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# ON ALMOST $\omega_1$-$n$-SIMPLY PRESENTED ABELIAN $p$-GROUPS

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Abstract. We define and investigate the class of almost $\omega_1$-$n$-simply presented $p$-torsion abelian groups, which class properly contains the subclasses of almost $n$-simply presented groups and $\omega_1$-$n$-simply presented groups, respectively. The obtained results generalize those obtained by us in Korean J. Math. (2014) and J. Algebra Appl. (2015).

## 1. Introduction and background

In what follows, all considered groups are additive $p$-primary abelian, where $p$ is a fixed prime integer. Also, let $n \geqslant 0$ be a nonnegative integer. As usual, for any ordinal $\alpha$, the symbol $p^\alpha G$ denotes the *$p^\alpha$-th power subgroup* of $G$, that is, the subgroup of $G$ consisting of all elements with heights $\geqslant \alpha$. In the case when $\alpha = \omega$, $p^\omega G$ is just called the *first Ulm subgroup* of $G$. We say that the group $G$ is *separable* if $p^\omega G = \{0\}$.

All unexplained notions and notations are mainly standard and follow essentially those from [**5**] and [**6**]. For the specific terminology, we provide the reader with the following:

• [**9**] A reduced group $G$ is said to be *almost totally projective* if it has a collection $\mathcal{C}$ consisting of nice subgroups of $G$ satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leqslant j$ $(i, j \in I)$, then $\bigcup_{i\in I} H_i \in \mathcal{C}$;

(3) If $K$ is a countable subgroup of $G$, then there is $L \in \mathcal{C}$ (that is, a nice subgroup $L$ of $G$) such that $K \subseteq L$ and $L$ is countable.

If $G$ is separable, it is known as an *almost direct sum of cyclic groups*.

• A group $G$ is said to be *almost simply presented* if it is the direct sum of a divisible group and an almost totally projective group.

The last concept can be generalized for any natural number $n$ as follows:

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• [**2**] A group $G$ is said to be *almost $n$-simply presented* if there is $L \leqslant G[p^n]$ with $G/L$ almost simply presented.

If $L$ is nice in $G$, the latter group will be called *nicely almost $n$-simply presented.*

• [**3**] A group $G$ is said to be *almost $p^{\omega+n}$-projective* if there is $P \leqslant G[p^n]$ with $G/P$ an almost direct sum of cyclic groups.

• [**3**] A group $G$ is said to be *almost $\omega_1$-$p^{\omega+n}$-projective* if there is a countable subgroup $C \leqslant G$ such that $G/C$ is almost $p^{\omega+n}$-projective.

These two concepts can be extended in the sense of [**4**] like this:

Definition 1.1. A group $G$ is called *almost $\omega_1$-$n$-simply presented* if there exists a countable subgroup $K$ of $G$ such that $G/K$ is almost $n$-simply presented. In addition, if $K$ is finite, $G$ is said to be *almost $\omega$-$n$-simply presented.*

When $K$ is a priory chosen to be nice in $G$, one may state the following:

Definition 1.2. The group $G$ is called *nicely almost $\omega_1$-$n$-simply presented* if there exists a countable nice subgroup $N$ of $G$ such that $G/N$ is almost $n$-simply presented. In addition, if $K$ is finite, $G$ is said to be *nicely almost $\omega$-$n$-simply presented.*

Apparently, almost $\omega_1$-$p^{\omega+n}$-projective groups and almost $\omega$-$n$-simply presented groups are themselves nicely almost $\omega_1$-$n$-simply presented and nicely almost $\omega$-$n$-simply presented, respectively. Besides, owing to Theorem 2.25 of [**2**], nicely almost $\omega_1$-$n$-simply presented groups with countable first Ulm subgroup are almost $n$-simply presented (compare with Corollary 3.1 listed below).

On the other vein, Hill and Megibben gave in [**7**] the definition of *a c.c. group* as a group $G$ such that $p^\omega(G/C)$ is countable whenever $C \leqslant G$ is a countable subgroup. Since $p^\omega G/(p^\omega G \cap C) \cong (p^\omega G + C)/C \subseteq p^\omega(G/C)$ is countable, it easily follows that $p^\omega G$ must be countable as well.

