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THE INDEX OF PRODUCT SYSTEMS OF HILBERT MODULES: TWO EQUIVALENT DEFINITIONS

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Abstract. We prove that a conditionally completely positive definite kernel, as the generator of completely positive definite (CPD) semigroup associated with a continuous set of units for a product system over a C^*-algebra $\mathcal{B}$, allows a construction of a Hilbert $\mathcal{B}-\mathcal{B}$ module. That construction is used to define the index of the initial product system. It is proved that such definition is equivalent to the one previously given by Kečkić and Vujošević [*On the index of product systems of Hilbert modules*, Filomat, to appear, ArXiv:1111.1935v1 [math.OA] 8 Nov 2011]. Also, it is pointed out that the new definition of the index corresponds to the one given earlier by Arveson (in the case $\mathcal{B} = \mathbb{C}$).

1. Introduction

Product systems over $\mathbb{C}$ have been studied during last several decades in connection with E_0-semigroups acting on a type I factor. Although the main problem of classification of all nonisomorphic product systems is still open, this theory is well developed. The reader is referred to [**2**] and references therein. In the present century there are some significant results that generalize this theory to product systems over a C^*-algebra $\mathcal{B}$, either in connection with E_0 semigroups (see [**8, 10**]) or in connection with quantum probability dynamics (see [**4, 3, 9**]).

There are many difficulties in generalizing the notion of the index of a product system introduced in [**1**] to this more general concept. Up to our knowledge there are attempts in this direction done in [**11**] and recently in [**5**].

Here we give another definition of the index of product systems of Hilbert $\mathcal{B}$–$\mathcal{B}$ modules and show that it is equivalent to the one previously given in [**5**]. Also, we point out that the new definition of index corresponds to the one given by Arveson (in the case $\mathcal{B} = \mathbb{C}$).

Throughout the paper $\mathcal{B}$ will denote a unital C^*-algebra and 1 will denote its unit.

The rest of Section 1 is devoted to basic definitions.

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DEFINITION 1.1. a) A product system over C^*-algebra $\mathcal{B}$ is a family $(E_t)_{t\geqslant 0}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$ modules, with $E_0 \cong \mathcal{B}$, and a family of (unitary) isomorphisms $\varphi_{t,s}$: $E_t \otimes E_s \to E_{t+s}$, where $\otimes$ stands for the so called inner tensor product obtained by identifications $ub\otimes v \sim u\otimes bv$, $u\otimes vb \sim (u\otimes v)b$, $bu\otimes v \sim b(u\otimes v)$, ($u \in E_t$, $v \in E_s$, $b \in \mathcal{B}$) and then completing in the inner product $\langle u \otimes v, u_1 \otimes v_1\rangle = \langle v, \langle u, u_1\rangle\, v_1\rangle$;

b) Unit on E is a family $u_t \in E_t$, $t \geqslant 0$, such that $u_0 = 1$ and $\varphi_{t,s}(u_t \otimes u_s) = u_{t+s}$, which will be abbreviated to $u_t\otimes u_s = u_{t+s}$. A unit u_t is unital if $\langle u_t, u_t\rangle = 1$. It is central if for all $b \in \mathcal{B}$ and all $t \geqslant 0$ there holds $bu_t = u_tb$;

DEFINITION 1.2. Two units u_t and v_t give rise to the family of mappings $\mathcal{K}_t^{u,v}$: $\mathcal{B} \to \mathcal{B}$, given by $\mathcal{K}_t^{u,v}(b) = \langle u_t, bv_t\rangle$. All $\mathcal{K}_t^{u,v}$ are bounded $\mathbb{C}$-linear operators on $\mathcal{B}$, and this family forms a semigroup. The set of units S is continuous if the corresponding semigroup $(K_t^{\xi,\eta})_{\xi,\eta\in S}$ (with respect to Schur multiplying) is uniformly continuous. A single unit u_t is uniformly continuous, or briefly just continuous, if the set $\{u\}$ is continuous, that is, the corresponding family $\mathcal{K}_t^{u,u}$ is continuous in the norm of the space $B(\mathcal{B})$ (the algebra of all bounded $\mathbb{C}$-linear operators on $\mathcal{B}$).

