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ON SCALING AND REGULAR VARIATION

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Abstract. We survey scaling arguments, both asymptotic (involving regular variation) and exact (involving self-similarity), in various areas of mathematical analysis and mathematical physics.

1. Scaling and Fechner's law

There is a sizeable body of theory to the effect that, where two related physically meaningful functions f and g have no natural scale in which to measure their units, and are reasonably smooth, then their relationship is given by a power law:

$$f = cg^{\alpha}. \tag{F}$$

This is known as *Fechner's law* (Gustav Fechner (1801–1887) in 1860). Our motivation here is to survey the links between this and other results on scaling with the theory of regular variation (our source for which is Bingham–Goldie–Teugels [BGT]).

We work with arguments $x \uparrow \infty$ (where $x \downarrow 0$ is more convenient, replace x by $1/x$). Suppose that our f and g are increasing and unbounded, and satisfy some (unknown) functional relationship of the form

$$f(x) = \phi(g(x)) : \qquad f = \phi \circ g.$$

If there is no natural scale, then at least asymptotically, this relationship should be scale-independent regarding x. That is, one should have

$$f(\lambda x) \sim \psi(\lambda) f(x) \qquad (x \to \infty) \qquad \forall\, \lambda > 0, \tag{RV}$$

for some $\psi(.) > 0$. That is, f should be *regularly varying*, in Karamata's sense [BGT]:

$$f \in R.$$

So subject to some smoothness (see [BGT, §1.4] for measurability, or the Baire property, and see e.g., [BinO1], [BinO2], [Ost] for generalisations), one has by the characterisation theorem of regular variation

$$\psi(x) \equiv x^{\rho}$$

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for some ρ (here $\geqslant 0$ as f and g are increasing and unbounded).

Similarly, from $g = f \circ \phi^{\leftarrow}$,

$$g \in R.$$

Then as $\phi = f \circ g^{\leftarrow}$, $f, g \in R$ give

$$\phi \in R : \qquad \phi(x) = \ell(x)x^{\alpha}$$

for some α ($\geqslant 0$ here) and *slowly varying* ℓ (i.e., $\ell(\lambda x) \sim \ell(x)$ for all $\lambda > 0$).

The classically important special case is ℓ constant, $\ell \equiv c$:

$$\phi(x) = cx^{\alpha}; \qquad f(x) = cg(x)^{\alpha} : \qquad f = cg^{\alpha},$$

giving Fechner's law.

For illustration, we give an instance of this, drawn from athletics times: for aerobic running (say, 800m to the marathon), time t and distance d show Fechner dependence:

$$t = cd^{\alpha}.$$

Here c (time per unit distance) reflects the quality of the athlete, while α is approximately constant between athletes [BinF, Ex. 3.37 and §8.2.3].

Here we view regular variation as *asymptotic scaling*. If one requires exact scaling, one encounters the related but more specific field of self-similarity and fractals; see below.

2. Kendall's theorem

The double occurrence of f in (RV) is not always convenient (particularly in generalisations where one may not be able to divide). Kendall's theorem below gives an alternative approach, which is often useful ([Ken]; [BGT, Th. 1.9.2]). Take a sequence x_n with

$$\limsup x_n = \infty, \qquad \limsup x_{n+1}/x_n = 1$$

(the prototypical case $x_n \equiv n$ will suffice here). If f is smooth as above, and

$$\text{(K)} \qquad a_n f(\lambda x_n) \to g(\lambda) \in (0, \infty) \quad (n \to \infty) \quad \forall \lambda \in S$$

for some sequence $a_n \to \infty$ and interval $S = (a, b)$ with $0 < a < b < \infty$,[1] then f is regularly varying: $f \in R$. The scaling sequence $a = (a_n)$ is also regularly varying (for the sequential aspects of regular variation, see [BGT, §1.9], [BinO3]).

Kendall's theorem also underlies the definition of a *regularly varying measure* (Hult et al., [Hul]). The law of a random vector $\mathbf{X}$ is called regularly varying if

$$nP(\mathbf{X}/a_n \in .) \to \mu(.) \qquad (n \to \infty)$$

for some measure μ; again, a_n is then regularly varying.

[1] Intervals suffice for us here, but one can work more generally, with non-null sets using measure theory, or non-meagre sets working topologically.

3. Regular variation: sums and products

The characteristic function is the natural tool for handling addition of independent random variables: it transforms the corresponding operation of *convolution* of measures into the much simpler operation of *multiplication* of characteristic functions. This holds on the line $\mathbb{R}$ and in Euclidean space $\mathbb{R}^d$ (Fourier–Stieltjes transform of measures), in Hilbert space and in Banach space (characteristic functional). Restricting attention for simplicity to the case where no centring is needed (centre at means where means exist): convergence in distribution, after scaling, of a sum $X_1 + \cdots + X_n$ of independent copies from a probability distribution F to a non-degenerate limit, in distribution,

$$(X_1 + \cdots + X_n)/a_n \to_d G \qquad (n \to \infty),$$

corresponds to

$$(\phi(t/a_n))^n \to \psi(t),$$

writing ψ for the characteristic function (CF) of the limit law G. Since ϕ is continuous and $\phi(0) = 1$, this says