Our purpose here is to give a systematic study of the defined above two group classes as thereby we somewhat settle Problem 3 in [**2**]. Our work is organized as follows: In the next section we state and prove some preliminary technical claims and a background material. After that, in the third section, we proceed by proving the most of our basic results as we leave some specific of them in the fourth section. We close in the final section with some left-open problems of interest and importance.

## 2. Preliminaries and backgrounds

The following two technicalities are pivotal for the “niceness" property (cf. [**2**] too).

Lemma 2.1. *Suppose that $\alpha$ is an ordinal, and that $G$ and $F$ are groups where $F$ is finite. Then the following formula is fulfilled:*

$$p^\alpha(G+F) = p^\alpha G + [F \cap p^\alpha(G+F)] \subseteq p^\alpha G + F.$$

Proof. We will use a transfinite induction on $\alpha$. First, if $\alpha - 1$ exists, we have

$$\begin{aligned}p^\alpha(G+F) &= p(p^{\alpha-1}(G+F)) = p(p^{\alpha-1}G+[F\cap p^{\alpha-1}(G+F)])\\ &= p(p^{\alpha-1}G)+p(F\cap p^{\alpha-1}(G+F))\\ &\subseteq p^\alpha G+[F\cap p(p^{\alpha-1}(G+F))] = p^\alpha G+[F\cap p^\alpha(G+F)].\end{aligned}$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha-1$ does not exist, we have that

$$p^\alpha(G+F)=\bigcap_{\beta<\alpha}(p^\beta(G+F))\subseteq\bigcap_{\beta<\alpha}(p^\beta G+F)=\bigcap_{\beta<\alpha}p^\beta G+F=p^\alpha G+F.$$

In fact, the second sign "=" follows like this: Given $x\in\bigcap_{\beta<\alpha}(p^\beta G+F)$, we write that $x=g_{\beta_1}+f_1=\dots=g_{\beta_s}+f_s=\dots$ where $f_1,\dots,f_s\in F$ are the all elements of $F$; $g_{\beta_1}\in p^{\beta_1}G,\dots,g_{\beta_s}\in p^{\beta_s}G$ with $\beta_1<\dots<\beta_s<\cdots$.

Since $F$ is finite, while the number of equalities is infinite due to the infinite cardinality of $\alpha$, we infer that $g_{\beta_s}\in p^\beta G$ for any ordinal $\beta<\alpha$ which means that $g_{\beta_s}\in\bigcap_{\beta<\alpha}p^\beta G=p^\alpha G$. Thus $x\in\bigcap_{\beta<\alpha}p^\beta G+F=p^\alpha G+F$, as claimed. Furthermore, $p^\alpha(G+F)\subseteq(p^\alpha G+F)\cap p^\alpha(G+F)=p^\alpha G+[F\cap p^\alpha(G+F)]$ which is obviously equivalent to an equality. □

Lemma 2.2. *Let $N$ be a nice subgroup of a group $G$. Then*
(i) *$N+R$ is nice in $G$ for every finite subgroup $R\leqslant G$;*
(ii) *$N$ is nice in $G+F$ for each finite group $F$.*

Proof. (i) For any limit ordinal $\gamma$, we deduce that

$$\bigcap_{\delta<\gamma}(N+R+p^\delta G)\subseteq R+\bigcap_{\delta<\gamma}(N+p^\delta G)=R+N+p^\gamma G$$

as required. Indeed, the relation "$\subseteq$" follows like this: Given $x\in\bigcap_{\delta<\gamma}(N+R+p^\delta G)$, we write $x=a_1+r_1+g_1=\dots=a_s+r_s+g_s=\dots=a_k+r_1+g_k=\dots$, where $a_1,\dots,a_k\in N$; $r_1,\dots,r_k\in R$; $g_1\in p^{\delta_1}G,\dots,g_k\in p^{\delta_k}G$ with $\delta_1<\dots<\delta_k$. So $a_1+g_1=\dots=a_k+g_k=\dots\in\bigcap_{\delta<\gamma}(N+p^\delta G)$ and hence $x\in R+\bigcap_{\delta<\gamma}(N+p^\delta G)$, as required.

