{n+1} = S_{n+1}(\theta) - Q_{n+1}(\pi^{-1})$ and comparing with (3.8), we have

$$\begin{aligned}S_{n+1}(\theta) &= \theta S_n(\theta) - S_{n-r}(\theta),\\ Q_{n+1}(\pi^{-1}) &= (\pi+\pi^{-r})Q_n(\pi^{-1}) - Q_{n-r}(\pi^{-1}).\end{aligned} \tag{3.9}$$

Taking the substitution $t = \pi^{-1}$ and using (3.9) it is easy to deduce that $Q_n(t)$ satisfies the recurrence

$$tQ_{n+1}(t) = (1+t^{r+1})Q_n(t) - tQ_{n-r}(t). \tag{3.10}$$

Lemma 3.2. *We have $Q_n(0) = 0$ for all $n > 0$.*

Proof. Let $m = r+1$. First assume $1 \leqslant n \leqslant r$. With regard to Lema 3.1 we have

$$\begin{aligned}Q_n(t) &= t^{rn}((1+t^{-m})^n - t^{-mn})\\ &= t^{rn}\Big(1 + \binom{n}{1}t^{-m} + \cdots + \binom{n}{n-1}t^{-m(n-1)}\Big).\end{aligned}$$

As $rn - m(n-1) = m-n > 0$, it follows that $t \mid Q_n(t)$. Further,

$$\begin{aligned}Q_m(t) &= t^{rm}((1+t^{-m})^m - t^{-m^2} - mt^{-m^2+m})\\ &= t^{rm}\Big(1 + \binom{m}{1}t^{-m} + \cdots + \binom{m}{m-2}t^{-m(m-2)}\Big).\end{aligned}$$

As $-m^2+2m+rm = m > 0$, it follows that $t \mid Q_n(t)$.

Therefore, we proved that $t \mid Q_n(t)$ for $n \leqslant m$, i.e., $Q_n(0) = 0$. For $n > m$ we use the recurrence (3.10). We see immediately that $t \mid Q_n(t)$. □

Corollary 3.1. *The constant term of $Q_n(t)$ is equal to 0.*

Proof of Theorem 3.1. The proof immediately follows by induction from recurrence relations 3.9, derivations 3.7, 3.8 and the previous corollary. □

Now we deliver the explicit forms of the polynomials $S_n(x)$ and $Q_n(t)$.

Proposition 3.2. *Let r be a positive integer, $m = r+1$ and assume a sequence of polynomials $S_n(x) \in \mathbb{Z}[x]$ satisfies:*

(a) $S_{n+1}(x) = xS_n(x) - S_{n-r}(x)$
(b) $S_l(x) = x^l$, $1 \leqslant l \leqslant r$, *and* $S_m(x) = x^m - m$.

Then for all positive integers n

$$S_n(x) = \sum_{0\leqslant k\leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} x^{n-mk}. \tag{3.11}$$

Proof. The characteristic equation of the recurrence (a) is

$$\lambda^m - x\lambda^{m-1} + 1 = 0. \tag{3.12}$$

It is easy to see that equation (3.12) has no multiple roots, hence if $\lambda_1, \lambda_2, \dots, \lambda_m$ are roots of (3.12) then

$$S_n(x) = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_m\lambda_m^n,$$

for some unique constants $c_1, c_2, \dots, c_m$. As any linear combination of the n-th powers of $\lambda_1, \lambda_2, \dots, \lambda_m$ satisfies recurrence (3.11), due to the initial conditions (b) and the uniqueness of the constants $c_1, c_2, \dots, c_m$, according to Proposition 3.1 we have $c_i = 1$, $i \leqslant m$, and

$$S_n(x) = s_n(-x, 1) = \sum_{0\leqslant k\leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} x^{n-mk}. \tag{3.13}$$ □

We note that for the most applications the explicit form 3.13 of the polynomials $S_n(t)$ and the recurrence 3.2 are sufficient. However, if one wants to find the polynomials $Q_n(t)$, it is possible and convenient to introduce new polynomials $R_n(t)$ and $h_n(t)$ with integer coefficients related to $Q_n(t)$ in the following way

$$\begin{aligned} Q_n(t^{-1}) &= t^{-rn}R_n(t), \quad n \in N,\\ R_n(t) &= h_n(t^m), \quad n \in N. \end{aligned} \tag{3.14}$$

We also note that the polynomials $S_n(x)$ are related to the so called incomplete polynomials, see [**17**], and to the orthogonal polynomials on the radial rays in the complex plane which were introduced by Milovanović in [**15**] and studied in details in [**16**].