$$n \log \phi(t/a_n) \sim n[1 - \phi(t/a_n)] \to \log \psi(t).$$

On the positive half-line, this yields regular variation (at the origin) of $1 - \phi(.)$ immediately, by Kendall's theorem, regular variation of $a = (a_n)$), and that

$$\log \psi(t) = ct^\alpha.$$

To pass from regular variation of $1 - \phi$ at zero to that of the tail of the probability distribution, one needs a Tauberian argument; for details, see e.g., [BGT, §8.3]. The laws G are *stable laws*, or (as we have restricted here to no centring for simplicity) the *strictly stable* laws; they have a *scaling property.* The laws F of the X_n belong to the *domain of attraction* of G. One necessarily has $0 < \alpha \leq 2$; $\alpha = 2$ corresponds to the *Gaussian* case, as in the classical central limit theorem.

On the line, or in higher dimensions, one needs a polar decomposition

$$x = (\|x\|, x/\|x\|) = (r, \theta),$$

say. One can let $r \uparrow \infty$ as above; the limiting conclusion now involves the point θ on the unit sphere Σ. One obtains a conclusion of the form

$$(*) \qquad nP(\|X\| > x, X/\|X\| \in .) \to \nu(.) \qquad (n \to \infty).$$

The limit ν is a measure on Σ, called the *spectral measure*; it shows how mass is distributed between directions (ν is indeed a measure: the setwise limit of measures is a measure, by the Vitali–Hahn–Saks theorem; see e.g., Doob [Doo, III.10, IX.10]). On the line, $\Sigma = \{\pm 1\}$, and one obtains both regular variation of the tail and *tail balance* [BGT, §8.3]. In higher dimensions, (*) defines a *regularly varying measure*; see §2, [Hul].

The theory here may be carried through in Euclidean space, Hilbert space or Banach space. The Euclidean case is considered in detail by Meerschaert and Scheffler [MeeS], using spectral decomposition of matrices. The Hilbert-space case was considered by Varadhan [Var1] in 1962, and the Banach-space case by Kuelbs

[Kue] in 1973; for textbook accounts, see e.g., Vakhania [Vak], Vakhania et al. [VakTC], Ledoux and Talagrand [LedT].

Where one has a (not necessarily commutative) *group* structure rather than a vector space, one needs local compactness to develop a comparable theory, so as to have a Haar measure. A fine account of the theory up to 1977 is in Heyer [Hey], but the analogues of the results above on stability and domains of attraction came later. See e.g., Hazod [Haz] in 1986, Born [Bor] in 1989, and for a textbook account, Hazod and Siebert [HazS]. In brief: one can reduce to the special case of *simply connected nilpotent Lie groups*.

4. Regularly variation: maxima and extremes

One often has to work in a setting where there is no natural origin or unit; one then has to choose a location and scale (as when measuring temperature, for example). Arbitrary choices of location and scale are often irrelevant; in probability theory, one often works to within *type* (that is, modulo location and scale). If one does so, the stable laws on the line (§3) have two parameters, the *index* $\alpha \in (0,2]$ and the *skewness* parameter $\beta \in [-1,1]$ reflecting the tail balance.

If one works with *maxima* rather than with sums, the situation is similar but simpler. One obtains a *one-parameter* family of limit laws, by the Fisher–Tippett theorem of 1928 (Weibull, Fréchet and Gumbel laws, usually combined nowadays into the *generalised Pareto laws*); see e.g., [BGT, §8.13.2], and Coles [Col, §4.2.1] for the generalised Pareto laws. For, the distribution function of the maximum M_n of n independent copies is the nth power of the *distribution function*, F say: M_n has distribution function F^n (cf. the nth power ϕ^n of the CF for sums). Writing $\bar{F} := 1 - F$, one has as $x \to \infty$

$$\log F^n(a_n x) = n \log(1 - \bar{F}(a_n x)) \sim n\bar{F}(a_n x),$$

and the assumption that M_n/a_n has a limit law G yields

$$n\bar{F}(a_n x) \to \log G(x) \qquad (n \to \infty).$$

By Kendall's theorem, this both gives regular variation of $\bar{F}$ and identifies $\log G$ as a power, whence the functional form of G; see e.g., [BGT, 8.13.2].

Both sums and maxima were treated by B. V. Gnedenko (1912–1995); for his life and work, see e.g., [Bin3] and the references cited there.

5. The Legendre–Fenchel transform and large deviations

The Legendre transform dates from the work of A. M. Legendre (1752–1833) in his work of 1805 on the determination of the orbits of comets. It is used to pass from the Lagrangian to the Hamiltonian formulation of mechanics [Gol, §7.1], and in, e.g., the calculus of variations (see e.g., Courant and Hilbert [Cou, Vol. I, IV.9.2, Vol. II, I.6]).

The Legendre transform was extended to its modern form, and linked with convex analysis, by Werner Fenchel (1905–1988) in 1946. The Legendre–Fenchel transform f^* of a convex function f satisfies $f^{**} = f$ if f is closed and proper; see Rockafellar [Roc, §12]. The term *Young conjugate* is also used in this connection.

The Legendre–Fenchel transform behaves well under regular variation: if $\alpha, \beta > 1$ are *conjugate indices*,

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then $f \in R_\alpha$ implies $f^* \in R_\beta$ (Bingham and Teugels, 1975: [BGT, Th.,1.8.10]). So for closed proper convex f, this is an equivalence.