(ii) Since $N$ is nice in $G$, we may write $\bigcap_{\delta<\gamma}[N+p^\delta G]=N+p^\gamma G$ for every limit ordinal $\gamma$. Furthermore, with Lemma 2.1 at hand, we subsequently deduce that

$$\begin{aligned}\bigcap_{\delta<\gamma}[N+p^\delta(G+F)] &= \bigcap_{\delta<\gamma}[N+p^\delta G+(F\cap p^\delta(G+F))]\\ &\subseteq\bigcap_{\delta<\gamma}(N+p^\delta G)+[F\cap p^\gamma(G+F)]\\ &= N+p^\gamma G+[F\cap p^\gamma(G+F)]=N+p^\gamma(G+F).\end{aligned}$$

The inclusion "$\subseteq$" follows thus: Given $x\in\cap_{\delta<\gamma}[N+p^\delta G+(F\cap p^\delta(G+F))]$, we write $x=a_1+g_1+f_1=\dots=a_s+g_s+f_s=\dots=a_k+g_k+f_1=\dots$, where $a_1,\dots,a_k\in N$; $g_1\in p^{\delta_1}G,\dots,g_k\in p^{\delta_k}G$; $f_1\in F\cap p^{\delta_1}(G+F),\dots,f_k\in F\cap p^{\delta_k}(G+F)$ with $\delta_1<\dots<\delta_k$. Hence $a_1+g_1=\dots=a_k+g_k=\dots\in\bigcap_{\delta<\gamma}(N+p^\delta G)$ and because the number of the $f_i$'s ($1 \leqslant i \leqslant k$) is finite whereas the number of equalities is not, we can deduce that $f_1 \in \bigcap_{\delta<\gamma}(F \cap p^\delta(G+F)) = F \cap p^\gamma(G+F)$, as needed. $\square$

The following can be seen as Proposition 2.20 from [**2**]. It is listed here only for the sake of completeness and the readers' convenience.

Lemma 2.3. *If $T$ is almost $n$-simply presented and $G/T$ is countable, then $G$ is almost $n$-simply presented.*

Proof. Write $G = T + K$, where $K \leqslant G$ is countable. With bullet three listed above at hand, there exists $P \leqslant T[p^n]$ such that $T/P$ is almost simply presented. Furthermore, $G/P = (T/P) + (K+P)/P$, where $(K+P)/P \cong K/(K \cap P)$ is countable. Thus Theorem 1 of [**1**] (see [**8**] too) can be employed to show that $G/P$ is almost simply presented, as required. $\square$

The next statement appears as Proposition 2.23 (b) in [**2**] in the case of almost $n$-simply presented groups. We here extend it even in the situation of nicely almost $n$-simply presented groups.

Lemma 2.4. *If $S$ is a subgroup of a group $G$ such that $G/S$ is finite, then $G$ is (nicely) almost $n$-simply presented if and only if $S$ is (nicely) almost $n$-simply presented.*

Proof. Write $G = S + F$, where $F \leqslant G$ is finite. Suppose first that $S$ is nicely almost $n$-simply presented. With bullet three quoted above in hand, there is $Z \leqslant S[p^n]$ which is nice in $S$ such that $S/Z$ is almost simply presented. We therefore have that $G/Z = [S/Z] + [(F+Z)/Z]$, where $(F+Z)/Z \cong F/(F \cap Z)$ is finite. Again by virtue of Theorem 1 in [**1**], $G/Z$ should be almost simply presented. But $Z$ is nice in $G$ utilizing Lemma 2.2 (ii), as required.

Reciprocally, let $G$ be nicely almost $n$-simply presented. Since $p^tG = p^tS$ for some $t \in \mathbb{N}$, and in Lemma 2.3 (iii) of [**2**] it was established that any group $A$ is nicely almost $n$-simply presented if and only if so is $p^tA$, one may derive that $S$ is nicely almost $n$-simply presented. Actually, this idea also provides a new verification of the sufficiency, considered above.

