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ON THE CONVERGENCE OF A CLASS OF THE REGULARIZATION METHODS FOR ILL-POSED QUADRATIC MINIMIZATION PROBLEMS WITH CONSTRAINT

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Abstract. We study a class of regularization methods for solving least-squares ill-posed problem with a convex constraint. Convergence and convergence rate results are proven for the problems which satisfy so called power source condition. All the results are obtained under the assumptions that, instead of exact initial data, only their approximations are known.

1. Introduction

We consider an ill-posed quadratic minimization problem

$$J(u) = \frac{1}{2}\|Au - f\|^2 \to \inf, \quad u \in U, \tag{1.1}$$

where $A : H \to F$ is a bounded linear operator mapping between Hilbert spaces H and F, $U \subseteq H$ is a closed convex set and $f \in F$ is fixed. We will deal with this problem assuming that the sets U_* and U_∞ of the solutions of the given problem and of the corresponding problem without constraints

$$J(u) = \frac{1}{2}\|Au - f\|^2 \to \inf, \quad u \in H, \tag{1.2}$$

are nonempty. In this case the problems

$$\|u\|^2 \to \inf, \quad u \in U_*, \quad \|u\|^2 \to \inf, \quad u \in U_\infty \tag{1.3}$$

have unique solutions, which we will denote by u_* and u_∞ and call *normal solutions* of problems (1.1) and (1.2).

Let us observe that problem (1.1) is equivalent to the variational inequality

$$\text{find } u \in U \text{ such that } \langle A^*Au - A^*f, v - u\rangle \geqslant 0, \ \forall v \in U.$$

while the operator equation (which is equivalent to the problem (1.2)) $A^*Au = A^*f$, is a specific case of the previous variational inequality for $U = H$.

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Problems (1.1) and (1.2) in the literature (see [**9**, **11**, **12**, **18**]) are regularly studied under the assumptions that, instead of the exact operator A and instead of the element f, one actually deals with their approximations $A_\eta \in \mathcal{L}(H, F)$, $f_\delta \in F$, such that

$$\|A - A_\eta\| \leqslant \eta, \quad \|f - f_\delta\| \leqslant \delta, \quad \|A_\eta\|^2 \leqslant a, \tag{1.4}$$

where $\eta, delta > 0$ are small positive real numbers and $a > 0$.

In general, problems (1.1) and (1.2) are ill-posed. Therefore, it is necessary to apply some methods of regularization [**1**, **5**, **18**, **19**] that will produce good and stable sequence of approximate solutions of the problems.

Methods of regularization considered here are based on a modification of the family of regularizing functions from [**18**]. Note that the Tikhonov method and the iterated Tikhonov method of regularization belong to this class of methods.

The bounds of the accuracy of the regularization methods for solving ill-posed problems (1.1) and (1.2) are usually obtained for classes of problems defined by certain conditions concerning their normal solutions. The conditions that we discuss here and that were discussed in the cited papers belong to the class of so-called source conditions or sourcewise representable conditions. The well known conditions of this type are power source conditions that were widely used in [**18**] for obtaining the estimates of the convergence rate of regularization methods for solving linear operator equations.

Our main goal is to construct the methods of regularization of some classes of ill-posed problems (1.1) (minimization of quadratic function with convex constraints), with the convergence rate that can be compared with the convergence rate of regularization of linear operator equation (see for example [**18**]). However, it is not possible to obtain such convergence rates without additional conditions. In [**11**] an example was constructed that shows the rate of convergence, depends on the boundary of the set $U \subseteq R^2$, and it can be arbitrarily slow.

The paper is organized as follows. In Section 2 we present and discuss the roles and the meanings of the source conditions for problems (1.1) and (1.2) and describe one class of regularization methods. The key relation of the method for solving (1.2 is presented in two forms: as a variational inequality and as a problem of minimization. The proposed methods are inspired by the regularization methods for solving linear operator equation without constraints, studied in [**1**, **5**, **17**, **18**] and developed e.g., in [**3**, **7**, **8**, **13**–**18**], in a way that is different from the one presented here. Taking into account the presence of the constraints, for the functions that define the regularization method, we had to include new conditions, and to use new techniques other than those used in [**9**–**12**].

The main results of the paper are contained in Section 3 where we prove convergence of the regularization method for ill-posed problems (1.1). In two theorems we show that the typical (power) source condition or projected source condition is sufficient for the convergence rate (of regularized solutions of (1.1) to normal solution) of the same order as in the case of minimization problem (1.2) without constraints.

2. Source conditions and regularization method

2.1. Power source conditions and unconstrained regularization. The first source conditions we consider here is the so-called power source condition or power sourcewise representable condition for problem (1.2), which can be presented in the form

$$u_\infty = |A|^p h_*, \text{ where } h_* \in H, \ |A|^p = (A^*A)^{p/2}, \ p > 0. \tag{2.1}$$

The previous condition seems quite natural, if we have in mind that $u_\infty \in \overline{R(A^*)}$, where $R(A^*)$ is the range of the operator A and $\overline{R(A^*)}$ its closure in norm of the space H. It means that the solution u_∞ is densely surrounded by the elements from $R(A^*)$.

In [**1**, **5**, **15**, **18**] as approximations of the normal solution u_∞ of problem (1.2), the following was used

$$u_\alpha = g_\alpha(A_\eta^* A_\eta)A_\eta^* f_\delta \tag{2.2}$$

that are generated by the family of Borel measurable functions $g_\alpha : [0, a] \mapsto \mathbb{R}$ satisfying the conditions

$$\begin{gathered} \sup_{0\leqslant t\leqslant a} |g_\alpha(t)| \leqslant M/\alpha, \quad M > 0, \\ \sup_{0\leqslant t\leqslant a} t^p|1 - tg_\alpha(t)| \leqslant \gamma_p\alpha^p, \quad \alpha > 0, \quad \gamma_p = \text{const}, \quad 0 \leqslant p \leqslant p_0, \quad 0 < p_0. \end{gathered} \tag{2.3}$$

Here, a is the constant from (1.4). Number p_0 is called qualification of the family $\{g_\alpha : \alpha > 0\}$ and it has an important role. For example, (see [**19**, p. 100]) if the conditions (1.4), (2.3) and (2.1) are satisfied, then for

$$\alpha = \alpha(\eta, \delta) = d(\eta + \delta)^{\frac{2}{p+2}}, \quad d = \text{const} > 0,$$

one has

$$\|u_\alpha - u_\infty\| \leqslant \text{const}(\eta + \delta)^{\frac{p}{p+2}}, \quad 0 < p \leqslant 2p_0.$$

2.2. Projected source condition and constrained regularization. At first, we need to address the question of the meaning of the source condition [**15**]

$$u_* = |A|^p h_*, \text{ where } h_* \in H, \ |A|^p = (A^*A)^{p/2}, \ p > 0. \tag{2.4}$$

When studying the problem (1.1) of minimization with constraints, it is necessary to have in mind that, in general, the normal solution u_* does not satisfy the condition

$$u_* \in \overline{R(A)^*} = \mathcal{N}(A)^\perp, \tag{2.5}$$

that was fulfilled for the problem without constraints (in this case $u_* = u_\infty$). This fact was important for the justification of the source condition (2.1). Hence, in case (1.1) of minimization with constraints, it is necessary to describe a class of the problems whose normal solutions satisfy the condition (2.4) or to find a more adequate source condition with similar meaning. We will follow these steps to prove a lemma.

