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IRRATIONALITY MEASURES FOR CONTINUED FRACTIONS WITH ARITHMETIC FUNCTIONS

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Abstract. Let $f(n)$ or the base-2 logarithm of $f(n)$ be either $d(n)$ (the divisor function), $\sigma(n)$ (the divisor-sum function), $\varphi(n)$ (the Euler totient function), $\omega(n)$ (the number of distinct prime factors of n) or $\Omega(n)$ (the total number of prime factors of n). We present good lower bounds for $\left|\frac{M}{N}-\alpha\right|$ in terms of N, where $\alpha = [0; f(1), f(2), \ldots]$.

1. Introduction and notations

For $\alpha \in \mathbb{R}$ and $N \in \mathbb{Z}_+$ denote $J_\alpha(N) = N\|N\alpha\|$, where $\|\cdot\|$ means the distance to the nearest integer. The function $J_\alpha(N)$ is connected to the rougher concept of the irrationality exponent $\mu(\alpha)$, the infimum of exponents μ such that $J(N) \leqslant N^{2-\mu}$ holds for infinitely many N. For almost all α we have $\mu(\alpha) = 2$, although $\mu(\alpha) = 1$ for rational numbers and $\mu(\alpha) \in (2, \infty]$ for a zero-measure subset of irrational numbers. For more information, see [**1**] for example. In all of our examples we have the usual $\mu(\alpha) = 2$ but we go further by studying the more refined function $J_\alpha(N)$. For irrational α we are interested in finding lower bounds $J_\alpha(N) \geqslant f(N)$ for $N \geqslant N_0$. To emphasize that our results are in some sense sharp we also give bounds $J_\alpha(N) \leqslant g(N)$ holding for infinitely many N. Throughout the work, this kind of pair of bounds is denoted by

$$J_\alpha(N) \in \big(f(N), g(N)\big\rangle.$$

Another short-hand notation used in the statement of Theorem 2.1 is to write

$$L_k(N) = \underbrace{\log\log\cdots\log}_{k \text{ times}} N.$$

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Because of the law of best approximations, the simple continued fraction expansion of α is ideal for bounding $J_\alpha(N)$. Recall that if $\alpha = [a_0; a_1, \ldots]$ is the simple continued fraction expansion of the irrational number α and $p_n/q_n = [a_0; a_1, \ldots, a_n]$ is the n-th convergent for each $n \in \mathbb{N}$, we have the recursion formulae

$$\begin{aligned} &p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1, \\ &p_{n+2} = a_{n+2} p_{n+1} + p_n, \quad q_{n+2} = a_{n+2} q_{n+1} + q_n \end{aligned} \tag{1.1}$$

for $n \in \mathbb{N}$, and the estimates

$$\frac{1}{q_n^2(a_{n+1}+2)} < \left| a - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2 a_{n+1}} \tag{1.2}$$

for $n \in \mathbb{N}$. We shall use the notation $[0; \overline{f(j)}]_{j=1}^\infty = [0; f(1), f(2), \ldots]$, where $f \colon \mathbb{Z}_+ \to \mathbb{Z}_+$ is a function. For more details on continued fractions please see the book of Hardy and Wright [**6**] for example.

The case where the asymptotic geometric mean of the sequence $\{a_j\}_{j=0}^\infty$ tends to infinity is easiest to deal with, although it is untypical in the metric sense. In that case the lowest behavior of the function $J_\alpha(N)$ is basically governed by the maximal order and the asymptotic geometric mean of the sequence $\{a_j\}_{j=0}^\infty$. In our examples we have chosen $a_n = f(n)$ or $a_n = 2^{f(n)}$ with some arithmetic function $f(n)$, because maximal orders of arithmetic functions have been extensively studied by Landau, Ramanujan, Nicolas, etc. (see [**2, 6, 7, 8, 9**] for example). On the other hand, the behavior of the asymptotic geometric mean of arithmetic functions is generally not known. This is not a big problem however; as long as there is enough error in the maximal order of $f(n)$, it suffices to bound the asymptotic geometric mean by asymptotic arithmetic mean, which again is usually known (more typically called the average order of $f(n)$, see [**6, 10**] for example). And of course in the cases $a_n = 2^{f(n)}$ the base-2 logarithm of the geometric mean of a_n simply corresponds to the arithmetic mean of $f(n)$.