As it can be seen in [**3**], for a (uniformly) continuous set of units $\mathcal{U}$, there can be formed a uniformly continuous completely positive definite semigroup (CPD-semigroup further on) $\mathcal{K} = (\mathcal{K}_t)_{t\in\mathbb{R}_+}$.

Denote by $\mathcal{L} = \frac{d}{dt}\mathcal{K}\mid_{t=0}$ the generator of CPD-semigroup $\mathcal{K}$. It is well known [**3**] that $\mathcal{L}$ is conditionally completely positive definite, that is, for all finite n-tuples $x_1, \dots, x_n \in \mathcal{U}$ and for all a_j, $b_j \in \mathcal{B}$ there holds

$$\sum_{j=1}^{n} a_jb_j = 0 \Longrightarrow \sum_{i,j=1}^{n} b_i^*\mathcal{L}^{x_i,x_j}(a_i^*a_j)b_j \geqslant 0. \tag{1.1}$$

Also,

$$\mathcal{L}^{y,x}(b) = \mathcal{L}^{x,y}(b^*)^*. \tag{1.2}$$

It is known that $\mathcal{K}$ is uniquely determined by $\mathcal{L}$. More precisely, $\mathcal{K}$ can be recovered from $\mathcal{L}$ by $\mathcal{K} = e^{t\mathcal{L}}$ using the Schur product, i.e., $\mathcal{K}_t^{x,y}(b) = \langle x_t, by_t\rangle = (\exp t\mathcal{L}^{x,y})(b)$.

REMARK 1.1. One should distinguish the continuous set of units from the set of continuous units. In the second case only $\mathcal{K}_t^{\xi,\xi}$ should be uniformly continuous for $\xi \in S$, whereas in the first case all $\mathcal{K}_t^{\xi,\eta}$ should be uniformly continuous.

In Section 2 the auxiliary statements, that are necessary for the proofs of the main result, are listed. In Section 3 another definition of the index of product systems of Hilbert $\mathcal{B} - \mathcal{B}$ modules is obtained and the equivalency with the one previously given in [**5**] is proved. Also, it is pointed out that the new definition of the index corresponds to the one given by Arveson (in the case $\mathcal{B} = \mathbb{C}$).

2. Preliminary results

In [**6**], Liebscher and Skeide introduce an interesting way to obtain new units in a given product system. The results are stated in Lemma 3.1, Proposition 3.3 and Lemma 3.4 of the mentioned paper and here they are quoted as

PROPOSITION 2.1. a) *Suppose that a continuous set S of units generates a product system E. Let $t \mapsto y_t \in E_t$ be a mapping (not necessarily unit), with K and $K_\xi \in B(\mathcal{B})$ $(\xi \in S)$ such that for all $b \in \mathcal{B}$*

$$\langle y_t, b y_t\rangle = b + tK(b) + O(t^2) \quad \textit{and} \quad \langle y_t, b\xi_t\rangle = b + tK_\xi(b) + O(t^2).$$

Then there exists a product system $F \supseteq E$ and a unit ζ such that $S \cup \{\zeta\}$ is continuous and $\mathcal{L}^{\zeta,\zeta} = K$, $L^{\zeta,\xi} = K_\xi$.

b) *The following three conditions are equivalent.*

(1) *$\zeta \in E$;*
(2) *ζ can be obtained as the norm limit of the sequence $(y_{t/n})^{\otimes n}$;*
(3) *$\lim_{n\to\infty}\langle \zeta_t, (y_{t/n})^{\otimes n}\rangle = \langle \zeta_t, \zeta_t\rangle$.*

REMARK 2.1. In [**6**], a more general limit over the filter of all partitions of segment $[0,t]$ was considered instead of $\lim_{n\to\infty}(y_{t/n})^{\otimes n}$. However, such a general context is not necessary here.

The previous proposition is used in [**5**, Proposition 2.3] to obtain new units in a product system in the following way.