Generally, in what follows, we will denote by $\Pi_X(x)$ (or by $\Pi_X x$) the metric projection of the point x on a closed convex set $X \subseteq H$.

Lemma 2.1. *Normal solution u_* of problem (1.1) belongs to $\overline{\Pi_U(R(A^*))}$.*

Proof. The function $F_\alpha(u) = \frac{1}{2}\|Au - f\|^2 + \frac{1}{2}\alpha\|u\|^2$ is strongly convex, such that the problem $F_\alpha(u) \mapsto \inf$, $u \in U$ has a unique solution, which we denote by v_α. It is well known that $v_\alpha \to u_*$ as $\alpha \to 0$. The corresponding variational inequality $\langle F'_\alpha(v_\alpha), u - v_\alpha\rangle \geqslant 0$, $\forall u \in U$ can be written in the form $\langle v_\alpha - (A^*f - A^*Au_\alpha)/\alpha, u - v_\alpha\rangle \geqslant 0$, $\forall u \in U$. It is well known ([**17**] or [**19**, p. 183]) that then $v_\alpha = \Pi_U((A^*f - A^*Au_\alpha)/\alpha) \in \Pi_U(R(A^*))$. From here, allowing that $\alpha \to 0$, we obtain that $u_* \in \overline{\Pi_U(R(A^*))}$. □

In what follows we will also study the accuracy of the regularization methods on the class of problems of the type (1.1) with normal solutions satisfying the so-called projected source condition:

$$u_* = \Pi_U(|A|^p h_*), \quad h_* \in H, \quad p > 0. \tag{2.6}$$

Note that the source conditions for the linear operator equations with constraints, that are coordinated with the Tikhonov method of regularization were considered in [**11**].

The following example indicates one class of problems (1.1) whose normal solutions satisfy (2.5).

Example 2.1. If $U \subseteq H$ is a closed convex which has exactly one hyperplane of support at each boundary point and the conditions $u_* \in \Pi_U\overline{(R(A^*))}$ and $U_* \cap U_\infty = \emptyset$, are satisfied, then (2.5) holds.

Proof. From $U_* \cap U_\infty = \emptyset$ it follows that u_* belongs to the boundary of the set U_* and that $J'(u_*) = A^*Au_* - A^*f \neq 0$. Let $u_* = \Pi_U(h)$, $h \in \overline{R(A^*)}$ and denote $c = u_* - h$. Then $\langle c, u - u_*\rangle = \langle u_* - h, u - u_*\rangle \geqslant 0$, $\forall u \in U$. It follows from here that the hyperplane $H_c := \{x \in H : \langle c, x\rangle = \langle c, u_*\rangle\}$ is a hyperplane of support of U at u_*. From the inequality $\langle A^*Au_* - A^*f, u - u_*\rangle = \langle J'(u_*), u - u_*\rangle \geqslant 0$, $\forall u \in U$, it follows that $H_{J'(u_*)} = \{x \in H : \langle J'(u_*), x\rangle = \langle J'(u_*), u_*\rangle\}$ is also a hyperplane of support of U at u_*. Having in mind that U has only one supporting hiperplane at u_*, we can now conclude that the vector $A^*Au_* - A^*f = J'(u_*)$ is collinear with $c : J'(u_*) = A^*Au_* - A^*f = \gamma c$, $\gamma \neq 0$. It means that $c \in R(A^*) \subseteq \overline{R(A^*)}$. Now $u_* \in \overline{R(A^*)}$ immediately follows from here and from equality $u_* = c + h$, $h \in \overline{R(A^*)}$. □

Now we can describe a class of regularization methods whose accuracy we study on a class of the problems whose normal solutions satisfy (2.1). Later, we will also consider condition (2.6).

Suppose that the Borel measurable functions $g_\alpha : [0, a] \mapsto \mathbb{R}$, $a > 0$ satisfy the conditions

$$1 - \tan_\alpha(t) \geqslant 0, \;\; t \in [0, a], \tag{2.7}$$

$$\frac{1}{t + \beta\alpha} \leqslant g_\alpha(t) \leqslant \frac{1}{\beta\alpha}, \;\; t \in [0, a], \quad \beta > 0, \tag{2.8}$$

(2.9)
$$\exists p_0>0 \text{ such that } \forall p\in[0,p_0] \sup_{0\leqslant t\leqslant a} t^p(1-\tan_\alpha(t))\leqslant \gamma\alpha^p,\ \alpha>0,\ \text{where } \gamma=\gamma(p_0).$$

Notice that, in this case, the functions satisfying conditions (2.7)–(2.9) also satisfy conditions (2.3). In addition, the family of the functions $g_\alpha(t)=(t+\alpha)^{-1}$ and $g_\alpha(t)=\sum_{j=0}^{m-1}\frac{\alpha^j}{(t+\alpha)^j}=t^{-1}(1-(1+t)^{-m})$ (that defines the Tikhonov methods of regularization and its iterated variant) also satisfy these conditions.

We prove some properties of the operators $g_\alpha(A_\eta^*A_\eta)$ and $g_\alpha^{-1}(A_\eta^*A_\eta)$, where the function $t\mapsto 1/g_\alpha(t)$ is denoted by $(g_\alpha(t))^{-1}$.

Firstly, let us observe that $g_\alpha(A_\eta^*A_\eta):H\mapsto H$ is a self-adjoint positive definite operator. Hence there exist $g_\alpha^{1/2}(A_\eta^*A_\eta)$ and $g_\alpha^{-1/2}(A_\eta^*A_\eta)$ and they are also self-adjoint and positive definite operators.