Finally we note that we have already introduced bounds like (2.2) as an example in our currently unpublished work [**4**]. However, the result of (2.2) is slightly sharper and presented here for completeness.

2. Results

The following theorem contains our bounds in all of the ten examples. Note that the lower bounds are always asymptotically bigger than any negative power of N, implying that in each case the irrationality exponent is 2.

THEOREM 2.1. *Let $d(n)$ be the number of positive divisors of n. Then*

$$J_{[0;\overline{d(j)}]_{j=1}^\infty}(N) \in \left(2^{-\frac{L_2(N)(L_3^2(N)+L_3(N)+4.7624)}{L_3^3(N)}}, 2^{-\frac{L_2(N)}{L_3(N)-1+O(L_4(N)/L_3(N))}} \right\rangle, \tag{2.1}$$

$$(2.2)\quad J_{[0;\overline{2^{d(j)}}]_{j=1}^{\infty}}(N) \in \left\langle 2^{-2^{\frac{L_2(N)(L_3^2(N)+L_3(N)+4.7624)}{L_3^3(N)}}}, 2^{-2^{\frac{L_2(N)}{L_3(N)-1+O(L_4(N)/L_3(N))}}} \right\rangle,$$

where N is big enough.

Let $\sigma(n)$ be the sum of positive divisors of n. Then

$$(2.3)\qquad J_{[0;\overline{\sigma(j)}]_{j=1}^{\infty}}(N) \in \left\langle \frac{L_3(N)(L_2(N)-L_3(N))}{L_1(N)(e^{\gamma}L_3^2(N)+0.6483)}, \frac{L_2(N)(1+o(1))}{e^{\gamma}L_1(N)L_3(N)} \right\rangle,$$

$$(2.4)\quad J_{[0;\overline{2^{\sigma(j)}}]_{j=1}^{\infty}}(N) \in \left\langle 2^{-\frac{2\sqrt{3L_1(N)}(e^{\gamma}(L_3(N)-L_1(2))L_3(N)+0.6483)}{\pi\sqrt{L_1(2)}L_3(N)}}, 2^{-\frac{2e^{\gamma}}{\pi}\sqrt{\frac{3L_1(N)}{L_1(2)}}L_3(N)(1+o(1))} \right\rangle,$$

where γ is the Euler–Mascheroni constant and N is big enough.

Let $\varphi(n)$ be the number of positive integers less than and prime to n, denote

$$A = \sum_{p \text{ prime}} \frac{1}{p}\log\left(1+\frac{1}{p}\right),$$

let $z(x)$ stand for the inverse of the function $y(x) = x\log x$, and define $z_0(x) = x$ and $z_{k+1}(x) = x/\log(z_k(x))$ for each k (see [**3**])*. Then*

$$(2.5)\quad J_{[0;\overline{\varphi(j)}]_{j=1}^{\infty}}(N) \in \left\langle \frac{2L_1(N)L_1(z(e^{A-1}L_1(N))) + L_2^2(N) - 2L_1(N)L_3(N) + O(L_2(N))}{2L_1^2(N)+4L_1(N)}, \frac{2L_1(N)L_1(z(e^{A-1}L_1(N))) + L_2^2(N) + O(L_2(N))}{2L_1^2(N)} \right\rangle,$$

or if one prefers the use of elementary functions only then one can replace $L_1(z(e^{A-1}L_1(N)))$ by $L_1(z_k(e^{A-1}L_1(N))) + O(L_3(N)/L_2^k(N))$ for any k, and

$$(2.6)\quad J_{[0;\overline{2^{\varphi(j)}}]_{j=1}^{\infty}}(N) \in \left\langle 2^{-\pi\sqrt{\frac{L_1(N)}{3\log 2}}+O(L_2^{2/3}(N)L_3^{3/4}(N))}, 2^{-\pi\sqrt{\frac{L_1(N)}{3\log 2}}+O(L_2^{2/3}(N)L_3^{3/4}(N))} \right\rangle.$$

Let $\omega(n)$ be the number of different prime factors of n, counted without multiplicities. Then

$$(2.7)\quad J_{[0;\overline{\omega(j)}]_{j=1}^{\infty}}(N) \in \left\langle \frac{L_3^3(N)}{L_2(N)(L_3^2(N)+L_3(N)+2.89727)}, \frac{L_3(N)-1+O\left(\frac{L_4(N)}{L_3(N)}\right)}{L_2(N)} \right\rangle,$$

$$(2.8)\quad J_{[0;\overline{2^{\omega(j)}}]_{j=1}^{\infty}}(N) \in \left(2^{-\frac{L_2(N)(L_3^2(N)+L_3(N)+2.89727)}{L_3^3(N)}}, 2^{-\frac{L_2(N)}{L_3(N)-1+O(L_4(N)/L_3(N))}}\right\rangle,$$

where N is big enough.