The same method works for almost $n$-simply presented groups as well. $\square$

So, we are now coming to the following (see also Proposition 2.23 (a) from [**2**]).

Lemma 2.5. *A group $G$ is almost $n$-simply presented if and only if $G/F$ is almost $n$-simply presented for some finite subgroup $F$ of $G$.*

Proof. The "and only if" direction was proved as Corollary 2.19 in [**2**].

To treat the "if" one, write $(G/F)/(A/F) \cong G/A$ is almost simply presented for some $A \leqslant G$ such that $p^nA \subseteq F \subseteq A$. Since $p^nA$ is finite, it is a routine technical exercise to check that $A = L + A[p^n]$ for some finite $L \leqslant A$. Furthermore, $G/A \cong (G/A[p^n])/(A/A[p^n])$ being almost simply presented with finite $A/A[p^n] \cong L/L[p^n]$ implies with the help of [**1**] (see [**8**] or [**9**] as well) that $G/A[p^n]$ is almost simply presented, as required. $\square$

With the last assertion at hand, one observes that almost $\omega$-$n$-simply presented groups are exactly the almost $n$-simply presented ones.

## 3. Main results

We begin this section with some different characterizations of almost $\omega_1$-$n$-simply presented groups. The first major result is the following:

THEOREM 3.1. *The following points are equivalent:*

(i) *$G$ is almost $\omega_1$-$n$-simply presented;*
(ii) *$G/(C \oplus L)$ is almost simply presented, where $C$ is a countable subgroup of $G$ and $L$ is a $p^n$-bounded subgroup of $G$;*
(iii) *$G/L$ is almost $\omega_1$-simply presented for some $L \leqslant G[p^n]$.*

PROOF. (i) $\Leftrightarrow$ (ii). Foremost, letting point (i) be fulfilled, given $G/K$ is almost $n$-simply presented for some countable subgroup $K \leqslant G$. Thus there is $A/K$ with $A \leqslant G$ and $p^n A \subseteq K$ such that $G/A$ is almost simply presented. But it is well known that $A = C \oplus L$, and hence (ii) holds.

Conversely, assume that clause (ii) is true. Thus $G/(C \oplus L) \cong [G/C]/[(C \oplus L)/C]$ is almost simply presented, where $(C \oplus L)/C \cong L$ is $p^n$-bounded. Therefore $G/C$ is almost $n$-simply presented, as required.

(ii) $\Leftrightarrow$ (iii). First, assuming that point (ii) is valid, we see that $G/(C \oplus L) \cong [G/L]/[(C \oplus L)/L]$ is almost simply presented, where $(C \oplus L)/L \cong C$ is countable. So, $G/L$ is almost $\omega_1$-simply presented.

Reciprocally, let clause (iii) be true, so given $G/L$ is almost $\omega_1$-simply presented for some $p^n$-bounded subgroup $L$. Hence there is a countable subgroup $B/L$ with $B \leqslant G$ such that $(G/L)/(B/L) \cong G/B$ is almost simply presented. Besides, $B = L + K$ for some countable $K \leqslant B$. Since $p^n L = \{0\}$, we write $L = L_1 \oplus L_2$, where $L_2$ is countable and $L \cap K \subseteq L_2$. Observe that $B = L_1 + (K + L_2)$, where $L_1$ is $p^n$-bounded and $K + L_2$ is countable. Moreover, $L_1 \cap (K + L_2) = \{0\}$; indeed take $a = b + c$, where $a \in L_1$, $b \in K$ and $c \in L_2$. Furthermore, $a - c \in L \cap K \subseteq L_2$, whence $a \in L_1 \cap L_2 = \{0\}$ and so $a = 0$. Finally, $B = L_1 \oplus (K + L_2)$ and thus $G/(C \oplus M)$ is almost simply presented for the countable $C = K + L_2$ and the $p^n$-bounded $M = L_1$, as stated. □

We continue with some other structural affirmations.