Lemma 2.2. *If conditions* (2.7) *and* (2.8) *are satisfied, then the following estimates hold:*

$$\|g_\alpha(A_\eta^*A_\eta)\|\leqslant 1/\beta\alpha,\quad \|g_\alpha(A_\eta^*A_\eta)-A_\eta^*A_\eta\|\leqslant\alpha\beta, \tag{2.10}$$

$$\max\{\|A_\eta u\|^2;\beta\alpha\|u\|^2\}\leqslant\langle g_\alpha^{-1}(A_\eta^*A_\eta)u,u\rangle\leqslant\|A_\eta u\|^2+\beta\alpha\|u\|^2, \tag{2.11}$$

$$\|A_\eta g_\alpha(A_\eta^*A_\eta)A_\eta^*\|\leqslant 1,\quad \|g_\alpha^{1/2}(A_\eta^*A_\eta)A_\eta^*\|\leqslant 1. \tag{2.12}$$

Proof. The first inequality in (2.10) follows from (2.8):

$$\|g_\alpha(A_\eta^*A_\eta)\|\leqslant\sup_{0\leqslant t\leqslant a} g_\alpha(t)\leqslant 1/\beta\alpha.$$

while the second one from $\|g_\alpha^{-1}(A_\eta^*A_\eta)-A_\eta^*A_\eta\|\leqslant\sup_{0\leqslant t\leqslant a}|(g_\alpha(t))^{-1}-t|\leqslant\alpha\beta$. Let us prove the first inequality in (2.11). Based on (2.8), we have that $\beta\alpha\leqslant(g_\alpha(t))^{-1}\leqslant t+\beta\alpha$. Therefore, we have $\beta\alpha\|u\|^2\leqslant\langle g_\alpha^{-1}(A_\eta^*A_\eta)u,u\rangle$. From $1-\tan_\alpha(t)\geqslant 0$ it follows that $t\leqslant g_\alpha^{-1}(t)$, wherefrom we obtain $\langle g_\alpha^{-1}(A_\eta^*A_\eta)u,u\rangle\geqslant\langle A_\eta^*A_\eta u,u\rangle=\|A_\eta u\|^2$. Furthermore, the second inequality in (2.11) is a consequence of the inequality $(g_\alpha(t))^{-1}\leqslant t+\beta\alpha$.

Let us prove (2.12). Using the equality $A_\eta g_\alpha(A_\eta^*A_\eta)A_\eta^*=A_\eta A_\eta^*g_\alpha(A_\eta A_\eta^*)$, we have $\|A_\eta g_\alpha(A_\eta^*A_\eta)A_\eta^*\|=\|A_\eta A_\eta^*g_\alpha(A_\eta A_\eta^*)\|\leqslant\sup_{0\leqslant t\leqslant a}tg_\alpha(t)\leqslant 1$. The second inequality in (2.12) is a consequence of the previous one:

$$\begin{aligned}\|g_\alpha^{1/2}(A_\eta^*A_\eta)A_\eta^*u\|^2&=\langle g_\alpha^{1/2}(A_\eta^*A_\eta)A_\eta^*u,g_\alpha^{1/2}(A_\eta^*A_\eta)A_\eta^*u\rangle\\ &\langle g_\alpha(A_\eta^*A_\eta)A_\eta^*u,A_\eta^*u\rangle=\langle A_\eta g_\alpha(A_\eta^*A_\eta)A_\eta^*u,u\rangle\leqslant\|u\|^2.\end{aligned}$$

Consequently, $\|g_\alpha^{1/2}(A_\eta^*A_\eta)A_\eta^*\|\leqslant 1$. □

As an approximation of the normal solution of problem (1.1), one can take the unique solution $u_\alpha\in U$ of the variational inequality

$$\left\langle g_\alpha^{-1}(A_\eta^*A_\eta)u_\alpha-A_\eta^*f_\delta,u-u_\alpha\right\rangle\geqslant 0,\quad\forall u\in U, \tag{2.13}$$

where the functions g_α satisfy conditions (2.7)–(2.9).

In the cases of Tikhonov regularization and the Tikhonov iterated regularization, variational inequalities (2.13) become $\left\langle(A_\eta^*A_\eta+\alpha I)u_\alpha-A_\eta^*f_\delta,u-u_\alpha\right\rangle\geqslant 0$,

$\forall u \in U$, and $\big\langle (A_\eta^* A_\eta + \alpha I)u_{n,\alpha} - \alpha u_{n-1,\alpha} - A_\eta^* f_\delta, u - u_{n,\alpha}\big\rangle \geqslant 0$, $\forall u \in U$, $n = 1, \dots, m$, $u_{0,\alpha} = 0$, $u_\alpha = u_{m,\alpha}$.

Problem (2.13) has a unique solution, since the operator $G_\alpha(u) = g_\alpha^{-1}(A_\eta^* A_\eta)u - A\eta^* f_\delta$ is strongly monotone:

$$\langle G_\alpha(u) - G_\alpha(v), u - v\rangle = \big\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u - v), u - v\big\rangle \geqslant \beta\alpha\|u - v\|^2.$$

Note that $u_\alpha^* = g_\alpha(A_\eta^* A_\eta)A_\eta^* f_\delta$ in (2.2) (which is an approximate solution of problem (1.2)), is a unique solution of the equation $G_\alpha(u) = 0$.

It is easy to see that G_α is a potential operator: $G(u) = T_\alpha'(u)$, where

$$T_\alpha(u) := \big\|g_\alpha^{-1/2}(A_\eta^* A_\eta)u - g_\alpha^{1/2}(A_\eta^* A_\eta)A_\eta^* f_\delta\big\|^2.$$

Consequently, variational inequality (2.13) is equivalent to the minimization problem $T_\alpha(u) \to \inf$, $u \in U$.

3. Convergence of the regularization methods

In this section we prove the convergence $u_\alpha \to u_*$ as $\alpha \to 0$ and derive an estimate of the rate of convergence in the case when condition (2.4) (or (2.6) is satisfied. Firstly, we prove some auxiliary results.

Lemma 3.1. *If conditions* (1.4) *and* (2.8) *are satisfied, and if* $\frac{\eta}{\alpha} \to 0$ *as* $\alpha \to 0$, *then*

$$\frac{1}{\beta\alpha}\big\|A_\eta^* A_\eta\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \to 0 \text{ as } \alpha \to 0, \tag{3.1}$$

$$\frac{1}{\beta\alpha}\big\|\big(g_\alpha^{-1}(A_\eta^* A_\eta) - \beta\alpha\big)\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \to 0 \text{ as } \alpha \to 0. \tag{3.2}$$

$$\frac{1}{\beta\alpha}\big\|\big(g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta\big)\big\| \leqslant C, \quad C = const. \tag{3.3}$$

Proof. Taking into account that the equality $A\big(I - \Pi_{\overline{R(A^*)}}\big)u_* = 0$, (3.1) can be obtained as an immediate consequence of (1.4) and of the lemma, namely, bearing in mind the conditions of the lemma, we can obtain

$$\begin{aligned}\frac{1}{\beta\alpha}\big\|A_\eta^* A_\eta\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| &= \frac{1}{\beta\alpha}\big\|A_\eta^*(A_\eta - A)\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \\ &\leqslant \frac{\eta}{\beta\alpha}\|A_\eta^*\|\big\|\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \to 0 \text{ as } \alpha \to 0.\end{aligned}$$

Convergence (3.2) is also a consequence of (1.4) and of the conditions of the lemma. From $\beta\alpha \leqslant (g_\alpha(t))^{-1} \leqslant t + \beta\alpha$ it follows $0 \leqslant (g_\alpha(t))^{-1} - \beta\alpha \leqslant t$. Therefore, we have

$$\begin{aligned}&\frac{1}{\beta\alpha}\big\|\big(g_\alpha^{-1}(A_\eta^* A_\eta) - \beta\alpha I\big)\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \leqslant \frac{1}{\beta\alpha}\big\|A_\eta^*\big\|\big\|A_\eta\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \\ &= \frac{1}{\beta\alpha}\|A_\eta^*\|\|(A_\eta - A)\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \leqslant \frac{a\eta}{\beta\alpha}\big\|\big(I - \Pi_{\overline{R(A^*)}}\big)u_*\big\| \to 0 \text{ as } \alpha \to 0.\end{aligned}$$

Finally, (3.3) is a consequence of the inequality $\|(g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta)\| \leqslant \sup_{0 \leqslant t \leqslant a}\{1/(g_\alpha(t) - t)\} \leqslant \alpha\beta$. □

In the proof of the convergence $u_\alpha \to u_*$, we shall also use the following lemma whose proofs can be found in [**18**, pp. 93 (Lemma 1.2), 99 (Lemma 2.2), 109 (Lemma 3.2) and 114 (inequality 3.60)].