Let $\Omega(n)$ be the number of different prime factors of n, counted with multiplicities, and denote

$$B = 1 - \log\log 2 + \int_2^\infty \frac{\sum_{p\leqslant t}\frac{1}{p}\log p - \log t\,dt}{t(\log t)^2} + \sum_{p \text{ prime}} \frac{1}{p(p-1)}.$$

Then

$$(2.9)\quad J_{[0;\overline{\Omega(j)}]_{j=1}^{\infty}}(N) \in \left(\frac{L_1(2)}{L_2(N)-L_5(N)+2L_1(2)+o(1)}, \frac{L_1(2)}{L_2(N)-L_5(N)+o(1/L_4(N))}\right\rangle,$$

$$(2.10)\quad J_{[0;\overline{2^{\Omega(j)}}]_{j=1}^{\infty}}(N) \in \left(\frac{L_1(2)(L_3(N)+B+o(1))}{L_1(N)}, \frac{L_1(2)(L_3(N)+B+o(1))}{L_1(N)}\right\rangle.$$

Proof of Equation (2.1). The recursive formula (1.1) gives us

$$q_n = a_n q_{n-1}\left(1+\frac{q_{n-2}}{a_n q_{n-1}}\right) = \cdots = \prod_{j=1}^{n} a_j \prod_{j=1}^{n-1}\left(1+\frac{q_{j-1}}{a_{j+1}q_j}\right) = e^{O(n)}\prod_{j=1}^{n} a_j,$$

since $1+q_{j-1}/(a_{j+1}q_j) \in (1,2)$ for all j. By using the asymptotic arithmetic mean

$$(2.11)\qquad \frac{1}{n}\sum_{j=1}^{n} d(j) = \log n + O(1)$$

(Theorem 320 on page 347 of [**6**]) and the arithmetic–geometric inequality we get an upper bound $\log q_n \leqslant n\log\log n + O(n)$. On the other hand, the recursive formula (1.1) implies a trivial lower bound $\log q_n \geqslant bn$ with some positive constant b. Now we can solve n from those inequalities to get

$$\begin{aligned} n &\geqslant \frac{\log q_n}{\log\log n + O(1)} \geqslant \frac{\log q_n}{\log\log(\frac{1}{b}\log q_n)+O(1)} = \frac{\log q_n}{\log(\log\log q_n - \log b)+O(1)} \\ &= \frac{\log q_n}{\log\log\log q_n + O(1/\log\log q_n) + O(1)} = \frac{\log q_n}{\log\log\log q_n + O(1)} \end{aligned}$$

and $n \leqslant \log q_n/b$.

To show that the lower bound of the claim always holds we use a result

$$(2.12)\qquad d(n) \leqslant 2^{\frac{\log n}{\log\log n}\left(1+\frac{1}{\log\log n}+\frac{4.7623\ldots}{(\log\log n)^2}\right)}$$

of Robin (Proposition 8 in [**8**] with the constant rounded up to 4.7624; the fact that the constant is in fact strictly smaller is implicitly in [**9**]) and the left inequality in (1.2), and we simplify this. Note that rounding the constant 4.7623... up to 4.7634 causes an error big enough to make many other terms, including the ones with b, negligible.

To show that the upper bound of the claim holds infinitely often we use a slightly weaker estimate. It is easy to show with the estimate $p_n \sim n \log n$ of the n-th prime number (Theorem 8 on page 12 of [**6**]) and Abel's partial summation formula

$$\sum_{j=1}^{n} a_j b_j = b_n \sum_{j=1}^{n} a_j - \sum_{j=1}^{n-1} (b_{j+1} - b_j) \sum_{i=1}^{j} a_i, \tag{2.13}$$

that when n is a product of the first primes we have

$$d(n) = 2^{\frac{\log n}{\log\log n - 1 + O(\log\log\log n / \log\log n)}}. \tag{2.14}$$