PROPOSITION 3.1. *If $G$ is nicely almost $\omega_1$-$n$-simply presented, then $G/p^\omega G$ is almost $p^{\omega+n}$-projective. In particular, separable nicely almost $\omega_1$-$n$-simply presented groups are always almost $p^{\omega+n}$-projective.*

PROOF. According to Corollary 2.8 of [**3**], one observes that the quotient $(G/N)/p^\omega(G/N) \cong G/(p^\omega G + N)$ is almost $p^{\omega+n}$-projective, where $N$ is a countable nice subgroup of $G$. But $G/(p^\omega G + N) \cong [G/p^\omega G]/[(p^\omega G + N)/p^\omega G]$, where it is obvious that $(p^\omega G + N)/p^\omega G \cong N/(N \cap p^\omega G)$ is countable. Henceforth, we apply Proposition 2.10 from [**3**] to get the first claim. The second part is its trivial consequence. □

COROLLARY 3.1. *Suppose $G$ is a group for which $p^\omega G$ is countable. Then $G$ is nicely almost $\omega_1$-$n$-simply presented if and only if $G$ is almost $\omega_1$-$p^{\omega+n}$-projective.*

PROOF. In view of Proposition 3.1, the factor-group $G/p^\omega G$ is almost $p^{\omega+n}$-projective. Hence, in virtue of [**3**], $G$ is almost $\omega_1$-$p^{\omega+n}$-projective, as expected.

The reverse implication is obvious. □

PROPOSITION 3.2. *If $G$ is both an almost $\omega_1$-$n$-simply presented group and a c.c. group, then $G/p^\omega G$ is almost $p^{\omega+n}$-projective.*

PROOF. Let $G/K$ be an almost $n$-simply presented group, where $K$ is a countable subgroup of $G$. Therefore, we apply Corollary 2.8 from [**3**] to show that

$$(G/K)/p^\omega(G/K) \cong G/\Big[\bigcap_{i<\omega}(p^iG+K)\Big] \cong (G/p^\omega G)/\Big[\bigcap_{i<\omega}(p^iG+K)/p^\omega G\Big]$$

is almost $p^{\omega+n}$-projective. Since $[\bigcap_{i<\omega}(p^iG+K)]/p^\omega G$ is countable, again Proposition 2.10 in [**3**] applies to get that $G/p^\omega G$ remains almost $p^{\omega+n}$-projective, as desired. □

As two immediate consequences, we deduce:

COROLLARY 3.2. *Suppose $G$ is a c.c. group. Then $G$ is almost $\omega_1$-$n$-simply presented if and only if $G$ is almost $\omega_1$-$p^{\omega+n}$-projective.*

PROOF. The sufficiency being elementary, we deal with the necessity. Since c.c. groups are obviously with countable first Ulm subgroup, Proposition 3.2 allows us to conclude with the help of [**3**] that $G$ is almost $\omega_1$-$p^{\omega+n}$-projective, as stated. □

We recollect that a group is termed *weakly $\omega_1$-separable* if it is a separable c.c. group. So, we directly obtain:

COROLLARY 3.3. *Suppose $G$ is a weakly $\omega_1$-separable group. Then $G$ is almost $\omega_1$-$n$-simply presented if and only if $G$ is almost $p^{\omega+n}$-projective.*

Furthermore, we come to the following.

PROPOSITION 3.3. *Suppose that $A$ is a group with a countable subgroup $L$. Then $A$ is almost $\omega_1$-$n$-simply presented if and only if $A/L$ is almost $\omega_1$-$n$-simply presented.*

PROOF. First, let us assume that $A$ be almost $\omega_1$-$n$-simply presented, hence $A/K$ is almost $n$-simply presented for some countable $K \leqslant A$. But

$$[A/L]/[(L+K)/L] \cong A/(L+K) \cong [A/K]/[(L+K)/K],$$

where the last factor-group $[A/K]/[(L+K)/K]$ is almost $n$-simply presented by Proposition 2.18 (or Corollary 2.19) of [**2**] since $(L+K)/K$ is countable. Therefore, $[A/L]/[(L+K)/L]$ is almost $n$-simply presented with countable $(L+K)/L \cong K/(K\cap L)$, as wanted.