LEMMA 3.2. (a) *If* $\|A - A_\eta\| \leqslant \eta$, *then*

$$(\forall p > 0)\ \||A_\eta|^p - |A|^p\| \leqslant c(1 + |\ln \eta|)\eta^{\min\{1,p\}},\ c = c(p) \equiv \text{const}.$$

(b) *If, in addition, the conditions* $\|A_\eta\|^2 \leqslant a$, $\|A\|^2 \leqslant a$, *and* (2.9) *are satisfied, then*
(b_1) $\|A_\eta K_{\alpha\eta}(|A|^p - |A_\eta|^p)h_*\| \leqslant \gamma\eta,\ \gamma = \gamma(\alpha, p)$ *as* $\alpha \to 0$,
(b_2) $(\forall x \in \overline{R(A^*)})\ \|K_{\alpha\eta}x\| \to 0$ *as* $\alpha, \eta \to 0$,
(b_3) $(\forall x \in \overline{R(A^*)})(\forall p_0 > 1/2)\ \|K_{\alpha\eta}x\| \to 0,\ \frac{1}{\sqrt{\alpha}}\|A_\eta K_{\alpha\eta}x\| \to 0$
as $\alpha, \eta \to 0$, *where* $K_{\alpha\eta} = I - A_\eta^* A_\eta g_\alpha(A_\eta^* A_\eta)$ *and* γ_p *is the constant from* (2.9)
(b_4) $\|A_\eta K_{\alpha\eta}|A|^p h_*\| \leqslant c_2\alpha^{(p+1)/2}\|h_*\|,\ c_2 = c_2(p), 0 \leqslant p \leqslant 2p_0 - 1$.

Now, we can prove the main results of the paper. They are related to the convergence of the methods of regularization.

THEOREM 3.1. *Suppose conditions* (1.4) *and* (2.7)–(2.9) *are satisfied.*
(a_1) *If* $Au_* = f$ *and the parameter* α *in* (2.13) *is chosen such that* $\alpha = \alpha(\eta, \delta) \to 0$ *and* $\frac{\eta+\delta}{\alpha} \to 0$ *as* $\eta, \delta \to 0$, *then* $u_\alpha \to u_*$ *as* $\eta, \delta \to 0$.
(a_2) *If* $Au_* \neq f$ *and* $\alpha = \alpha(\eta, \delta) \to 0$ *and* $\frac{\eta^2}{\alpha^3} + \frac{\delta}{\alpha} \to 0$ *as* $\eta, \delta \to 0$, *then* $u_\alpha \to u_*$ *as* $\eta, \delta \to 0$.

(b) *If in addition, condition* (2.4) *is satisfied, then*
(b_1) *in the case of* $Au_* = f$, *we have that*

$$\|u_\alpha - u_*\| = O\big((\eta + \delta)/\sqrt{\alpha} + \alpha^{p/2}\big) + o\big(\eta^{p/(p+2)}\big), \tag{3.4}$$

$$\|A_\eta(u_\alpha - u_*)\| = O\big(\eta + \delta + \alpha^{\frac{p+1}{2}}\big) + \sqrt{\alpha} o(\eta^{\frac{p}{p+2}}),\ 0 \leqslant p \leqslant 2p_0 - 1,\ p_0 > \frac{1}{2}. \tag{3.5}$$

(b_2) *in the case of* $Au_* \neq f$ *we have that*

$$\|u_\alpha - u_*\| = O\Big(\frac{\eta}{\alpha} + \frac{\delta}{\sqrt{\alpha}} + \alpha^{\frac{p}{2}}\Big) + o\big(\eta^{\frac{p}{p+2}}\big), \tag{3.6}$$

$$\|A_\eta(u_\alpha - u_*)\| = O\Big(\frac{\eta}{\sqrt{\alpha}} + \delta + \alpha^{\frac{p+1}{2}}\Big) + \sqrt{\alpha}\, o\big(\eta^{\frac{p}{p+2}}\big),\ 0 \leqslant p \leqslant 2p_0 - 1,\ p_0 > \frac{1}{2}. \tag{3.7}$$

PROOF. Let us begin with the variational inequality (2.13) that for $u = u_*$ gives $\langle g_\alpha^{-1}(A_\eta^* A_\eta)u_\alpha - A_\eta^* f_\delta, u_* - u_\alpha\rangle \geqslant 0$. From here, taking into account the inequality $\langle A^*Au_* - A^*f, u_* - u_\alpha\rangle \leqslant 0$, and the equality

$$A_\eta^* f_\delta = A_\eta^* A_\eta u_* + (A^*f - A^*Au_*) + A_\eta^*[(A - A_\eta)u_* + (f_\delta - f)] + (A^* - A_\eta^*)(Au_* - f),$$

we obtain

$$\begin{aligned}
&\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_\alpha - u_*), u_\alpha - u_*\rangle \leqslant \langle g_\alpha^{-1}(A_\eta^* A_\eta)u_*, u_* - u_\alpha\rangle - \langle A_\eta^* f_\delta, u_* - u_\alpha\rangle \\
&\leqslant \langle g_\alpha^{-1}(A_\eta^* A_\eta)u_*, u_* - u_\alpha\rangle - \langle A_\eta^* A_\eta u_*, u_* - u_\alpha\rangle \\
&\quad + \langle A_\eta^*[(A_\eta - A)u_* + (f - f_\delta)], u_* - u_\alpha\rangle + \langle (A_\eta^* - A^*)(Au_* - f), u_* - u_\alpha\rangle \\
&= \langle (g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta)u_*, u_* - u_\alpha\rangle + \langle (A_\eta - A)u_* + (f - f_\delta), A_\eta(u_* - u_\alpha)\rangle \\
&\quad + \langle (A_\eta^* - A^*)(Au_* - f), u_* - u_\alpha\rangle.
\end{aligned}$$