Laplace's method for the asymptotic behaviour of the Laplace transform $\hat{f}$ of a function f is well known (e.g., it provides a convenient way to Stirling's formula, via a Gamma integral). The Legendre–Fenchel transform is used in Tauberian theorems of exponential type for the Laplace transform. It links regular variation of $\log \hat{f}$ with that of $\log f$ under weaker conditions than are needed to link the asymptotic behaviour of $\hat{f}$ and f via Laplace's method. For the three main cases, the Tauberian theorems of Kohlbecker, Kasahara and de Bruijn, see [BGT, §4.12], [Bin1], [BinO4].

The Legendre–Fenchel transform is widely used in thermodynamics, to pass between the four thermodynamic potentials (internal energy U; Gibbs free energy G; Helmholtz free energy A (for Arbeit, or F, for 'free' and 'function'); enthalpy H (for 'heat')). For an account of the role of convexity in this area, and in particular the work of Gibbs, see Wightman [Wig, xxii]; cf. [Sim2], [Sim3]. The relevant case of the exponential Tauberian theorem for Laplace transforms was obtained in this connection by Minlos and Povsner in 1968; see e.g., [Bin1].

In probability theory, the Legendre–Fenchel transform is used in the theory of large deviations [Ell1], [Ell2]; [Dup]. The *rate function* I is the Legendre–Fenchel transform of the scaled limit of the cumulant generating function, as in the relation displayed on the cover of [Dup]:

$$-\frac{1}{n} \log \int_x e^{-nh} d\theta^n \to \inf_{x \in X} \{h(x) + I(x)\}.$$

The prototype of this result is *Cramér's theorem*. The original result is due to Cramér in 1938 [Cra, Th. 6]; the formulation in terms of the Legendre–Fenchel transform (not by that name) is due to Chernoff in 1952; the modern form is in Varadhan [Var2]; for details on the history, see [Ell2, VII.7].

Kasahara's Tauberian theorem dates from 1978, before the appearance of a number of the now-standard sources on large deviations c. 1984, and the account in [BGT] dates from 1987, soon afterwards. The link with large deviations was explored by Yuji Kasahara and Nobuko Kosugi in 2002 [KasK1], [KasK2], and the area was surveyed by the present writer in 2008 [Bin1].

Bose–Einstein condensation (see e.g., Griffin et al. [Grif]) has been studied using large-deviations methods; see van den Berg, Lewis et al. [vdB1], [vdB2], Lebowitz et al. [Leb].

All this fits well with the needs of critical phenomena; see Section 8 below.

6. Scaling and dimensional analysis

Dimensional analysis (DA) goes back at least to Rayleigh in 1878 [Ray, I,54, II, 429]. Scaling arguments lie at the heart of key results such as the Buckingham Pi theorem. See e.g., Gibbings [Gib], Huntley [Hun].

Sometimes, striking quantitative results can be obtained by dimensional arguments. We mention four examples.

a. *The Holtsmark* 3/2 *law.* The symmetric stable law with index 3/2 (CF $\exp\{-|t|^{3/2}\}$) was proposed by the Norwegian physicist J. Holtsmark (1894–1975) in 1919 to model the distribution of galaxies in space; see e.g., [**Fel**, VI.1,2]. The 3 comes from 3 dimensions; the 2 is from Newton's inverse square law of gravity.

b. *Kolmogorov's* 5/3 *power law in turbulence.* This is due to A. N. Kolmogorov (1903–1987) in 1941 [**Kol2, Kol3**]; we return to turbulence below.

c. *Taylor's* 5*th power law.* During World War II, and later the Cold War, it was a matter of national security to have reliable ways of estimating the yields of warheads exploded in nuclear weapons tests. G. I. (Sir Geoffrey) Taylor (1886–1975) used dimensional analysis to link the yield to the speed of propagation of the shock wave [Tay1], [Tay2]. He estimated the yield of the Los Alamos test of 1945 from photographs of the shock wave published openly. His work was classified, and not published until 1950.

d. *Lighthill's* 8*th power law.* Sir James Lighthill (1924–1988) showed that the acoustic power radiated from a jet aircraft is proportional to the 8th power of the speed [Deb].

7. Fractals; Self-similarity

Fractals exhibit scaling, but not regular variation–as with the paths of Brownian motion, for example. For background on fractals, see e.g., [Fal], [Barn] and a number of books by B. B. Mandelbrot (1924–2010), e.g., [Man].

Self-similarity is a probabilistic version of scaling; it is treated at length in [EmbM]. We note *Lamperti's theorem*: that for processes X for which $\{X(\lambda t)/b(\lambda)\}$ converges to a non-degenerate limit process $T = \{T_t\}$, the norming function b is regularly varying and the limit process T is self-similar; see e.g., [BGT, §8.5]. Thus regular variation is intrinsically present, via the norming function, in a vast class of limit theorems in probability theory.

The prototype of self-similar processes, Brownian motion (BM), is contained within the family of *fractional Brownian motion* (fBM) (Kolmogorov in 1940 [Kol1]; for stochastic calculus for fBM, see Biagini et al. [Bia]).