The result follows after using the right inequality of (1.2) and simplifying. □

Proof of Equation (2.2). From the asymptotic arithmetic mean (2.11) we deduce

$$\frac{\log q_n}{\log 2} = n \log n + O(n) = n \log n + O(n \log\log n).$$

Because of the error in bounds (2.12) and (2.14), making our error bigger like this will not matter, but instead simplifies things. Solving for n gives

$$n = \frac{\log q_n}{\log 2 \log\log q_n + O(\log\log\log q_n)},$$

and after simplifying, the claims follow from estimations (2.12), (2.14) and (1.2). □

Proof of Equation (2.3). This time we bound the asymptotic geometric mean by the asymptotic arithmetic mean

$$\frac{1}{n} \sum_{j=1}^{n} \sigma(j) = \frac{\pi^2}{12} n + O(\log n) \tag{2.15}$$

(Theorem 324 on page 351 of [**6**]) from above, and by the trivial estimate $\sigma(j) \geqslant j+1$ together with Stirling's formula from below to show that $\log q_n = n \log n + O(n)$. Solving for n yields

$$n = \frac{\log q_n}{\log\log q_n - \log\log\log q_n + O(1)}.$$

To verify our claim on the lower bound we use the estimate

$$\sigma(n) \leqslant n \Big(e^{\gamma} \log\log n + \frac{0.6482\ldots}{\log\log n} \Big) \tag{2.16}$$

of Nicolas (Proposition 11 in [**8**]), where γ is the Euler–Mascheroni constant, and inequality (1.2).

For the upper bound we use Grönwall's theorem

$$\limsup_{n \to \infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma} \tag{2.17}$$

(equation (25) in [**2**]), implying that $\sigma(n) = e^{\gamma} n \log\log n(1+o(1))$ for infinitely many values of n, and inequality (1.2). $\square$

Proof of Equation (2.4). Now the asymptotic arithmetic mean (2.15) directly gives us the estimate

$$\log q_n = \frac{\pi^2 \log 2}{12} n^2 + O(n \log n).$$

Solving for n yields

$$n = \frac{2}{\pi}\sqrt{\frac{3\log q_n}{\log 2}} + O(\log\log q_n).$$

After simplifying, the claims follow from estimations (2.16), (2.17) and (1.2) again. $\square$

Proof of Equation (2.5). We happen to know exactly how the function $\varphi(n)$ behaves at its biggest. Still we are allowed to have some slack in the estimates of the convergents because of the difficulty of solving n in terms of q_n.

We write

$$\prod_{j=1}^{n} \varphi(j) = \prod_{j=1}^{n} j \prod_{\substack{p \text{ prime} \\ p \mid j}} \Big(1 - \frac{1}{p}\Big) = n! \prod_{\substack{p \text{ prime} \\ p \leqslant n}} \Big(1 - \frac{1}{p}\Big)^{\lfloor n/p \rfloor}.$$

Let us deal with the product over the primes first. By using the prime number theorem (Theorem 6 on page 10 of [**6**]) and partial integration, we see that

$$\sum_{\substack{p \text{ prime} \\ p \leqslant n}} \frac{1}{p} \log\Big(1 - \frac{1}{p}\Big) = A + O\Big(\frac{1}{n \log n}\Big),$$

where

$$A = \sum_{p \text{ prime}} \frac{1}{p} \log\Big(1 - \frac{1}{p}\Big)$$

is a known (negative) constant. So estimating $\lfloor n/p \rfloor \leqslant n/p$ gives

$$\log\bigg(\prod_{\substack{p \text{ prime} \\ p \leqslant n}} \Big(1 - \frac{1}{p}\Big)^{\lfloor n/p \rfloor}\bigg) \geqslant An + O\Big(\frac{1}{\log n}\Big). \tag{2.18}$$

On the other hand, by Mertens's theorem

$$\prod_{\substack{p \text{ prime} \\ p \leqslant n}} \Big(1 - \frac{1}{p}\Big) = \frac{1 + o(1)}{e^{\gamma} \log n}$$

(Theorem on 429 on page 466 of [**6**]), where γ is the Euler–Mascheroni constant, estimating $\lfloor n/p \rfloor \geqslant n/p - 1$ gives

$$\log\bigg(\prod_{\substack{p \text{ prime} \\ p \leqslant n}} \Big(1 - \frac{1}{p}\Big)^{\lfloor n/p \rfloor}\bigg) \leqslant An + \log\log n + \gamma + o(1). \tag{2.19}$$