Reciprocally, let us now $A/L$ be almost $\omega_1$-$n$-simply presented, and so let $C/L$ be a countable subgroup of $A/L$ for some $C \leqslant A$ such that $(A/L)/(C/L) \cong A/C$ is almost $n$-simply presented. Observing that $C$ is of necessity countable, we deduce via Definition 1.1 that $A$ is almost $\omega_1$-$n$-simply presented, as formulated. □

As an easy consequence, we deduce:

Corollary 3.4. *Suppose $A$ is a group such that $p^\alpha A$ is countable for some ordinal $\alpha$. Then $A$ is almost $\omega_1$-$n$-simply presented if and only if $A/p^\alpha A$ is almost $\omega_1$-$n$-simply presented.*

Proposition 3.4. *Let $A$ be a group with a subgroup $G$ such that $A/G$ is countable. Then $A$ is almost $\omega_1$-$n$-simply presented if and only if $G$ is almost $\omega_1$-$n$-simply presented.*

Proof. Write $A = G+C$ where $C$ is countable and assume that $G$ is almost $\omega_1$-$n$-simply presented. Now, Definition 1.1 insures that there is a countable subgroup $K$ such that $G/K$ is almost $n$-simply presented. Consequently, $A/K = (G/K) + (C+K)/K$. Employing Lemma 2.3, $A/K$ is almost $n$-simply presented since $(C+K)/K$ is obviously countable. This gives that $A$ is almost $\omega_1$-$n$-simply presented, as desired.

Conversely, let us assume that $A$ is almost $\omega_1$-$n$-simply presented. Now, Proposition 3.3 guarantees that $(G+C)/C \cong G/(G\cap C)$ is almost $\omega_1$-$n$-simply presented. But $G\cap C$ is countable and again Proposition 3.3 will work to get that $G$ is almost $\omega_1$-$n$-simply presented, as wanted. □

We are now ready to prove the following central result:

Theorem 3.2. *The class of almost $\omega_1$-$n$-simply presented groups is closed under the formation of $\omega_1$-bijections, and is the smallest class containing almost $n$-simply presented groups with this property.*

*In other words, if $f : G \to A$ is an $\omega_1$-bijective homomorphism and $G$ is an almost $\omega_1$-$n$-simply presented group, then $A$ is an almost $\omega_1$-$n$-simply presented group, and thus almost $\omega_1$-$n$-simply presented groups form the minimal class of groups possessing that property.*

Proof. The first part follows by [**10**, Lemma 1.9] accomplished with Propositions 3.3 and 3.4.

For the second one, that it is the minimal class possessing that property, we making use Proposition 1.10 of [**10**] and Theorem 3.1. □

Proposition 3.5. *Suppose $A$ is a group with a finite subgroup $F$. Then $A$ is nicely almost $\omega_1$-$n$-simply presented if and only if $A/F$ is nicely almost $\omega_1$-$n$-simply presented.*

Proof. Assume first that $A$ is nicely almost $\omega_1$-$n$-simply presented, i.e., there is a countable nice subgroup $N$ such that $A/N$ is almost $n$-simply presented. Observing as above that

$$[A/F]/[(F+N)/F] \cong A/(F+N) \cong [A/N]/[(F+N)/N],$$

and that $[A/N]/[(F+N)/N]$ is almost $n$-simply presented, it follows that $A/F$ is nicely almost $\omega_1$-$n$-simply presented, because $(F+N)/F$ is countable and nice in $A/F$ in accordance with Lemma 2.2 (i).