8. Critical phenomena

The area of critical phenomena is concerned with *phase transitions*–between solid, liquid and gas, between ferromagnetism and its absence, and similarly for superconductivity, superfluidity etc. Near a phase transition, theory and observation suggest that the details of the system become irrelevant; what is decisive can be expressed in terms of *powers* of the difference between the relevant physical quantity at its present value and at the critical value. Such insensitivity to detail is called *universality*. Scaling arguments are used to pass between the system in the small and in the large: e.g., in thermodynamics, to pass to the *thermodynamic limit* (finite systems cannot exhibit a phase transition, which is a bulk or collective phenomenon). The technique used to handle this is that of the *renormalisation group (RG)* (Kenneth G. Wilson in 1971). For background, see e.g., Barenblatt [Bar1], [Bar2], Stanley [Sta1], [Sta2], Cardy [Car], Domb [Dom], Zinn-Justin [Zin2].

In brief: while disorder is local, order is global; thus an ordered system such as a ferromagnet below its critical temperature shows *long-range dependence (LRD)*[2]. If one stresses an ordered regime enough, one would expect it to become eventually disordered. This does indeed happen, and it happens abruptly: in physical systems, such as a ferromagnet heated to its critical temperature (the *Curie point* T_c: Pierre Curie (1859–1906) in 1895); or (in the reverse direction) a conductor cooled close enough to absolute zero to become superconducting; or a social structure such as an orderly queue or a military formation, which may if stressed enough suddenly crack and dissolve into a panic-stricken mob.

In statistical physics, from the work of Gibbs, the distribution of energy in equilibrium is exponential (see e.g., [Geo], (0.2)). A non-exponential distribution indicates non-equilibrium; the prototypical situation of this is a power-law distribution for the approach to a critical point, as above.

The *Ising model* of ferromagnetism is the prototype of a tractable model of such a critical phenomenon (with zero magnetic field, in two dimensions: one dimension does not show a phase transition, three or more dimensions is not solved). It was solved by L. Onsager (1903–1976) in 1944; see e.g., Simon [Sim2], II.6. Onsager found the critical temperature T_c above which magnetism is lost, and found a $1/8$ power law for the magnetisation near the critical temperature.

One of the most studied critical phenomena is *percolation* [Gri1], modelling diffusion through a random medium, where edges between vertices (on some graph or lattice) are open (to passage of some fluid, say) with some probability $p \in (0, 1)$. For p small enough, fluid cannot 'escape to infinity', but remains localized; for p large enough, it can. These *subcritical* and *supercritical* regimes are separated by some *critical point*, $p = p_c$ (on the square lattice in the plane, $p_c = \frac{1}{2}$ by Kesten's theorem of 1980). As one approaches criticality from below, interest centres on the 'incipient infinite cluster'. This is a model of the optical phenomenon of *critical opalescence* (of which one can see videos on the Internet). Smoluchowski linked this with large density fluctuations in 1908; Einstein linked this quantitatively with Rayleigh scattering in 1910. Chapter 9 of [Gri1] is devoted to a non-rigorous survey (power laws and critical exponents; scaling theory; renormalization; the incipient infinite cluster), Chapter 10 to rigorous results.

Crucially important is the rate at which correlations between states of the system decay with distance. LRD corresponds to the ordered phase, and to slow decay of correlations, its absence to disorder and rapid decay of correlations. Crucial here are *correlation inequalities*; see e.g., [Sim1]. The prototypical case of rapid decay

[2]The term is used in rather different senses in the physics literature and that of mathematics and statistics; we follow the latter here.

is exponential decay; for some models, this happens iff the correlation function is summable. See e.g., [Gri2], §9.2, 9.3, [Gri3], Th. 8.7.

If the nearest-neighbour interactions of the classical Ising model are replaced by long-range interactions, a phase transition is possible even in one dimension. See e.g., Grimmett [Gri1] §12.3 and the references cited there.

There is a sense in which the Ising model dominates related but less tractable models, such as the Potts and random-cluster models. So the fact that the Ising model is exactly solvable yields useful bounds for these; see [Gri2], §§3.7, 9.3.

The extreme-value theory of §4 occurs in this context also: '... at low temperatures, a disordered system will preferentially occupy its low-energy states, which are random variables, because of the disordered nature of the problem' ([BouM], cf. [FyoDR]).

The link between long-range dependence and correlation decay will recur below with time series and long memory.

9. Time series and finance

Mandelbrot devoted Volume E of his Selecta to *Fractals and scaling in finance* [Man]. He was one of the first to take the view that *power law* behaviour–tail decay like a power, as in the Student t-distributions, rather than the extremely fast quadratic-exponential decay rate of Gaussian tails –was likely to be typical, particularly for financial data.

In financial return data, one sees a range of tail decay, depending on the length Δ of the *return interval*. For long return intervals (monthly, say–though the rule of thumb is that 16 trading days suffice), one sees *aggregational Gaussianity*, the classical central limit theorem applies, and tail decay is Gaussian (log-density decays quadratically). For intermediate returns (daily returns, say), tail decay is much slower: typically, log-densities decay linearly, as with the generalised hyperbolic (GH) distributions (see e.g. [BinK] §2.12). Finally, for high-frequency data ('tick data', which can be every few seconds), power-law decay is typical (the limit $\Delta \downarrow 0$ is analogous to the limit $T \uparrow T_c$ for a ferromagnet, etc). Such power-law tails, or regularly varying tails, are often called *heavy tails (HT)*, as they have infinite moments of order $> \alpha$, when the tail is regularly varying with index $-\alpha$.