Next note that by recursive formula (1.1) and Landau's theorem

$$\liminf_{n\to\infty} \frac{\varphi(n)\log\log n}{n} = \frac{1}{e^\gamma}$$

(pages 217–219 of [**7**]), where γ is the Euler–Mascheroni constant again, we see

$$\prod_{j=1}^{n-1}\left(1+\frac{q_{j-1}}{\varphi(j+1)q_j}\right) = O(1)$$

as a beginning of a converging product. Finally we use Stirling's formula and bounds (2.18) and (2.19) to get

$$\begin{aligned}\log q_n &\leqslant n\log n + (A-1)n + \tfrac{1}{2}\log(2\pi n) + \log\log n + O(1)\\ \log q_n &\geqslant n\log n + (A-1)n + \tfrac{1}{2}\log(2\pi n) + O(1).\end{aligned}$$

In any case we have at least

$$n = \frac{\log q_n}{\log n + A - 1 + O(1/n\cdot\log n)},$$

which is equivalent to

$$e^{A-1}n\log(e^{A-1}n) = e^{A-1}\log q_n + O(\log n).$$

Now we apply the function $z(x)$ (the inverse of $x\log x$) and the logarithm function to both sides of the equation to get

$$\log n + A - 1 = \log z\left(e^{A-1}\log q_n + O(\log n)\right).$$

The mean-value theorem implies

$$\begin{aligned}\log n + A - 1 &= \log z\left(e^{A-1}\log q_n + O(\log n)\right)\\ &= \log\left(z(e^{A-1}\log q_n) + O\Big(\frac{\log n}{\log z(\log q_n)}\Big)\right)\\ &= \log z(e^{A-1}\log q_n) + O\Big(\frac{\log n}{\log q_n}\Big),\end{aligned}$$

Using this we can solve n:

$$\begin{aligned} n &\geqslant \frac{\log q_n}{\log z(e^{A-1}\log q_n) + \frac{(\log\log q_n)^2}{2\log q_n} + O\big(\frac{\log\log q_n}{\log q_n}\big)}\\ n &\leqslant \frac{\log q_n}{\log z(e^{A-1}\log q_n) + \frac{(\log\log q_n)^2}{2\log q_n} - \log\log\log q_n + O\big(\frac{\log\log q_n}{\log q_n}\big)}.\end{aligned}$$

Now the claims follow from using (1.2), since $\varphi(n) \leqslant n-1$ with equality whenever n is a prime.

We can also derive bounds that use only elementary functions but have bigger error. By using the fact that $\log z(e^{A-1}\log q_n)$ as well as $\log z_k(e^{A-1}\log q_n)$ for any

k is $O(\log\log q_n)$, we see that

$$\begin{aligned}\log z(e^{A-1}\log q_n) &= \log\Big(\frac{e^{A-1}\log q_n}{\log z(e^{A-1}\log q_n)}\Big)\\ &= A-1+\log\log q_n + O(\log\log\log q_n)\\ &= \log z_0(e^{A-1}\log q_n) + O(\log\log\log q_n)\end{aligned}$$

and inductively

$$\begin{aligned}&\log z(e^{A-1}\log q_n) = \log\Big(\frac{e^{A-1}\log q_n}{\log z(e^{A-1}\log q_n)}\Big)\\ &= \log\Bigg(\frac{e^{A-1}\log q_n}{\log z_{k-1}(e^{A-1}\log q_n) + O\big(\frac{\log\log\log q_n}{(\log\log q_n)^{k-1}}\big)}\Bigg)\\ &= A-1+\log\log q_n - \log\log z_{k-1}(e^{A-1}\log q_n) + O\Big(\frac{\log\log\log q_n}{(\log\log q_n)^k}\Big)\\ &= \log z_k(e^{A-1}\log q_n) + O\Big(\frac{\log\log\log q_n}{(\log\log q_n)^k}\Big)\end{aligned}$$

for any k. □

Proof of Equation (2.6). From the asymptotic arithmetic mean

$$\frac{1}{n}\sum_{j=1}^{n}\varphi(j) = \frac{3}{\pi^2}n + O\Big((\log n)^{\frac{2}{3}}(\log\log n)^{\frac{3}{4}}\Big)$$

(see [**10**]) we get

$$\log q_n = \frac{3\log 2}{\pi^2}n^2 + O\Big(n(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{3}{4}}\Big).$$