From here, using the conditions of the theorem, and the conditions on functions g_α, and ε-inequality $\langle a, b\rangle \leqslant \frac{\varepsilon}{2}\|a\|^2 + \frac{1}{(2\varepsilon)}\|b\|^2$, we obtain

$$\begin{aligned}
\frac{1}{2}\alpha\beta\|u_\alpha - u_*\|^2 + \frac{1}{2}\|A_\eta(u_\alpha - u_*)\|^2 &\leqslant \max\{\alpha\beta\|u_\alpha - u_*\|^2; \|A_\eta(u_\alpha - u_*)\|^2\} \\
&\leqslant \langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_\alpha - u_*), u_\alpha - u_*\rangle \\
&\leqslant \frac{\varepsilon_1}{2}\|u_* - u_\alpha\|^2 + \frac{1}{2\varepsilon_1}\|g_\alpha(A_\eta^* A_\eta) - A_\eta^* A_\eta\|^2\|u_*\|^2 \\
&\quad + \frac{\varepsilon_2}{2}\|A_\eta(u_* - u_\alpha)\|^2 + \frac{1}{\varepsilon_2}\|A_\eta - A\|^2\|u_*\|^2 + \frac{\varepsilon_2}{2}\|A_\eta(u_* - u_\alpha)\|^2 \\
&\quad + \frac{1}{\varepsilon_2}\|f - f_\delta\|^2 + \frac{\varepsilon_3}{2}\|u_* - u_\alpha\|^2 + \frac{1}{2\varepsilon_3}\|Au_* - f\|^2\|A_\eta^* - A^*\|^2 \\
&\leqslant \frac{\varepsilon_1}{2}\|u_* - u_\alpha\|^2 + \frac{1}{2\varepsilon_1}(\alpha\beta)^2\|u_*\|^2 + \varepsilon_2\|A_\eta(u_* - u_\alpha)\|^2 \\
&\quad + \frac{1}{\varepsilon_2}\eta^2\|u_*\|^2 + \frac{1}{\varepsilon_2}\delta^2 + \frac{\varepsilon_3}{2}\|u_* - u_\alpha\|^2 + \frac{1}{2\varepsilon_3}\|Au_* - f\|^2\eta^2.
\end{aligned}$$

For $\varepsilon_1 = \varepsilon_3 = \frac{\alpha\beta}{2}, \varepsilon_2 = \frac{1}{4}$, we have

$$\|A_\eta(u_\alpha - u_*)\|^2 \leqslant 4\alpha\beta\|u_*\|^2 + 4\eta^2\|u_*\|^2 + 8\delta^2 + \frac{\eta^2}{\alpha\beta}\|Au_* - f\|^2,$$

and consequently, for $\alpha \sim \eta$,

$$\|A_\eta(u_\alpha - u_*)\| = O\Big(\sqrt{\alpha} + \eta + \delta + \frac{\eta}{\sqrt{\alpha}}\Big) = O\big(\sqrt{\alpha} + \sqrt{\eta} + \delta\big).$$

For $\varepsilon_1 = \varepsilon_3 = \frac{\alpha\beta}{4}, \varepsilon_2 = \frac{1}{2}$, we have

$$\|u_\alpha - u_*\|^2 \leqslant 8\|u_*\|^2 + 2\frac{\eta^2}{\alpha\beta}\|u_*\|^2 + 2\frac{\delta^2}{\alpha\beta} + 8\frac{\eta^2}{\alpha^2\beta^2}\|Au_* - f\|^2$$

It means that (u_α) remains bounded as $\alpha \to 0$.

Now, according to the equality $u_* = Pu_* + (I - P)u_*$, where $P = \Pi - \overline{R(A^*)}$ is the (orthogonal) projection onto the subspace $\overline{R(A^*)}$, and bearing in mind that $g_\alpha^{1/2}(A_\eta^* A_\eta)$ and $g_\alpha^{-1/2}(A_\eta^* A_\eta)$ are self-adjoint operators, the previous inequality can be written in the form

$$
\begin{aligned}
&\big\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_* - u_\alpha), u_* - u_\alpha \big\rangle = \|g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\|^2 \\
&\leqslant \big\langle g_\alpha^{-1/2}(A_\eta^* A_\eta) K_{\alpha\eta} P u_*, g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\big\rangle \\
&\quad + \big\langle (g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta)(I-P)u_*, u_* - u_\alpha \big\rangle \\
&\quad + \big\langle g_\alpha^{1/2}(A_\eta^* A_\eta) A_\eta^* \big[(A_\eta - A)u_* + (f - f_\delta)\big], g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\big\rangle \\
&\quad + \big\langle g_\alpha^{1/2}(A_\eta^* A_\eta)(A_\eta^* - A^*)(Au_* - f), g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\big\rangle,
\end{aligned}
$$

where $K_{\alpha\eta} = I - A_\eta^* A_\eta g_\alpha(A_\eta^* A_\eta)$. From here, using again ε-inequality, we get

$$
\begin{aligned}
&\big\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_* - u_\alpha), u_* - u_\alpha \big\rangle = \|g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\|^2 \\
&\leqslant \frac{3}{2}\|g_\alpha^{-1/2}(A_\eta^* A_\eta) K_{\alpha\eta} P u_*\|^2 + \|g_\alpha^{1/2}(A_\eta^* A_\eta) A_\eta^* \big[(A_\eta - A)u_* + (f - f_\delta)\big]^2 \\
&\quad + \|g_\alpha^{1/2}(A_\eta^* A_\eta)(A_\eta^* - A^*)(Au_* - f)\|^2 + \frac{1}{2}\|g_\alpha^{-1/2}(A_\eta^* A_\eta)(u_* - u_\alpha)\|^2 \\
&\quad + \big\langle (g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta)(I-P)u_*, u_* - u_\alpha \big\rangle.
\end{aligned}
$$

or

$$
\begin{aligned}
&\big\langle g_\alpha^{-1}(A_\eta^* A_\eta)(u_* - u_\alpha), u_* - u_\alpha \big\rangle \\
&\leqslant 3\|g_\alpha^{-1/2}(A_\eta^* A_\eta) K_{\alpha\eta} P u_*\|^2 + \|g_\alpha^{1/2}(A_\eta^* A_\eta) A_\eta^* \big[(A_\eta - A)u_* + (f - f_\delta)\big]^2 \\
&\quad + \big\{\|g_\alpha^{1/2}(A_\eta^* A_\eta)(A_\eta^* - A^*)(Au_* - f)\|\big\}^2 \\
&\quad + 2\big\langle (g_\alpha^{-1}(A_\eta^* A_\eta) - A_\eta^* A_\eta)(I-P)u_*, u_* - u_\alpha \big\rangle.
\end{aligned}
$$

Furthermore, as an immediate consequence of (2.12), we have

$$
\|g_\alpha^{-1/2}(A_\eta^* A_\eta)u\|^2 = \langle g_\alpha^{-1}(A_\eta^* A_\eta)u, u\rangle \leqslant \big(\|A_\eta u\| + \sqrt{\alpha\beta}\,\|u\|\big)^2. \tag{3.8}
$$