Suppose that a continuous set S of units generates a product system E. Let $x^j \in S$, $\varkappa_j \in \mathcal{B}$, $j = 1, \dots, n$ such that $\sum \varkappa_j = 1$. Then the functions $t \mapsto \sum_{j=1}^n \varkappa_j x_t^j$ and $t \mapsto \sum_{j=1}^n x_t^j \varkappa_j$ satisfy all the assumptions of Proposition 2.1 and the resulting units, denoted by $\varkappa_1 x^1 \boxplus \cdots \boxplus \varkappa_n x^n$ and $x^1\varkappa_1 \boxplus \cdots \boxplus x^n\varkappa_n$, belong to E. For example, the kernels of $\zeta = \varkappa_1 x^1 \boxplus \varkappa_2 x^2 \boxplus \varkappa_3 x^3$ are

$$\begin{aligned}\mathcal{L}^{\zeta,\zeta} = {} & \mathcal{L}^{x^1,x^1} L_{\varkappa_1^*} R_{\varkappa_1} + \mathcal{L}^{x^1,x^2} L_{\varkappa_1^*} R_{\varkappa_2} + \mathcal{L}^{x^1,x^3} L_{\varkappa_1^*} R_{\varkappa_3} + \mathcal{L}^{x^2,x^1} L_{\varkappa_2^*} R_{\varkappa_1} \\ & + \mathcal{L}^{x^2,x^2} L_{\varkappa_2^*} R_{\varkappa_2} + \mathcal{L}^{x^2,x^3} L_{\varkappa_2^*} R_{\varkappa_3} + \mathcal{L}^{x^3,x^1} L_{\varkappa_3^*} R_{\varkappa_1} \\ & + \mathcal{L}^{x^3,x^2} L_{\varkappa_3^*} R_{\varkappa_2} + \mathcal{L}^{x^3,x^3} L_{\varkappa_3^*} R_{\varkappa_3}, \\ \mathcal{L}^{\zeta,\xi} = {} & \mathcal{L}^{x^1,\xi} L_{\varkappa_1^*} + \mathcal{L}^{x^2,\xi} L_{\varkappa_2^*} + \mathcal{L}^{x^3,\xi} L_{\varkappa_3^*},\end{aligned} \tag{2.1}$$

where $L_b, R_b : \mathcal{B} \to \mathcal{B}$ are the left and the right multiplication operators for $b \in \mathcal{B}$. Proposition 3.1 from [**5**] is quoted here as

PROPOSITION 2.2. *Let $\mathcal{U}$ be the set of all continuous units on a product system E. The relation ρ on $\mathcal{U}$ defined by*

$$x\,\rho\,y \Leftrightarrow \{x, y\} \textit{ is a continuous set}$$

is an equivalence relation.

Thus, the set of all continuous units on some product system can be decomposed into mutually disjoint collection of maximal continuous sets.

Let E be a product system over a unital C^*-algebra $\mathcal{B}$ with at least one continuous unit. (In view of [**9**, Definition 4.4] this means that E is non type III product system.) Further, let ω be an arbitrary continuous unit in E and let $\mathcal{U} = \mathcal{U}_\omega$ be the set of all uniformly continuous units that are equivalent to ω. (That refers to the equivalence relation ρ on $\mathcal{U}$ defined in Proposition 2.2.) As it can be seen in [**5**], the addition and multiplication by $b \in \mathcal{B}$ on $\mathcal{U}_\omega$ are defined by

$$x + y = x \boxplus y \boxplus -\omega, \quad b \cdot x = bx \boxplus (1-b)\omega, \quad x \cdot b = xb \boxplus \omega(1-b), \tag{2.2}$$