Conversely, given that $A/F$ is nicely almost $\omega_1$-$n$-simply presented, so there exists a countable nice subgroup $C/F$ of $A/F$ with $C \leqslant A$ such that the factor-group $(A/F)/(C/F) \cong A/C$ is almost $n$-simply presented. Since $F$ is nice in $A$, one can see that $C$ is countable and nice in $A$ (see [**5**]), whence Definition 1.2 gives the claim. □

Proposition 3.6. *Let $A$ be a group with a subgroup $G$ such that $A/G$ is finite. Then $A$ is almost $\omega_1$-$p^{\omega+n}$-projective if and only if $G$ is almost $\omega_1$-$p^{\omega+n}$-projective.*

Proof. Repeating the same method as in Proposition 3.3 combined with Proposition 3.5, we complete the arguments. □

We are now ready to establish the following main result:

Theorem 3.3. *The class of nicely almost $\omega_1$-$n$-simply presented groups is closed under taking $\omega$-bijections.*

Proof. Follows by our discussion in Section 2 (see again [**10**, Lemma 1.9]) along with Propositions 3.5 and 3.6. □

Proposition 3.7. *Let $A$ be a group with a countable nice subgroup $N$. If $A/N$ is nicely almost $\omega_1$-$n$-simply presented, then so is $A$.*

## 4. Nunke-like theorems

We will now prove some versions of Nunke-esque results for the new group classes defined in the introductory section. Generalizing the given above concept of a c.c. group, introduced in [**7**], one may define the following notion:

Definition 4.1. Let $\lambda$ be an ordinal. A group $G$ is said to be $\lambda$-*countably* if for any countable subgroup $K \leqslant G$ the quotient $p^\lambda(G/K)/(p^\lambda G + K)/K = p^\lambda((G/K)/(p^\lambda G + K)/K)$ is countable.

Theorem 4.1. *Suppose $G$ is a $\lambda + n$-countably group for some ordinal $\lambda$ such that $p^\lambda G$ is almost $n$-simply presented. Then $G$ is almost $\omega_1$-$n$-simply presented if and only if $G/p^{\lambda+n}G$ is almost $\omega_1$-$n$-simply presented.*

Proof. ($\Rightarrow$) Given $G/K$ is an almost $n$-simply presented group for some countable $K \leqslant G$. Consequently, Theorem 2.7 (a) in [**2**] forces that

$$\begin{aligned}(G/K)/p^{\lambda+n}(G/K) &\cong [(G/K)/(p^{\lambda+n}G + K)/K]/[p^{\lambda+n}(G/K)/(p^{\lambda+n}G + K)/K] \\ &= [(G/K)/(p^{\lambda+n}G + K)/K]/p^{\lambda+n}((G/K)/(p^{\lambda+n}G + K)/K)\end{aligned}$$

is also almost $n$-simply presented. Because of the countability of the quotient $p^{\lambda+n}(G/K)/[(p^{\lambda+n}G + K)/K] = p^{\lambda+n}((G/K)/(p^{\lambda+n}G + K)/K)$, a simple appeal to Theorem 2.10 of [**2**] leads to almost $n$-simply presentness of

$$[G/K]/[(p^{\lambda+n}G + K)/K] \cong G/(p^{\lambda+n}G + K) \cong [G/p^{\lambda+n}G]/[(p^{\lambda+n}G + K)/p^{\lambda+n}G].$$

And since $(p^{\lambda+n}G + K)/p^{\lambda+n}G \cong K/(K \cap p^{\lambda+n}G)$ is countable, we are done.

($\Leftarrow$) Let $(G/p^{\lambda+n}G)/(C/p^{\lambda+n}G) \cong G/C$ be almost $n$-simply presented for some countable $C/p^{\lambda+n}G$ with $C \leqslant G$. Write $C = p^{\lambda+n}G + K$ for some countable subgroup $K$. So, $G/(p^{\lambda+n}G + K) \cong [G/K]/[(p^{\lambda+n}G + K)/K]$ is almost $n$-simply presented; note that this gives with the aid of Theorem 2.7 (a)from [**2**] that $p^{\lambda+n}((G/K)/(p^{\lambda+n}G+K)/K) = p^{\lambda+n}(G/K)/(p^{\lambda+n}G+K)/K$ is almost $n$-simply presented – however we have by assumption the more restrictive condition that this quotient is countable. Moreover, again Theorem 2.7(a) or Corollary 2.19 in [**2**] applies to conclude that