For time series, the long-range dependence (LRD) in space is replaced by *long memory*. See Beran [Ber], [Bin2]. Some of the extensive theoretical and empirical background here is discussed by Mikosch [Mik], in the context of financial time series. In the GARCH case, stationarity requires restrictions on the parameters. These involve Liapounov exponents (and Kingman's subadditive ergodic theorem). These can be calculated explicitly in the GARCH(1,1) case, which is usually all that is needed in practice and to which we confine ourselves here for simplicity ([Mik], §5.3.2, 5.5.2, 5.6.1). For stationary GARCH(1,1), one has heavy (regularly varying) tails and exponentially decreasing correlations ([BinM], §4).

Tauberian theorems of exponential type have found financial application, in the asymptotics of volatility surfaces. In the benchmark Black-Scholes model, volatility is constant; in reality, it varies; graphs show a shape suggesting a smile, whence the term volatility smile. Smile asymptotics concern the tails of such curves, and here exponential Tauberian theorems are used, under the name 'Bingham's lemma'. See Benaim and Friz [Ben1], [Ben2], Benaim, Friz and Lee [Ben3].

10. Turbulence

Turbulence is believed to involve an energy cascade from larger to smaller scales, resulting in increasing intensity of eddies and vorticity. Kolmogorov's work of 1941 [Kol2], [Kol3], which was pioneering and influential, is regarded as an early success for scaling arguments. The area is extensive, and is notoriously complicated. We refer for more here to the Lighthill memorial volume, [Deb], to Frisch [Fri] and to Barenblatt and Chorin [BarC].

11. Telecommunications

There has been a great deal of interest in the statistical properties of internet traffic. The fundamental result is *Taqqu's theorem*, by which long memory (or LRD) is linked with heavy tails (HT), via infinite superpositions of On/Off processes. The proof uses the language of $M/G/\infty$ queues; see e.g., Abry et al. [Abr], Doukhan et al. [Dou].

12. Psychology

The *Fechnerian scaling* of §1 is used in psychology, in particular in the study of *just noticeable differences (jnds)*. See for example Dzhafarov [Dzh], Dzhafarov and Colonius [DzhC1], [DzhC2]. Here one needs multidimensional regular variation, as there may be stimuli of various kinds.

One familiar instance of power-law behaviour in perception is the use of decibels to measure perceived noise; this is on a logarithmic scale.

13. Measurement

There is a great deal to be said about measurement, in many different contexts (one being the question of how accurately one can measure such things as physical constants, in practice or in theory). For a monograph treatment, see e.g., Hand [Han]. In particular, Hand considers measurement in psychology (§12): Fechner's law, and later work by Stevens.

14. Econophysics

Based on a loose analogy between the economy and a physical system, or of the interaction of many economic agents and of many physical particles, a number of people have used physical language, insights and results in economic and financial contexts. See for example Mantegna and Stanley [Mant], Farmer and Geanakoplos [FarG], Thurner et al. [Thu].

The financial crash of 2007 (USA), 2008 (UK), ... has focussed attention on the whole question of *systemic stability* in finance, at global and national level. The links with critical phenomena (§8) in physics are relevant here.

15. Generalized functions

While we commonly work with functions (real-valued or otherwise), it is as well to remember that this typically involves a degree of idealisation, and that a generalized function or Schwartz distribution may yield a better model. As Gelfand and Vilenkin put it, 'However, every actual measurement is accomplished by means of an apparatus which has a certain inertia. So the reading given is not an instantaneous value, but rather an averaged value by a test function ϕ characterizing the apparatus' [GelV, III.1.2]. Here scaling arguments are crucial: pointwise relations are no longer meaningful, and one studies asymptotic behaviour by introducing scaling or translation parameters. For textbook accounts of the extensive theory here, see e.g., [VlaDZ], [PilSV], [EstK].

16. Other fields

1. *Quantum phase transitions; quantum field theory.* Quantum statistical mechanics shows phase transitions just as classical statistical mechanics does (see e.g., Sachdev [Sac], Carr [Carr]), and also scaling (see e.g., Continentino [Con]). There are many similarities between quantum field theory and equilibrium statistical mechanics (see e.g., [Sim2]). For critical phenomena in quantum field theory, see e.g., Zinn-Justin [Zin1].

2. *Continuum mechanics.* Presutti [Pres] is largely devoted to thermodynamic limits, scaling limits and the interface effects between phases.

3. *Self-organised criticality.* Prototypical examples here include *sandpile models*; see e.g., Preussner [Preu].

4. *Random graphs.* The connectivity properties of random graphs are analogous to those of the components in percolation. For a recent monograph treatment, including phase transitions, see e.g., Janson et al. [Jan].

5. *Fragmentation and coagulation.* For a study of random fragmentation and coagulation processes, including self-similarity, see Bertoin [Bert].

6. *Interacting particle systems.* For scaling limits of interacting particle systems and hydrodynamic limits, see Kipnis and Landim [Kip], Spohn [Spo], Varadhan [Var3].