Now, according to inequalities (2.10)–(2.12), (3.8), and taking into account conditions (1.4), we obtain

$$
\begin{aligned}
&\max\big\{\|A_\eta(u_\alpha - u_*)\|^2; \alpha\beta\|u_\alpha - u_*\|^2\big\} \\
&\leqslant \Big\{\|A_\eta K_{\alpha\eta} P u_*\| + \sqrt{\alpha\beta}\,\|K_{\alpha\eta} P u_*\| + \eta\|u_*\| + \delta + \frac{\eta}{\sqrt{\alpha\beta}}\|Au_* - f\|\Big\}^2 \\
&\quad + 2\big\langle \big(g_\alpha^{-1}(A_\eta^* A_\eta) - \alpha\beta I\big)(I-P)u_*, u_* - u_\alpha \big\rangle \\
&\quad + 2\alpha\beta\big\langle (I-P)u_*, u_* - u_\alpha\big\rangle - 2\big\langle A_\eta^* A_\eta(I-P)u_*, u_* - u_\alpha\big\rangle
\end{aligned} \tag{3.9}
$$

Because (u_α) is bounded, there exist a null-sequences (α_n), (η_n), and (δ_n), and $v_* \in U$, such that u_{α_n} converges weakly to v_* as $n \to +\infty$. In order to simplify the text, we will assume that u_α converges weakly to v_* as $\alpha \to 0$. Then, from (3.9), Lemmas 3.1 and 3.2, and from the conditions of the theorem, bearing in mind the lower-semicontinuity of the norm, it follows that

$$
\lim_{\alpha\to 0}\|A_\eta(u_* - u_\alpha\| = 0, \text{ i.e., } Av_* = Au_* \text{ and } \|u_* - v_*\|^2 \leqslant 2\big\langle \big(I - \Pi_{\overline{R(A^*)}}\big)u_*, u_* - v_*\big\rangle. \tag{3.10}
$$

The equality $Av_* = Au_*$ shows that v_* is also a solution of the problem (1.1) and that $v_* - u_* \in \mathrm{Ker}A$. Now, inequality (3.10) can be written as $\|u_* - v_*\|^2 \leqslant 2\langle u_*, u_* - v_*\rangle$.

Also, as a solution of minimization problem (1.3), u_* satisfies the variational inequality $\langle u_*, u_* - v_*\rangle \leqslant 0$. Hence, $v_* = u_*$ and consequently, u_α weakly converges to u_*. Finally, if in (3.9) $\eta \to 0$ and $\delta \to 0$, then we get that u_α strongly converges to u_*.

Inequality (3.7) remains to be proven. Let us suppose that condition (2.4) is satisfied. Then $(I-P)u_* = \big(I - \Pi_{\overline{R(A^*)}}\big)u_* = 0$ and inequality (3.9) can be written in the form

$$\max\left\{\frac{1}{\sqrt{\alpha\beta}}\|A_\eta(u_\alpha - u_*)\|; \|u_\alpha - u_*\|\right\} \leqslant \frac{1}{\sqrt{\alpha\beta}}\|A_\eta K_{\alpha\eta}|A|^p h_*\| + \|K_{\alpha\eta}|A|^p h_*\| + \frac{\eta\|u_*\| + \delta}{\sqrt{\alpha\beta}} + \frac{\eta\|Au_* - f\|}{\alpha\beta}.$$

Let us estimate $\frac{1}{\sqrt{\alpha\beta}}\|A_\eta K_{\alpha\eta} u_*\|$ and $\|K_{\alpha\eta}|A|^p h_*\|$. Applying the inequality (b_4) from Lemma 3.2, we obtain the estimate:

$$\frac{1}{\sqrt{\alpha\beta}}\|A_\eta K_{\alpha\eta}|A|^p h_*\| \leqslant \left(\varepsilon_{\alpha p}\frac{\eta}{\sqrt{\alpha\beta}} + \frac{\gamma_{\frac{p+1}{2}}}{\sqrt{\beta}}\alpha^{\frac{p}{2}}\right)\|h_*\|, \quad 0 < p \leqslant 2p_0 - 1.$$

In a similar way, we can obtain the inequality

$$\|K_{\alpha\eta}|A|^p h_*\| \leqslant \left[c_p(1 + |\ln\eta|)\eta^{\min\{1;p\}} + \gamma_{\frac{p}{2}}\alpha^{\frac{p}{2}}\right]\|h_*\|.$$

Then from (3.4) we have

$$\max\left\{\frac{1}{\sqrt{\alpha\beta}}\|A_\eta(u_\alpha - u_*)\|; \|u_\alpha - u_*\|\right\} \leqslant \left(\varepsilon_{\alpha p}\frac{\eta}{\sqrt{\alpha\beta}} + \frac{\gamma_{\frac{p+1}{2}}}{\sqrt{\beta}}\alpha^{\frac{p}{2}}\right)\|h_*\| + \left[c_p(1 + \ln\eta)\eta^{\min\{1;p\}} + \gamma_{\frac{p}{2}}\alpha^{\frac{p}{2}}\right]\|h_*\| + \frac{\eta\|u_*\| + \delta}{\sqrt{\alpha\beta}} + \frac{\eta\|Au_* - f\|}{\alpha\beta}.$$

Now, taking into account that $\frac{(1+|\ln\eta|)\eta^{\min\{1;p\}}}{\eta^{p/(p+2)}} \to 0$ as $\eta \to 0$, we obtain estimates (3.4)–(3.7). $\square$

To our knowledge, there are no regularization methods based on functions satisfying conditions (2.7)–(2.9) with qualification $p_0 > 1$, such that a theorem analogous to Theorem 3.1 holds when condition (2.4) is replaced by the projected source condition (2.6). At the same time, for the standard Tikhonov method of regularization

$$J_\alpha(u) = \frac{1}{2}\|A_\eta u - f_\delta\|^2 + \frac{1}{2}\alpha\|u\|^2. \tag{3.11}$$

generated by the functions $g_{T,\alpha}(t) = \frac{1}{t+\alpha}$, (that satisfy conditions (2.7)–(2.9) with qualification $p_0 = 1$), the estimate analogous to the estimate from Theorem 3.1 is true.