$$\begin{aligned}
&[(G/K)/(p^{\lambda+n}G + K)/K]/p^{\lambda+n}((G/K)/(p^{\lambda+n}G + K)/K) \\
&\cong [(G/K)/(p^{\lambda+n}G + K)/K]/[p^{\lambda+n}(G/K)/(p^{\lambda+n}G + K)/K] \\
&\qquad\cong (G/K)/p^{\lambda+n}(G/K)
\end{aligned}$$

is almost $n$-simply presented. But, on the other hand, by Theorem 2.7 (a) in [**2**] we deduce that $p^{\lambda+n}G$ is almost $n$-simply presented. Moreover, $(p^{\lambda+n}G+K)/K \cong p^{\lambda+n}G/(p^{\lambda+n}G \cap K)$, which means by Proposition 2.18 of [**2**] that the second term, and hence the first one, are almost $n$-simply presented because $p^{\lambda+n}G \cap K$ is countable. We therefore may apply Lemma 2.3 to derive that $p^{\lambda+n}(G/K)$ is almost $n$-simply presented. Finally, utilizing Theorem 2.10 in [**2**] to get after all that $G/K$ is almost $n$-simply presented, as expected. □

Notice that we have not used in the necessity the condition that $p^\lambda G$ is almost $n$-simply presented, so that what immediately arises is whether or not this limitation can be dropped off in the formulation of the theorem.

PROPOSITION 4.1. *Let $G$ be a group and $\alpha$ an ordinal. If $G$ is nicely almost $\omega_1$-$n$-simply presented, then $p^\alpha G$ and $G/p^\alpha G$ are nicely almost $\omega_1$-$n$-simply presented.*

PROOF. Let $G/N$ be almost $n$-simply presented for some countable nice subgroup $N$ of $G$. Hence, using [**2**], $p^\alpha(G/N) = (p^\alpha G + N)/N \cong p^\alpha G/(p^\alpha G \cap N)$ is almost $n$-simply presented, where $p^\alpha G \cap N$ is countable and nice in $p^\alpha G$ (cf. [**5**]).

Moreover, $(G/N)/p^\alpha(G/N) \cong G/(p^\alpha G + N) \cong [G/p^\alpha G]/[(p^\alpha G + N)/p^\alpha G]$ is almost $n$-simply presented, where $(p^\alpha G + N)/p^\alpha G \cong N/(p^\alpha G \cap N)$ is countable and $(p^\alpha G + N)/p^\alpha G$ is nice in $G/p^\alpha G$, because $N + p^\alpha G$ is so in $G$. □

PROPOSITION 4.2. *Suppose that $G/p^{\lambda+n}G$ is almost $n$-simply presented for some ordinal $\lambda$. Then $G$ is nicely almost $\omega_1$-$n$-simply presented if and only if $p^\lambda G$ is nicely almost $\omega_1$-$n$-simply presented.*

PROOF. The "only if" part follows by a direct application of Proposition 4.1.

As for the "if" part, let $p^\lambda G/Y = p^\lambda(G/Y)$ be almost $n$-simply presented for some nice countable subgroup $Y$. Hence $Y$ is also nice in $G$ (see, e.g., [**5**]), and besides $p^{\lambda+n}(G/Y)$ remains almost $n$-simply presented in conjunction with Theorem 2.7 (a) from [**2**]. But $G/p^{\lambda+n}G \cong (G/Y)/p^{\lambda+n}(G/Y)$ is almost $n$-simply presented by assumption. Now, the application of Theorem 2.10 of [**2**] leads us to $G/Y$ is almost $n$-simply presented, as wanted. □

## 5. Open problems

In closing, we shall state two left-open questions which remain unanswered.

PROBLEM 1. Does it follow that if $G$ is an almost $\omega_1$-$n$-simply presented group with $p^\omega G = \{0\}$, then it is almost $p^{\omega+n}$-projective?

PROBLEM 2. Let $\alpha$ be an ordinal. Does it follow that $G$ is (nicely) almost $\omega_1$-$n$-simply presented if and only if both $p^\alpha G$ and $G/p^\alpha G$ are (nicely) almost $\omega_1$-$n$-simply presented?

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