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References

[Abr] P. Abry, P. Borgnat, F. Ricciato, A. Scherrer and D. Veitch, *Revisiting an old friend: on the observability of the relation between long-range dependence and heavy tail*, Telecomm. Syst. **43** (2009), 147–165.

[Bar1] G. I. Barenblatt, *Scaling, Self-Similarity and Intermediate Asymptotics*, Cambridge University Press, 1996.

[Bar2] ———, *Scaling*, Cambridge University Press, 2002.

[BarC] G. I. Barenblatt, A. J. Chorin, *New perspectives in turbulence: scaling laws, asymptotics and intermittency*, SIAM Rev. **40** (1998), 265–291.

[Barn] M. Barnsley, *Fractals Everywhere*, Academic Press, 1988.

[Ben1] S. Benaim, P. K. Friz, *Regular variation and smile asymptotics*, Math. Finance **19** (2009), 1–12.

[Ben2] ———, *Smile asymptotics II: Models with known MGF*, J. Appl. Probab. **45** (2008), 16–32.

[Ben3] S. Benaim, P. K. Friz, R. Lee, *On the Black-Scholes implied volatility at extreme strikes*, Ch. 2; in: R. Cont (ed.), *Frontiers in Quantitative Finance: Volatility and Credit Risk Modelling*, Wiley, 2008.

[Ber] J. Beran, *Statistics for long-memory Processes*, Chapman & Hall, 1994.

[Bert] J. Bertoin, *Random Fragmentation and Coagulation Processes*, Cambridge University Press, 2006.

[Bia] F. Biagini, Y. Hu, B. Oksendal, T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer, 2008.

[Bin1] N. H. Bingham, *Tauberian theorems and large deviations*, Stochastics **80** (2008), 143–149 (Priscilla E. Greenwood Festschrift).

[Bin2] ———, *Szegő's theorem and its probabilistic descendants*, Probability Surveys **9** (2012), 287–324.

[Bin3] ———, *The worldwide influence of the work of B. V. Gnedenko*, Theory of Probability and its Applications **58**(1) (2014), 17–24 (B. V. Gnedenko Memorial Volume).

[BinF] N. H. Bingham, J. M. Fry, *Regression: Linear Models in Statistics*, Springer, 2010.

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*, Cambridge University Press, 1987 (2nd ed. 1989).

[BinK] N. H. Bingham, R. Kiesel, *Risk-neutral Valuation: Pricing and Hedging of Financial Derivatives*, 2nd ed., Cambridge University Press, 2004 (1st ed. 1998).

[BinM] N. H. Bingham, B. Missaoui, *Aspects of prediction*, J. Appl. Probab. **51A** (2014), 189–201.

[BinO1] N. H. Bingham, A. J. Ostaszewski, *Beyond Lebesgue and Baire: generic regular variation*, Colloq. Math. **116** (2009), 119–138.

[BinO2] ———, *Beyond Lebesgue and Baire II: bitopology and measure-category duality*, Colloq. Math. **121** (2010), 225–238.

[BinO3] ———, *Kingman, category and combinatorics*, *Probability and Mathematical Genetics* (Sir John Kingman Festschrift, ed. N. H. Bingham and C. M. Goldie), 135–168, London Math. Soc. Lecture Notes in Mathematics **378**, CUP, 2010.

[BinO4] ———, *Steinhaus theory and regular variation: De Bruijn and after*, Indag. Math. **24**(4) (2013), 679–692 (N. G. de Bruijn Memorial Issue).

[Bor] E. Born, *Projective Lie algebra bases of a locally compact group and uniform differentiability*, Math. Z. **200** (1989), 279–292.

[BouM] J.-P. Bochaud, M. Mezard, *Universality classes for extreme value statistics*, J. Phys. A **30** (1997), 7997.

[Car] J. L. Cardy, *Scaling and Renormalisation in Statistical Physics*, Cambridge University Press, 1996.

[Carr] L. D. Carr, *Understanding Quantum Phase Transitions*, CRC, 2011.

[Col] S. Coles, *An Introduction to Statistical Modelling of Extreme Values*, Springer, 2001.

[Con] M. A. Continentino, *Quantum Scaling in Many-Body Systems*, World Scientific, 2001.

[Cou] R. Courant, D. Hilbert, *Methods of mathematical physics*, Vol. I, Interscience, 1953; Vol. II, Interscience, 1962 (*Methoden der mathematischen Physik*, Springer, I, 1924/1930; II, 1938; US edition, Interscience, 1943).

[Cra] H. Cramér, *Sur un nouveau théorème-limite de la théorie des probabilités*, Actual. Sci. Industr. **736** (1938), 5–23 (*Harald Cramér, Collected Works* (ed. A. Martin-Löf), Springer, 1994, 895–913).

[Deb] L. Debnath, *Sir James Lighthill and Modern Fluid Mechanics*, Imperial College Press, 2008.

[Dom] C. Domb, *The Critical Point: A Historical Introduction to the Modern Theory of Critical Phenomena*, Taylor and Francis, 1996.

[Doo] J. L. Doob, *Measure Theory*, Grad. Texts Math. **143**, Springer, 1994.