and the kernels of $x + y$, $x \cdot a$, $a \cdot x$ are

$$\begin{aligned} \mathcal{L}^{x+y,\,x+y} &= \mathcal{L}^{x,\,x} + \mathcal{L}^{x,\,y} - \mathcal{L}^{x,\,\omega} + \mathcal{L}^{y,\,x} + \mathcal{L}^{y,\,y} - \mathcal{L}^{y,\,\omega} - \mathcal{L}^{\omega,\,x} - \mathcal{L}^{\omega,\,y} + \mathcal{L}^{\omega,\,\omega}, \\ \mathcal{L}^{x+y,\,\xi} &= \mathcal{L}^{x,\,\xi} + \mathcal{L}^{y,\,\xi} - \mathcal{L}^{\omega,\,\xi}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{L}^{x\cdot a,x\cdot a} &= a^*\mathcal{L}^{x,x}a + (1-a)^*\mathcal{L}^{\omega,x}a + a^*\mathcal{L}^{x,\omega}(1-a) + (1-a)^*\mathcal{L}^{\omega,\omega}(1-a), \\ \mathcal{L}^{x\cdot a,\xi} &= a^*\mathcal{L}^{x,\xi} + (1-a)^*\mathcal{L}^{\omega,\xi}, \qquad \xi \in \mathcal{U}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathcal{L}^{a\cdot x,a\cdot x} &= \mathcal{L}^{x,x}L_{a^*}R_a + \mathcal{L}^{\omega,x}L_{1-a^*}R_a + \mathcal{L}^{x,\omega}L_{a^*}R_{1-a} + \mathcal{L}^{\omega,\omega}L_{1-a^*}R_{1-a}, \\ \mathcal{L}^{a\cdot x,\xi} &= \mathcal{L}^{x,\xi}L_{a^*} + \mathcal{L}^{\omega,\xi}L_{1-a^*}, \qquad \xi \in \mathcal{U}, \end{aligned} \tag{2.5}$$

where $L_b, R_b : \mathcal{B} \to \mathcal{B}$ are the left and right multiplication operators for $b \in \mathcal{B}$.

Remark 2.2. For $x, y \in \mathcal{U}_\omega$, $x - y = x \boxplus (-y) \boxplus \omega$.

According to [**5**, Theorem 3.2], the set $\mathcal{U}$ with respect to the operations defined by (2.2) is a left–right $\mathcal{B}$–$\mathcal{B}$ module.

In [**5**] it was proved that the mapping $\langle\cdot,\cdot\rangle_1 : \mathcal{U} \times \mathcal{U} \to \mathcal{B}$ given by

$$\langle x, y\rangle_1 = (\mathcal{L}^{x,y} - \mathcal{L}^{x,\omega} - \mathcal{L}^{\omega,y} + \mathcal{L}^{\omega,\omega})(1) \tag{2.6}$$

(ω is the same as in (2.2)) is a $\mathcal{B}$-valued semi-inner product (in the sense that it can be degenerate, i.e., $\langle x, x\rangle_1 = 0$ need not imply $x = 0$) and that it satisfies all the customary properties:

(a) For all $x, y, z \in \mathcal{U}$, and $\alpha, \beta \in \mathbb{C}$ $\langle x, \alpha y + \beta z\rangle_1 = \alpha \langle x, y\rangle_1 + \beta \langle x, z\rangle_1$;
(b) For all $x, y \in \mathcal{U}$, $a \in \mathcal{B}$ $\langle x, y \cdot a\rangle_1 = \langle x, y\rangle_1 a$;
(c) For all $x, y \in \mathcal{U}$ $\langle x, y\rangle_1 = \langle y, x\rangle_1^*$;
(d) For all $x \in \mathcal{U}$ $\langle x, x\rangle_1 \geqslant 0$.

Also, the set $N = \{x \in \mathcal{U} \mid \langle x, x\rangle_1 = 0\}$ is a submodule of $\mathcal{U}$ and $\mathcal{U}/N$ is a pre-Hilbert left–right $\mathcal{B}$–$\mathcal{B}$ module.

3. The result

The definition of the index of the product system with at least one continuous unit, given in [**5**], is quoted here as

Definition 3.1. Let E be a product system, and let ω be a continuous unit on E. The index of a pair (E, ω) is the completion of pre-Hilbert left–right module $\mathcal{U}/_\sim$, where $\mathcal{U} = \mathcal{U}_\omega$ is the maximal continuous set of units containing ω, and $\sim$ is the equivalence relation defined by $x \sim y$ if and only if $x - y \in N$ where N is the set mentioned at the end of Section 2. Naturally, the index will be denoted by $\mathrm{ind}(E, \omega)$.