THEOREM 3.2. *If in* (3.11) *the condition* (2.6) *is satisfied, then the following bounds are true:*

(i) *If* $Au_* = f$ *then*

$$\|u_* - u_\alpha\| = O\left(\frac{\eta+\delta}{\sqrt{\alpha}} + \alpha^{\min\{1/2;p/2\}}\right) + o(\eta^{\frac{p}{p+2}}),\ 0 \leqslant p \leqslant 2p_0 - 1, p_0 > \frac{1}{2}$$

$$\|A_\eta(u_\alpha - u_*)\| = O\left(\eta + \delta + \alpha^{\min\{1;(p+1)/2}\right) + \sqrt{\alpha}o(\eta^{\frac{p}{p+2}}),\ 0 \leqslant p \leqslant 2p_0 - 1, p_0 > \frac{1}{2}$$

(ii) *If* $Au_* \neq f$ *and the parameter of regularization is choosen such that* $\alpha \geqslant 2\eta^2$, *then*

$$\|u_* - u_\alpha\| = O\left(\frac{\eta}{\alpha} + \frac{\delta}{\sqrt{\alpha}} + \alpha^{\min\{1/2;p/2\}}\right) + o(\eta^{\frac{p}{p+2}}),\ 0 \leqslant p \leqslant 2p_0 - 1, p_0 > \frac{1}{2}$$

$$\|A_\eta(u_\alpha - u_*)\| = O\left(\eta + \delta + \alpha^{\min\{1;(p+1)/2}\right) + \sqrt{\alpha}o(\eta^{\frac{p}{p+2}}),\ 0 \leqslant p \leqslant 2p_0 - 1, p_0 > \frac{1}{2}$$

PROOF. Here, by u_α we denote the unique solution of the problem (3.11). It is well known that $u_\alpha \to u_*$ as $\alpha \to 0$. Let us derive the estimate of the rate of convergence when condition (2.6) is fulfilled. We will begin with the variational inequality $\langle J'_\alpha(u_\alpha), u - u_\alpha\rangle \geqslant 0$, $\forall u \in U$, that for $u = u_*$ gives $\big\langle A^*_\eta A_\eta u_\alpha + \alpha u_\alpha - A^*_\eta f_\delta, u_* - u_\alpha\big\rangle \geqslant 0$. The above inequality can be written in the form

$$\alpha\|u_* - u_\alpha\|^2 + \|A_\eta(u_* - u_\alpha)\|^2 \leqslant \alpha\langle u_*, u_* - u_\alpha\rangle + \langle A^*_\eta A_\eta u_* - A^*_\eta f_\delta, u_* - u_\alpha\rangle. \tag{3.12}$$

Taking into account that $u_* = \Pi_U(|A|^p h_*)$ and using the ε-inequality and the equality $\alpha I = (A^*_\eta A_\eta + \alpha I)K_{\alpha\eta}$, where $K_{\alpha\eta} = I - (A^*_\eta A_\eta)(\alpha I + A^*_\eta A_\eta)^{-1}$, we have the following estimate of the first term in (3.12):

$$\begin{aligned}\alpha\langle u_*, u_* - u_\alpha\rangle &\leqslant \alpha\langle |A|^p h_*, u_* - u_\alpha\rangle = \langle(\alpha I + A^*_\eta A_\eta)K_{\alpha\eta}|A|^p h_*, u_* - u_\alpha\rangle \\ &= \langle A_\eta K_{\alpha\eta}|A|^p h_*, A_\eta(u_* - u_\alpha)\rangle + \alpha\langle K_{\alpha\eta}|A|^p h_*, u_* - u_\alpha\rangle \\ &\leqslant \frac{\varepsilon_1}{2}\|A_\eta K_{\alpha\eta}|A|^p h_*\|^2 + \frac{1}{2}\varepsilon_1\|A_\eta(u_* - u_\alpha)\|^2 + \frac{\alpha\varepsilon_2}{2}\|K_{\alpha\eta}|A|^p h_*\|^2 + \frac{\alpha}{2\varepsilon_2}\|u_* - u_\alpha\|^2.\end{aligned} \tag{3.13}$$

Furthermore, using the inequality $\langle A^*Au_* - A^*f, u_\alpha - u_*\rangle \geqslant 0$ and the conditions of approximation of A and f by A_η and f_δ, we have the estimate of the second term in (3.12):

$$\begin{aligned}&\langle A^*_\eta A_\eta u_* - A^*_\eta f_\delta, u_* - u_\alpha\rangle \leqslant \langle A^*_\eta A_\eta u_* - A^*_\eta f_\delta, u_* - u_\alpha\rangle - \langle A^*Au_* - A^*f, u_* - u_\alpha\rangle \\ &= \langle A^*_\eta(A_\eta - A)u_*, u_* - u_\alpha\rangle + \langle A^*_\eta(f - f_\delta), u_* - u_\alpha\rangle + \langle(A^*_\eta - A^*)(Au_* - f)u_*, u_* - u_\alpha\rangle.\end{aligned}$$

Now, using again ε-inequality, we have the estimates

$$\begin{aligned}&\langle A^*_\eta(A_\eta - A)u_*, u_* - u_\alpha\rangle + \langle A^*_\eta(f - f_\delta), u_* - u_\alpha\rangle \\ &\qquad = \langle(A_\eta - A)u_*, A_\eta(u_* - u_\alpha)\rangle + \langle f - f_\delta, A_\eta(u_* - u_\alpha)\rangle \\ &\leqslant \frac{\varepsilon_3}{2}\|(A_\eta - A)u_*\|^2 + \frac{1}{2\varepsilon_3}\|A_\eta(u_* - u_\alpha)\|^2 + \frac{\varepsilon_4}{2}\|f - f_\delta\|^2 + \frac{1}{2\varepsilon_2}\|A_\eta(u_* - u_\alpha)\|^2 \\ &\qquad \leqslant \frac{\varepsilon_1}{2}\eta^2\|u_*\|^2 + \frac{1}{2\varepsilon_1}\|A_\eta(u_* - u_\alpha)\|^2 + \frac{\varepsilon_2}{2}\delta^2 + \frac{1}{2\varepsilon_2}\|A_\eta(u_* - u_\alpha)\|^2\end{aligned} \tag{3.14}$$

Let us estimate $\|A_\eta K_{\alpha\eta}|A|^p h_*\|^2$ and $\|K_{\alpha\eta}|A|^p h_*\|^2$. Taking into account Lemma 1.3 and the fact that condition (2.9) in the Tikhonov method of regularization has the form (see [**1**, **18**])

$$\sup_{0\leqslant t\leqslant a^2} t^p|1-\tan_\alpha(t)| = \sup_{0\leqslant t\leqslant a^2} t^p\frac{\alpha}{t+\alpha} \leqslant \gamma_1\alpha^p, \quad p\in[0,1],$$

while for $p>1$, we have $\sup_{0\leqslant t\leqslant a^2} t^p|1-\tan_\alpha(t)| = \sup_{0\leqslant t\leqslant a^2} t^{p-1}t\frac{\alpha}{t+\alpha}\leqslant \gamma_2\alpha$. So, we have (see [**18**, p. 93])

$$\begin{aligned}\|A_\eta K_{\alpha\eta}|A|^p h_*\|^2 &\leqslant 2\|A_\eta K_{\alpha\eta}(|A|^p-|A_\eta|^p)h_*\|^2 + 2\|A_\eta K_{\alpha\eta}|A_\eta|^p h_*\|^2\\ &\leqslant 2\varepsilon_{\alpha p}\eta^2\|h_*\|^2 + 2\sup_{0\leqslant t\leqslant a}\Big[t^{\frac{p+1}{2}}\Big(1-\frac{t}{t+\alpha}\Big)\Big]^2\|h_*\|^2\\ &\leqslant 2(\varepsilon_{\alpha p}\eta^2 + c_p\alpha^{\min\{p+1;2\}})\|h_*\|^2,\ \varepsilon_{\alpha p}\to 0 \text{ as } \alpha\to 0.\end{aligned}\tag{3.15}$$