[Dou] P. Doukhan, G. Oppenheim, M. S. Taqqu (ed.), *Theory and Applications of long-Range Dependence*, Birkhäuser, Basel, 2003.

[Dup] P. Dupuis, R. S. Ellis, *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley, 1997.

[Dzh] E. N. Dzhafarov, *Multidimensional Fechnerian scaling: regular variation version*, J. Math. Psychology **46** (2002), 226–244.

[DzhC1] E. N. Dzhafarov, H. Colonius, *Fechnerian metrics in unidimensional and multidimensional stimulus spaces*, Psychological Bulletin & Review **6** (1999), 239–268.

[DzhC2] ———, *Multidimensional Fechnerian scaling: Basics*, J. Math. Psychology **45** (2001), 670–719.

[Ell1] R. S. Ellis, *Large deviations for a general class of random vectors*, Ann. Prob. **12** (1984), 1–12.

[Ell2] ———, *Entropy, Large Deviations and Statistical Mechanics*, Grundl. Math. Wiss. **271**, Springer, 1985.

[EmbM] P. Embrechts, M. Maejima, *Self-Similar Processes*, Princeton University Press, 2002.

[EstK] R. Estrada, R. P. Kanwal, *A Distributional Approach to Asymptotics: Theory and Applications*, Birkhäuser, 2002.

[Fal] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts Math. **85**, Cambridge University Press, 1985.

[FarG] J. D. Farmer, J. Geanakoplos, *The virtues and vices of equilibrium and the future of financial economics*, Complexity **14** (2009), 11–38.

[Fel] W. Feller, *An Introduction to Probability Theory and Its Applications, Volume II*, Wiley, 1971 (1st ed. 1966).

[Fri] U. Frisch, *Turbulence: the Legacy of A. N. Kolmogorov*, Cambridge University Press, 1995.

[FyoDR] Yu. V. Fyodorov, P. Le Doussal, A. Rosso, *Statistical mechanics of logarithmic REM: duality, freezing and extreme-value statistics of 1/f noise generated by Gaussian free fields*, J. Statist. Mech.: Theory and Experiment (2009), P10005, 32pp.

[GelV] I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions, Volume 4: Applications of Harmonic Analysis*, Academic Press, 1964.

[Geo] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, Walter de Gruyter, 1998 (2nd ed. 2011).

[Gib] J. C. Gibbings, *Dimensional Analysis*, Springer, 2011.

[Gol] H. Goldstein, *Classical Mechanics*, Addison-Wesley, 1959.

[Grif] A. Griffin, D. W. Snoke, S. Stringari (ed.), *Bose-Einstein Condensation*, Cambridge University Press, 1996.

[Gri1] G. R. Grimmett, *Percolation*, 2nd ed., Grundl. Math. Wiss. **321**, Springer, 1999 (1st ed. 1989).

[Gri2] ———, *The random-cluster model*, Grundl. Math. Wiss. **333**, Springer, 2006.

[Gri3] ———, *Flows and ferromagnets*; P.130–143; in: G. R. Grimmett, C. McDiarmid (eds.), *Combinatorics, Complexity and Chance: A Tribute to Dominic Welsh*, Oxford University Press, 2007, pp. 130–143

[Han] D. J. Hand, *Measurement: Theory and Practice. The World Through Quantification*, Arnold, London, 2004.

[Haz] W. Hazod, *Stable probability measures on groups and on vector spaces*; in: *Probability on Groups VIII*, Lect. Notes Math. **1210**, 1986, pp. 304–352.

[HazS] W. Hazod, E. Siebert, *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups: Structure Properties and Limit Theorems*, Kluwer, 2001.

[Hey] H. Heyer, *Probability measures on locally compact groups*, Erg. Math. **94**, Springer, 1977.

[Hul] H. Hult, F. Lindskog, T. Mikosch, G. Samorodnitsky, *Functional large deviations for multivariate regularly varying random walks*, Ann. Adv. Prob. **15** (2005), 2651–2680.

[Hun] H. E. Huntly, *Dimensional Analysis*, Dover, 1967.

[Isr] R. B. Israel, *Convexity in the Theory of Lattice Gases*, Princeton University Press, 1979.

[Jan] S. Janson, T. Luczak, A Ruciński, *Random Graphs*, Wiley, 2000.

[KasK1] Y. Kasahara, N. Kosugi, *Large deviations around the origin for sums of non-negative i.i.d. random variables*, Natur. Sci. Rep. Ochanomizu Univ. **51** (2002), 27–31.

[KasK2] ———, *Remarks on Tauberian theorem of exponential type and Fenchel-Legendre transform*, Osaka J. Math. **39** (2002), 613–620.

[Ken] D. G. Kendall, *Delphic semigroups, infinitely divisible regenerative phenomena and the arithmetic of p-functions*, Z. Wahrschein. **9** (1968), 163–195 (reprinted in *Stochastic Analysis* (ed. E. F. Harding, D. G. Kendall), 73–114, Wiley, 1973).

[Kip] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems*, Springer, 1999.

[Kol1] ———, *The Wiener spiral and other interesting curves in a Hilbert space*, Dokl. Akad. Nauk SSSR **26** (1940), 115–118.

[Kol2] ———, *The local structure of turbulence in an incompressible fluid with very large Reynolds number*, Dokl. Akad. Nauk SSSR **30** (1941), 477–479.