REMARK 3.1. If $\{\omega, \omega'\}$ is a continuous set, then $\mathrm{ind}(E,\omega) \cong \mathrm{ind}(E,\omega')$. Indeed, then $\mathcal{U}_\omega = \mathcal{U}_{\omega'}$ and the isometric isomorphism is given by translation $x \mapsto x \boxplus -\omega \boxplus \omega'$. Therefore, $\mathrm{ind}(E,\omega)$ is independent on the choice of ω in the same continuous set of units.

The index of the product systems with at least one continuous unit may also be defined in a different way and we prove that these two definitions are equivalent. In detail, let E be a product system and let $\mathcal{U}$ be a continuous set of units in E. Consider the $\mathcal{B}$-bimodule $\mathcal{BUB}$ where $\mathcal{BUB}$ is the set of all formal sums $\sum_i a_i x_i b_i$, $x_i \in \mathcal{U}$, $a_i, b_i \in \mathcal{B}$ with identification subject to the relations

$$(\lambda a)xb \sim ax(\lambda b)\ (\lambda \in \mathbb{C}),\ (a_1 + a_2)xb \sim a_1xb + a_2xb,\ ax(b_1 + b_2) \sim axb_1 + axb_2.$$

For $c \in \mathcal{B}$, $\big(\sum_i a_i x_i b_i\big)c = \sum_i a_i x_i (b_i c)$ and $c\big(\sum_i a_i x_i b_i\big) = \sum_i (ca_i)x_i b_i$. Also consider $\mathcal{B}$-subbimodule $(\mathcal{BUB})_0 = \big\{\sum_i a_i x_i b_i \in \mathcal{BUB} \mid \sum_i a_i b_i = 0\big\}$ and define the map $\langle\cdot,\cdot\rangle : (\mathcal{BUB})_0 \times (\mathcal{BUB})_0 \to \mathcal{B}$ by

$$\Big\langle \sum_i a_i x_i b_i, \sum_j a'_j x'_j b'_j \Big\rangle = \sum_{i,j} b_i^* \mathcal{L}^{x_i, x'_j}(a_i^* a'_j) b'_j. \tag{3.1}$$

LEMMA 3.1. *The map* (3.1) *satisfies the following properties:*

(a) *For all* $a_i, b_i, c_i, c'_i, d_i, d'_i \in \mathcal{B},\ x_i, y_i, y'_i \in \mathcal{U},\ \alpha, \beta \in \mathbb{C}$

$$\Big\langle \sum_i a_i x_i b_i, \alpha \sum_i c_i y_i d_i + \beta \sum_i c'_i y'_i d'_i \Big\rangle = \alpha \Big\langle \sum_i a_i x_i b_i, \sum_i c_i y_i d_i \Big\rangle + \beta \Big\langle \sum_i a_i x_i b_i, \sum_i c'_i y'_i d'_i \Big\rangle;$$

(b) *For all* $a_i, a'_i, b_i, b'_i \in \mathcal{B},\ x_i, x'_i \in \mathcal{U},\ c \in \mathcal{B}$

$$\Big\langle \sum_i a_i x_i b_i, \Big(\sum_i a'_i x'_i b'_i\Big) c \Big\rangle = \Big\langle \sum_i a_i x_i b_i, \sum_i a'_i x'_i b'_i \Big\rangle c;$$

(c) *For all* $a_i, a'_i, b_i, b'_i \in \mathcal{B},\ x_i, x'_i \in \mathcal{U}$

$$\Big\langle \sum_i a_i x_i b_i, \sum_i a'_i x'_i b'_i \Big\rangle = \Big\langle \sum_i a'_i x'_i b'_i, \sum_i a_i x_i b_i \Big\rangle^*;$$

(d) *For all* $a_i, b_i \in \mathcal{B},\ x_i \in \mathcal{U}$ $\big\langle \sum_i a_i x_i b_i, \sum_i a_i x_i b_i \big\rangle \geqslant 0$.