It is easy to prove the following proposition.

Proposition 3.3. *Let* r *be a nonnegative integer,* $m = r+1$, $\theta = \pi + \pi^{-r}$ *and* $1 \leqslant l \leqslant r$. *Then*

(a) $R_l(\pi) = (1+\pi^m)^l - \pi^{ml}$, $R_m(\pi) = (1+\pi^m)^m - \pi^{m^2} - m\pi^{m^2-m}$.
(b) $\deg(R_n(\pi)) < rn$, $n \in N$.
(c) $h_{n+1}(t) = (1+t)h_n(x) - t^r h_{n-r}(x)$
(d) $h_l(x) = (1+t)^l - t^l$, $1 \leqslant l \leqslant r$, *and* $h_m(x) = (1+t)^m - t^m - mt^{m-1}$.

From the next theorem and relations 3.14 immediately follows the explicit form of the polynomial $Q_n(t)$.

Theorem 3.2. *Assume that a sequence of polynomials* $h_n(t) \in \mathbb{Z}[t]$ *satisfies conditions* (c) *and* (d) *in the previous proposition. Then for all positive integers* n

$$h_n(t) = f_n(t) + g_n(t), \tag{3.15}$$

where

$$f_n(t) = (t+1)^n - t^n, \quad g_n(t) = \sum_{1 \leqslant k \leqslant n/m} (-1)^k \frac{n}{n-rk} \binom{n-rk}{k} (1+t)^{n-mk} t^{rk}.$$

PROOF. The characteristic equation of the recurrence (a) is

$$\lambda^m - (1+t)\lambda^{m-1} + t^r = 0.$$

The roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ of this equation are distinct and its one root is t, so we can take $\lambda_{r+1} = \lambda_m = t$. Let $u_n = \lambda_1^n + \lambda_2^n + \cdots + \lambda_r^n$ and $s_n = u_n + \lambda_m^n = u_n + t^n$. From Proposition 3.1, $s_n = (1+t)^n$ for $1 \leqslant n \leqslant r$ and $s_m = (1+t)^m - mt^{m-1}$. Hence, $u_n = (1+t)^n - t^n$ for $1 \leqslant n \leqslant r$ and $u_m = (1+t)^m - t^m - mt^{m-1}$. Also, u_n obviously satisfies the recurrence $u_{n+1}(t) = (1+t)u_n(t) - t^r u_{n-r}(t)$. Therefore, due to the the same initial conditions for u_n and h_n, by induction we have immediately $u_n = h_n$ for all n, i.e., $h_n(t) = s_n - t^n$. From Proposition 3.1, we have

$$s_n = \sum_{0 \leqslant k \leqslant n/m} (-1)^k \frac{n}{n-rk} \binom{n-rk}{k} (1+t)^{n-mk} t^{rk}.$$

wherefrom we immediately deliver (3.15). □

4. Gould inversion formula

As an application of the projection calculus and a the operator θ introduced in the previous section, we prove the Gould inversion formula, see [**7**] or [**8**]. This formula is a generalization of inversion formula 1.1 and it states

$$g_n = \sum_k \binom{n}{k} f_{n-mk} \ \Leftrightarrow \ f_n = \sum_{0 \leqslant k \leqslant n/m} (-1)^k \frac{n}{n-rk} \binom{n-rk}{k} g_{n-mk},$$

where $f, g \in \mathcal{F}$, r is a nonnegative integer and $m = r+1$. For the proof we use the same technique as in the case $r = 1$ where we used projection calculus, Tchebychev polynomials and their crucial properties 1.3 and 1.4. By use of this technique, the proof of the Gould formula directly follows from Theorem 3.1 and the explicit form 3.13 of the polynomial $S_n(x)$.