Solving for n yields

$$n = \pi\sqrt{\frac{\log q_n}{3\log 2}} + O\Big((\log\log q_n)^{\frac{2}{3}}(\log\log\log q_n)^{\frac{3}{4}}\Big),$$

and so the claim follows by using (1.2), since $\varphi(n) \leqslant n-1$, with equality whenever n is a prime. □

Proof of Equation (2.7). The asymptotic arithmetic mean

$$\frac{1}{n}\sum_{j=1}^{n}\omega(j) = \log\log n + O(1) \tag{2.20}$$

(Theorem 430 on page 472 of [**6**]) implies $\log q_n \leqslant n\log\log\log n + O(n)$, and trivially $\log q_n \geqslant bn$ with some positive constant b. Solving for n gives

$$\frac{\log q_n}{\log\log\log\log q_n + O(1)} \leqslant n \leqslant \frac{\log q_n}{b}.$$

To see that the lower bound of the claim always holds we use a result

$$\omega(n) \leqslant \frac{\log n}{\log\log n}\Big(1 + \frac{1}{\log\log n} + \frac{2.89726\ldots}{(\log\log n)^2}\Big) \tag{2.21}$$

of Nicolas (Proposition 5 in [**8**]) and inequality (1.2). For the upper bound we use the fact that when n is a product of first primes, we have

$$\omega(n) = \frac{\log n}{\log\log n - 1 + O\left(\frac{\log\log\log n}{\log\log n}\right)}. \tag{2.22}$$

The claims follow from this and (1.2). □

Proof of Equation (2.8). Now the asymptotic arithmetic mean (2.20) directly gives us the estimate

$$\frac{\log q_n}{\log 2} = n \log\log n + O(n),$$

from which we solve

$$n = \frac{\log q_n}{\log 2\ \log\log\log q_n + O(1)}.$$

The claims follow from inequalities (2.21), (2.22) and (1.2). □

Proof of Equation (2.9). Again we want to be sharper than usual because we know the exact worst-case behavior of $\Omega(n)$. We shall use a theorem of Hardy and Ramanujan (Theorem C′ in [**5**]), stating that whenever $f(n)$ is a function tending to infinity, we have

$$\log\log n - f(n)\sqrt{\log\log n} \leqslant \Omega(n) \leqslant \log\log n + f(n)\sqrt{\log\log n} \tag{2.23}$$

for almost all n. In particular, by choosing $f(n) = (\log\log n)^{\frac{1}{4}}$ and using Abel's summation formula (2.13) we get an estimate

$$\prod_{j=1}^{n-1}\left(1 + \frac{q_{j-1}}{a_{j+1}q_j}\right) \leqslant \prod_{j=1}^{n(1+o(1))}\left(1 + \frac{1}{\log\log j - (\log\log j)^{\frac{3}{4}}}\right) e^{o(n)} = e^{o(n)}. \tag{2.24}$$

Now this and the asymptotic arithmetic mean

$$\frac{1}{n}\sum_{j=1}^{n}\Omega(j) = \log\log n + B + o(1), \tag{2.25}$$

where

$$B = 1 - \log\log 2 + \int_2^\infty \frac{\sum_{p\leqslant t}\frac{1}{p}\log p - \log t\,dt}{t(\log t)^2} + \sum_{p\ \text{prime}} \frac{1}{p(p-1)}$$

is a known constant (Theorem 430 on page 472 of [**6**]) imply

$$\log q_n \leqslant n \log\log\log n + o(n).$$

As a lower bound we only get, by using (2.23), that

$$\log q_n \geqslant n \log\log\log n(1 + o(1)).$$

Solving for n yields

$$\frac{\log q_n}{\log\log\log\log q_n + o(1)} \leqslant n \leqslant \frac{\log q_n}{\log\log\log\log q_n(1 + o(1))}.$$

Because obviously $\Omega(n) \leqslant \log n/\log 2$, with equality whenever n is a power of 2, both claims now follow by using (1.2). $\square$

Proof of Equation (2.10). We may use the upper bound (2.24) and the asymptotic arithmetic mean (2.25) to get the estimate

$$\frac{\log q_n}{\log 2} = n \log\log n + Bn + o(n).$$

Now

$$n = \frac{\log q_n}{\log 2(\log\log\log q_n + B + o(1))},$$

and so the claim follows by using (1.2), since $\Omega(n) \leqslant \log n/\log 2$, with equality whenever n is a power of 2. $\square$

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