[Kol3] ———, *Dissipation of energy under locally isotropic turbulence*, Dokl. Akad. Nauk SSSR **32** (1941), 16–18.

[Kue] J. Kuelbs, *The invariance principle for Banach space valued random variables*, J. Multivariate Analysis **3** (1973), 161–172.

[Leb] J. L. Lebowitz, M. Lenci, H. Spohn, *Large deviations for ideal quantum systems*, J. Math. Phys. **41** (2000), 1224–1243.

[LedT] M. Ledoux, M. Talagrand, *Probability on Banach spaces: Isoperimetry and processes*, Erg. Math. (3) **23**, Springer, 1991 (paperback, Classics in Math., 2011).

[Man] B. B. Mandelbrot, *Fractals and scaling in finance: Discontinuity, concentration, risk*, Selecta Volume E, Springer, 1997.

[Mant] R. N. Mantegna, H. E. Stanley, *An Introduction to Econophysics: Correlation and Complexity in Finance*, Cambridge University Press, 2000.

[MeeS] M. M. Meerschaert, H.-P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, Wiley, 2001.

[Mik] Th. Mikosch, *Modelling dependence and tails of financial time series*, pp. 185–286; in: B. Finkenstädt, H. Rootzén (eds.), *Extreme Values in Finance, Telecommunications and the Environment*, Chapman and Hall, 2004.

[Ost] A. J. Ostaszewski, *Beyond Lebesgue and Baire III: Steinhaus' theorem and its descendants*, Topology Appl. **160** (2013), 1144–1154.

[PilSV] S. Pilipović, B. Stanković, J. Vindas, *Asymptotic Behaviour of Generalized Functions*, Series on Analysis, Applications and Computation 5, World Scientific, 2012.

[Pres] E. Presutti, *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Springer, 2009.

[Preu] G. Preussner, *Self-Organised Criticality: Theory, Models and Characterisation*, Cambridge University Press, 2012.

[Ray] J. W. S. Rayleigh, *The Theory of Sound*, I, II, Dover, 1945 (1st ed. 1878, 2nd ed. 1896).

[Roc] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[Sac] S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press, 1999 (2nd ed. 2011).

[Sim1] B. Simon, *Correlation inequalities and the decay of correlations in ferromagnets*, Comm. Math. Phys. **77** (1980), 111–126.

[Sim2] ———, *The Statistical Mechanics of Lattice Gases, Volume I*, Princeton University Press, 1993.

[Sim3] ———, *Convexity: An Analytic Viewpoint*, Cambridge Tracts in Math. **187**, Cambridge University Press, 2011.

[Spo] H. Spohn, *Large-Scale Dynamics of Interacting Particles*, Springer, 1991.

[Sta1] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena*, Oxford University Press, 1971.

[Sta2] ———, *Scaling, universality and renormalisation: Three pillars of modern critical phenomena*, Rev. Modern Phys. **71** (1999), 358–366.

[Tay1] G. I. Taylor, *The formation of a blast wave by a very intense explosion, I: Theoretical discussion*, Proc. Roy. Soc. A **201** (1950), 159Â–174.

[Tay2] ———, *The formation of a blast wave by a very intense explosion, II: The Atomic Explosion of 1945*, Proc. Roy. Soc. A **201** (1950), 175Â–186.

[Thu] S. Thurner, J. D. Farmer, J. Geanakoplos, *Leverage causes fat tails and clustered volatility*, Quantitative Finance **12** (2012), 695–707.

[Vak] N. N. Vakhania, *Probability on Linear Spaces*, North-Holland, 1981.

[VakTC] N. N. Vakhania, V. I. Tarieladze, S. A. Chobanyan, *Probability Distributions on Banach Spaces*, Reidel, 1987.

[vdB1] M. van den Berg, J. T. Lewis, M. Lunn, *On the general theory of Bose-Einstein condensation and the state of the free boson gas*, Helv. Phys. Acta **59** (1986), 1289–1310.

[vdB2] M. van den Berg, J. T. Lewis, J. V. Pulè, *A general theory of Bose-Einstein condensation*, Helv. Phys. Acta **59** (1986), 1271–1288.

[Var1] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhyā **24** (1962), 213–238.

[Var2] ———, *Large deviations and applications*, CBMS-NSF Reg. Conf. Ser. **46**, SIAM, 1984.

[Var3] ———, *Entropy methods in hydrodynamic scaling*; in: *Proc. Int. Congr. Math., Zürich 1994, Vol. 1*, Birkhäuser, Basel, 1995, pp. 196–208

[VlaDZ] V. S. Vladimirov, Yu. N. Drozhzhinov, B. I. Zav'yalov, *Tauberian Theorems for Generalized Functions*, Kluwer, Dordrecht, 1988 [Russian: Naukova Dumka, Kiev, 1986].

[Wig] A. S. Wightman, *Introduction. Convexity and the notion of equilibrium state in thermodynamics and statistical mechanics*, [Isr], ix-lxxxv.

[Zin1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed, 2002, Oxford University Press (1st ed. 1989, 2nd ed. 1993, 3rd ed. 1996).

[Zin2] ———, *Phase Transitions and Renormalization Group*, Oxford University Press, 2007.

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