Using linear functional identities (3.1) and (3.1) and explicit forms of the polynomials $S_n(\theta)$ and $Q_n(\pi^{-1})$, we can generalize the Gould inversion formula. Namely, we can find new inversion formulas for

$$g_n = \sum_k \binom{n}{k} f_{n-mk+l}, \quad 0 \leqslant l < m. \tag{4.1}$$

It is particularly simple to deliver the inversion formula for the case $m = 2$. To see this, assume $m = 2$ and let us introduce the new linear functional $\sigma_n = \pi\theta^n$. Then $\sigma_n = \sum_k \binom{n}{k} \pi^{n-2k+1}$. By (1.5), i.e., inversion formula $\pi^n = C_n(\theta) - \pi^{-n}$, we have $\pi^{n+1} = \pi C_n(\theta) - \pi^{1-n}$ and so

$$\pi^{n+1} = \sum_k (-1)^k \frac{n}{n-k} \binom{n-k}{k} \sigma_{n-2k+1} - \pi^{1-n}.$$

Hence we obtain the following inversion formula for (4.1), case $m = 2$, $l = 1$:

$$f_{n+1} = \sum (-1)^k \frac{n}{n-k}\binom{n-k}{k} g_{n-2k+1} - f_{1-n}. \tag{4.2}$$

We note that for a given sequence $g \in \mathcal{F}$, the sequence f is not uniquely determined by (4.2) as indices in g_i are shifted by one:

$$g_0 = f_1,\ g_1 = f_2 + f_0,\ g_2 = f_3 + 2f_1, \dots$$

while

$$f_1 = g_0,\ f_2 = g_1 - f_0,\ f_3 = g_2 - 2g_0, \dots$$

where f_0 is arbitrary. Also note that f_{1-n} vanishes for $n > 1$. In a similar manner we can obtain the inversion formula for (4.1) in the general case. So assume (4.1) and let us introduce the linear functional $\sigma_n = \pi^l \theta^n$. From Theorem 3.1 and Proposition 3.2 we have

$$\begin{aligned} \pi^{n+l} &= \pi^l S_n(\theta) - \pi^l Q_n(\pi^{-1}) \\ &= \sum_{0 \leqslant k \leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} \sigma_{n-mk+l} - \pi^l Q_n(\pi^{-1}). \end{aligned}$$

By (3.14), Lemma 3.2 and as $Q_n(0) = 0$

$$Q_n(t^{-1}) = t^{-rn} h_n(t^m) = c_n t^{-\lambda_n} + c'_n t^{-\lambda_n - m} + c''_n t^{-\lambda_n - 2m} + \cdots$$

where $1 \leqslant \lambda_n \leqslant m$. Hence $\pi^l Q_n(\pi^{-1}) = c_n \pi^{l-\lambda_n} + H_n(\pi^{-1})$ for some polynomial $H_n(t)$ with integer coefficients. Therefore we obtain the inversion formula for (4.1), $0 \leqslant l < m$:

$$f_{n+l} = \sum_{0 \leqslant k \leqslant n/m} (-1)^k \frac{n}{n-rk}\binom{n-rk}{k} g_{n-mk+l} - c_n f_{l-\lambda_n}.$$

We see that $f_0, f_1, \dots, f_l$ can be chosen arbitrarily. The coefficient c_n can be obtained from the representation of the polynomial h_n given by Theorem 3.2. Here we shall find the power λ_n:

Proposition 4.1. *We have $\lambda_n = m - \rho_m(n)$ where $\rho_m(n)$ is the remainder of division of n by m (remainder function).*

Proof. Note that $m = r + 1$. As $Q_n(t) = t^{rn} h_n(t^{-m})$ and $Q_n(0) = 0$, for powers t^{rn-lm} of terms in $Q_n(t)$ we have $rn - lm > 0$, so $l < rn/m$. First assume $m \mid n$. Then for the smallest power $t^{\lambda_n} = t^{rn-\bar{l}n}$ in $Q_n(t)$ we have $\bar{l} = rn/m - 1$, so $\lambda_n = rn - \bar{l}m = m = m - \rho_m(n)$. Assume $m \nmid n$. Then for $\bar{l} = [rn/m]$ we have $\bar{l} = [n - n/m] = n - 1 - [n/m]$, so $\lambda_n = rn - \bar{l}m = m - \rho_m(n)$. $\square$

In the similar way we can deliver various inversion formulas such as appearing in [**11**] by studying associated functionals in the Hopf algebra $\mathcal{A}$. For example, for delivering the inverse formula for $g_n = f_n + f_{n-1} + f_{n-2}$, see [**10**], one may use the functional $\theta = \pi + \pi^{-1} + \pi^{-2}$.

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