Furthermore, denoting $I-A^*A(\alpha I+A^*A)^{-1}=\alpha(\alpha I+A^*A)^{-1}$ by K_α, we have

$$\begin{aligned}\|K_{\alpha\eta}|A|^p h_*\|^2 &= \|(K_{\alpha\eta}-K_\alpha)|A|^p h_* + K_\alpha|A|^p h_*\|^2\\ &\leqslant 2\|(K_{\alpha\eta}-K_\alpha)|A|^p h_*\|^2 + 2\|K_\alpha|A|^p h_*\|^2.\end{aligned}\tag{3.16}$$

For the estimate of the first term from the right-hand side of this inequality, we use the equality

$$\|K_{\alpha\eta}-K_\alpha\| = \alpha\|(\alpha I+A_\eta^*A_\eta)^{-1}[A_\eta^*(A_\eta-A)+(A_\eta^*-A^*)A](\alpha I+A^*A)^{-1}\|.$$

and the inequality $\|B(\alpha I+B^*B)^{-1}\| = \|(\alpha I+B^*B)^{-1}B^*\|\leqslant\frac12\frac{1}{\sqrt{\alpha}}$ for $B=A$ and $B=A_\eta$. Taking into account the conditions of the theorem, we obtain

$$\|(K_{\alpha\eta}-K_\alpha)|A|^p h_*\|^2 \leqslant \frac{\alpha^2\eta^2\||A|^p h_*\|^2}{4\alpha^3} = \frac{\alpha^{-1}\eta^2\||A|^p h_*\|}{4}.\tag{3.17}$$

Second addition in (3.16) can be estimated by (see (3.15))

$$\|K_\alpha|A|^p h_*\|^2 \leqslant c_p'\alpha^{\min\{2;p\}}\|h_*\|^2.\tag{3.18}$$

From (3.16), (3.17), and (3.18), it can be obtained that

$$\|K_{\alpha\eta}|A|^p h_*\|^2 \leqslant \frac{\eta^2}{2\alpha}\||A|^p h_*\|^2 + c'\alpha^{\min\{2;p\}}\|h_*\|^2.\tag{3.19}$$

Therefore, using (3.13) and (3.14), estimate (3.12) can be written as

$$\begin{aligned}\alpha\|u_*-u_\alpha\|^2 + \|A_\eta(u_*-u_\alpha)\|^2 &\leqslant \frac{\varepsilon_1}{2}\|A_\eta K_{\alpha\eta}|A|^p h_*\|^2 + \frac{1}{2\varepsilon_1}\|A_\eta(u_*-u_\alpha)\|^2\\ &+ \frac{\alpha\varepsilon_2}{2}\|K_{\alpha\eta}|A|^p h_*\|^2 + \frac{\alpha}{2\varepsilon_2}\|u_*-u_\alpha\|^2 + \langle A_\eta^*(A_\eta-A)u_*, u_*-u_\alpha\rangle\\ &+ \langle A_\eta^*(f-f_\delta), u_*-u_\alpha\rangle + \langle (A_\eta^*-A^*)(Au_*-f)u_*, u_*-u_\alpha\rangle.\end{aligned}$$

or, using (3.14), (3.14) and (3.19),

$$
\begin{aligned}
(3.20)\quad \alpha\|u_*-u_\alpha\|^2+\|A_\eta(u_*-u_\alpha)\|^2 &\leqslant \frac{\varepsilon_1}{2}(\varepsilon_{\alpha p}\eta^2+c\alpha^{\min\{p+1;2\}})\|h_*\|^2\\
&+\frac{1}{2\varepsilon_1}\|A_\eta(u_*-u_\alpha)\|^2+\varepsilon_2 c'\alpha^{\min\{3,p+1\}}\|h_*\|^2+\frac{\varepsilon_2}{2}\frac{\||A|^p h_*\|^2}{2}\eta^2\\
&+\frac{\alpha}{2\varepsilon_2}\|u_*-u_\alpha\|^2+\frac{\varepsilon_3}{2}\|u_*\|^2\eta^2+\frac{1}{2\varepsilon_3}\|A_\eta(u_*-u_\alpha)\|^2\\
&+\frac{\varepsilon_4}{2}\delta^2+\frac{1}{2\varepsilon_4}\|A_\eta(u_*-u_\alpha)\|^2+\langle Au_*-f,(A_\eta-A)(u_*-u_\alpha)\rangle.
\end{aligned}
$$

We will distinguish between two possibilities:

(1) $Au_*=f$. Then using (3.12)–(3.16) and (3.20), for $\varepsilon_2=\frac{1}{2}$, $\varepsilon_3=\varepsilon_4=\varepsilon_1=2$, we obtain an estimate of the rate of convergence:

$$\|A_\eta(u_\alpha-u_*)\|=O(\eta+\delta+\alpha^{\min\{1;(p+1)/2\}})$$

while for $\varepsilon_2=\varepsilon_3=\varepsilon_4=1,\varepsilon_1=2$, we have

$$\alpha\|(u_\alpha-u_*)\|^2=O(\eta^2+\delta^2+\alpha^{\min\{2;p+1\}}),$$

i.e., $\|u_\alpha-u_*\|=O\big(\frac{\eta+\delta}{\sqrt{\alpha}}+\alpha^{\min\{1/2;p/2\}}\big)$.

(2) $Au_*\neq f$. Then the term $\langle Au_*-f,(A_\eta-A)(u_*-u_\alpha)\rangle$ can be estimated by

$$
\begin{aligned}
\langle Au_*-f,(A_\eta-A)(u_*-u_\alpha)\rangle &= \langle (A_\eta^*-A^*)(Au_*-f),u_*-u_\alpha\rangle\\
&\leqslant \|Au_*-f\|^2\frac{\eta^2}{2\alpha}+\frac{\alpha}{2}\|u_*-u_\alpha\|^2.
\end{aligned}
$$

Now, from (3.20), $\varepsilon_1=\varepsilon_3=\varepsilon_4=2$, $\varepsilon_2=1$, we obtain the estimates of the rate of convergence of the method

$$\|A_\eta(u_\alpha-u_*)\|=O\Big(\frac{\eta}{\sqrt{\alpha}}+\delta+\alpha^{\min\{1,(p+1)/2\}}\Big),$$

and (for $\varepsilon_2=\varepsilon_1=1$, $\varepsilon_3=\varepsilon_4=2$)

$$\|u_\alpha-u_*\|=O\Big(\frac{\eta}{\alpha}+\frac{\delta}{\sqrt{\alpha}}+\alpha^{\min\{1/2,p/2\}}\Big).$$

□

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