PROOF. (a), (b) are easy to check. For (c) use (1.2) and (d) follows since $\mathcal{L}$ is conditionally CPD (1.1). □

From the previous lemma, the Causchy–Schwartz inequality can be derived (see [**7**, Proposition 1.2.4]):

$$\Big\langle \sum_i a_i x_i b_i, \sum_i a'_i x'_i b'_i \Big\rangle \Big\langle \sum_i a_i x_i b_i, \sum_i a'_i x'_i b'_i \Big\rangle^* \leqslant \Big\langle \sum_i a_i x_i b_i, \sum_i a_i x_i b_i \Big\rangle \Big\| \Big\langle \sum_i a'_i x'_i b'_i, \sum_i a'_i x'_i b'_i \Big\rangle \Big\|.$$

It follows that the set $\mathcal{N} = \big\{\sum_i a_i x_i b_i \in (\mathcal{BUB})_0 \mid \big\langle \sum_i a_i x_i b_i, \sum_i a_i x_i b_i\big\rangle = 0\big\}$ is equal to $\big\{\sum_i a_i x_i b_i \in (\mathcal{BUB})_0 \mid \forall \sum_i a_i' x_i' b_i' \in (\mathcal{BUB})_0, \big\langle \sum_i a_i x_i b_i, \sum_i a_i' x_i' b_i'\big\rangle = 0\big\}$. So, $\mathcal{N}$ is a submodule of $(\mathcal{BUB})_0$ and $(\mathcal{BUB})_0/\mathcal{N}$ is a pre-Hilbert left–right $\mathcal{B}$–$\mathcal{B}$ module.

THEOREM 3.1. *Let E be a product system over a unital C^*-algebra $\mathcal{B}$. Let ω be an arbitrary continuous unit in E and let $\mathcal{U}$ be the maximal continuous set of units containing ω. The mapping $f : \mathcal{U}/_\sim \to (\mathcal{BUB})_0/\mathcal{N}$ defined by $f([y]) = y - \omega + \mathcal{N}$ is an isomorphism between pre-Hilbert $\mathcal{B}$–$\mathcal{B}$ module $\mathcal{U}/_\sim$ introduced in Definition* 3.1 *and pre-Hilbert $\mathcal{B}$–$\mathcal{B}$ module $(\mathcal{BUB})_0/\mathcal{N}$.*

PROOF. Let $y, y' \in \mathcal{U}$ and $y \sim y'$, i.e., $\langle y - y', y - y'\rangle_1 = 0$ (the substraction is as in Remark 2.2). For $1y1 + (-1)y'1 \in (\mathcal{BUB})_0$ we also write $y - y' \in (\mathcal{BUB})_0$. By (3.1) there holds

$$\langle y - y', y - y'\rangle = \mathcal{L}^{y,y}(1) - \mathcal{L}^{y,y'}(1) - \mathcal{L}^{y',y}(1) + \mathcal{L}^{y',y'}(1),$$

and also, by (2.6) and (2.1),

$$\begin{aligned}\langle y - y', y - y'\rangle_1 &= \langle y \boxplus (-y') \boxplus \omega, y \boxplus (-y') \boxplus \omega\rangle_1\\ &= \mathcal{L}^{y,y}(1) - \mathcal{L}^{y,y'}(1) - \mathcal{L}^{y',y}(1) + \mathcal{L}^{y',y'}(1).\end{aligned}$$

Therefore, $y - y' \in \mathcal{N}$ which means that f is well defined. Let $[y], [z] \in \mathcal{U}/_\sim$.

$$\begin{aligned}\langle f([y]), f([z])\rangle_{(\mathcal{BUB})_0/\mathcal{N}} &= \langle y - \omega + \mathcal{N}, z - \omega + \mathcal{N}\rangle_{(\mathcal{BUB})_0/\mathcal{N}} = \langle y - \omega, z - \omega\rangle\\ &= \mathcal{L}^{y,z}(1) - \mathcal{L}^{\omega,z}(1) - \mathcal{L}^{y,\omega}(1) + \mathcal{L}^{\omega,\omega}(1)\\ &= \langle y, z\rangle_1 = \langle [y], [z]\rangle_{\mathcal{U}/_\sim},\end{aligned}$$

so f is an isometry. For the surjectivity of f, it needs to be proved that for all $\sum_i a_i x_i b_i + \mathcal{N}$ in $(\mathcal{BUB})_0/\mathcal{N}$ there exists $[y] \in \mathcal{U}/_\sim$ such that $\sum_i a_i x_i b_i - y + \omega \in \mathcal{N}$. The mapping $t \mapsto \omega_t + \sum_i a_i x_{i,t} b_i$ satisfies all the assumptions of Proposition 2.1 and let us denote the resulting unit by ζ. The kernels of ζ are given by

$$\mathcal{L}^{\zeta,\zeta}(b) = \mathcal{L}^{\omega,\omega}(b) + \sum_i b_i^* \mathcal{L}^{x_i,\omega}(a_i^* b) + \sum_i \mathcal{L}^{\omega,x_i}(b a_i) b_i + \sum_{i,j} b_i^* \mathcal{L}^{x_i,x_j}(a_i^* b a_j) b_j,$$

$$\mathcal{L}^{\zeta,\xi}(b) = \mathcal{L}^{\omega,\xi}(b) + \sum_i b_i^* \mathcal{L}^{x_i,\xi}(a_i^* b), \quad \xi \in \mathcal{U},\ b \in \mathcal{B}. \tag{3.2}$$

By (3.1), (3.2), (1.2) it follows $\big\langle \sum_i a_i x_i b_i - \zeta + \omega, \sum_i a_i x_i b_i - \zeta + \omega\big\rangle = 0$. Therefore, $\sum_i a_i x_i b_i - \zeta + \omega \in \mathcal{N}$ and $f([\zeta]) = \sum_i a_i x_i b_i + \mathcal{N}$. Let $[x], [y] \in \mathcal{U}/_\sim$. Denote $\zeta = x + y \in \mathcal{U}$ (the addition is as in (2.2)). By (3.1), (2.3), (1.2) it follows that $\langle \zeta - x - y + \omega, \zeta - x - y + \omega\rangle = 0$ which means $\zeta - x - y + \omega \in \mathcal{N}$. Therefore,

$$f([x] + [y]) = f([x + y]) = \zeta - \omega + \mathcal{N} = x - \omega + y - \omega + \mathcal{N} = f([x]) + f([y]).$$

Let $[x] \in \mathcal{U}/_\sim$ and $b \in \mathcal{B}$. Denote $\eta = x \cdot b \in \mathcal{U}$ and $\mu = b \cdot x \in \mathcal{U}$ (the multiplication is as in (2.2)). By (3.1), (2.4), (2.5), (1.2) it follows that

$$\begin{aligned}\langle \eta - \omega - xb + \omega b, \eta - \omega - xb + \omega b\rangle &= 0,\\ \langle \mu - \omega - bx + b\omega, \mu - \omega - bx + b\omega\rangle &= 0,\end{aligned}$$

hence $\eta - \omega - xb + \omega b \in \mathcal{N}$ and $\mu - \omega - bx + b\omega \in \mathcal{N}$. Therefore,

$$f([x]\cdot b) = f([x\cdot b]) = \eta - \omega + \mathcal{N} = xb - \omega b + \mathcal{N} = f([x])b,$$
$$f(b\cdot [x]) = f([b\cdot x]) = \mu - \omega + \mathcal{N} = bx - b\omega + \mathcal{N} = bf([x]). \qquad \square$$

Corollary 3.1. *Let E be a product system over a unital C^*-algebra $\mathcal{B}$. Let ω be an arbitrary continuous unit in E and let $\mathcal{U}$ be the maximal continuous set of units containing ω. The index of E may also be defined as the completion of pre-Hilbert left-right $\mathcal{B}-\mathcal{B}$ module $(\mathcal{B}\mathcal{U}\mathcal{B})_0/\mathcal{N}$.*

Remark 3.2. Let E be an Arveson product system, i.e., E is a product system with $\mathcal{B} = \mathbb{C}$, and let $\mathcal{U}$ be the set of its units. As it can be found in [**2**], for $x, y \in \mathcal{U}$ there exists a unique complex number $c(x,y)$ satisfying $\langle x_t, y_t\rangle = e^{tc(x,y)}$. The function $c : \mathcal{U}\times\mathcal{U} \to \mathbb{C}$ is the covariance function of E. It is conditionally positive definite and there holds

$$\mathcal{L}^{x,y}(1) = \lim_{t\to 0}\frac{\langle x_t, y_t\rangle - 1}{t} = \lim_{t\to 0}\frac{e^{tc(x,y)} - 1}{t} = c(x,y). \tag{3.3}$$

Since all $\mathcal{L}^{x,y}$ are $\mathbb{C}$-linear, the $\mathcal{B}$-bimodule $\mathcal{B}\mathcal{U}\mathcal{B}$ is reduced to the complex vector space $\mathbb{C}\mathcal{U}$ consisting of all formal sums $\sum_i a_i x_i$ with $a_i \in \mathbb{C}$, $x_i \in \mathcal{U}$ and its $\mathcal{B}$-subbimodule $(\mathcal{B}\mathcal{U}\mathcal{B})_0$ is reduced to $(\mathbb{C}\mathcal{U})_0 = \{\sum_i a_i x_i \in \mathbb{C}\mathcal{U} \mid \sum_i a_i = 0\}$. Using (3.3), it follows

$$\begin{aligned}\Big\langle \sum_i a_i x_i, \sum_i a'_i x'_i\Big\rangle &= \sum_{i,j}\mathcal{L}^{x_i,x'_j}(\overline{a_i}a'_j)\\ &= \sum_{i,j}\mathcal{L}^{x_i,x'_j}(1)\overline{a_i}a'_j = \sum_{i,j} c(x_i,x'_j)\overline{a_i}a'_j.\end{aligned} \tag{3.4}$$

According to Corollary 3.1, the index of E is the completion of the inner product space $(\mathbb{C}\mathcal{U})_0/\mathcal{N}$ where $\mathcal{N} = \big\{\sum_i a_i x_i \in (\mathbb{C}\mathcal{U})_0 \mid \big\langle \sum_i a_i x_i, \sum_i a_i x_i\big\rangle = 0\big\}$. That definition of the index corresponds to the one previously given by Arveson in [**2**]. In detail, following the notation in [**2**], $\mathbb{C}_0\mathcal{U}$ is the complex vector space consisting of all finitely nonzero functions $f : \mathcal{U} \to \mathbb{C}$ satisfying $\sum_x f(x) = 0$. There is a mapping $\langle\cdot,\cdot\rangle : \mathbb{C}_0\mathcal{U}\times\mathbb{C}_0\mathcal{U} \to \mathbb{C}$ defined by

$$\langle f, g\rangle = \sum_{x,y\in\mathcal{U}} c(x,y)\overline{f(x)}g(y). \tag{3.5}$$

If $N = \big\{f \mid \langle f, f\rangle = \sum_{x,y} c(x,y)\overline{f(x)}f(y) = 0\big\}$, the mapping (3.5) is an inner product on $(\mathbb{C}_0\mathcal{U})/N$ and the index of E is defined as dimension of the completion of $(\mathbb{C}_0\mathcal{U})/N$. A basis for $\mathbb{C}_0\mathcal{U}$ is given by the set $\{\delta_x \mid x \in \mathcal{U}\}$ where $\delta_x(x) = 1$ and $\delta_x(y) = 0$, $\forall y \neq x$. The mapping $x \mapsto \delta_x$ is a bijection between $\mathcal{U}$ and the basis vectors $\{\delta_x \mid x \in \mathcal{U}\}$, hence $\mathcal{U}$ may be considered as a linearly independent basis for $\mathbb{C}_0\mathcal{U}$. Therefore, every $f \in \mathbb{C}_0\mathcal{U}$ may be written in the form $f = \sum_i a_i x_i$ where $a_i = f(x_i) \in \mathbb{C}$, $x_i \in \mathcal{U}$. Consequently, we may identify $(\mathbb{C}\mathcal{U})_0$ and $\mathbb{C}_0\mathcal{U}$, $\mathcal{N}$ and N and the mappings in (3.4) and (3.5).

The final conclusion is that, according to Corollary 3.1, the index of Arveson product system E may also be defined as the completion of the inner product space $\mathcal{U}/_{\sim}$, where $\sim$ is the equivalence relation introduced in Definition 3.1.

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