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DOMINATION NUMBER IN THE ANNIHILATING-IDEAL GRAPHS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and $\mathbb{A}(R)$ be the set of ideals with nonzero annihilator. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \smallsetminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this paper, we study the domination number of $\mathbb{AG}(R)$ and some connections between the domination numbers of annihilating-ideal graphs and zero-divisor graphs are given.

1. Introduction

The study of algebraic structures, using the properties of graphs, have become an exciting research topic in the past twenty years, leading to many interesting results and questions. There are many papers on assigning a graph to a ring, for instance see [**1, 3, 9, 11**].

Throughout this paper, all rings are assumed to be commutative rings with identity. By $\mathrm{Min}(R)$, $Z(R)$ and $\mathrm{Nil}(R)$ we denote the set of all minimal prime ideals of R, the set of all zero-divisors of R and the set of all nilpotent elements of R, respectively. The *socle* of ring R, denoted by $\mathrm{Soc}(R)$, is the sum of all minimal ideals of R. If there are no minimal ideals, this sum is defined to be zero. A prime ideal $\mathfrak{p}$ is said to be an *associated prime ideal* of a commutative Noetherian ring R, if there exists a nonzero element x in R such that $\mathfrak{p} = \mathrm{ann}(x)$. By $\mathrm{Ass}(R)$ we denote the set of all associated prime ideals of R. A ring R is said to be *reduced*, if it has no nonzero nilpotent element or equivalently $\bigcap_{P\in\mathrm{Min}(R)} P = 0$.

For every graph G, we denote by $V(G)$, the vertex set of G. A *bipartite graph* is a graph all of whose vertices can be partitioned into two parts U and V such that every edge joins a vertex in U to one in V. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If one of the parts is a singleton, then the graph is said to be a *star graph*. A subset

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D of $V(G)$ is called a *dominating set* if every vertex of G is either in D or adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A *total dominating set* of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a γ-*set* (γ_t-*set*).

Let R be a ring. The *zero-divisor graph* of R, $\Gamma(R)$, is a graph with the vertex set $Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of the zero-divisor graph was first introduced by Beck (see [**5**]). We call an ideal I of R, an *annihilating-ideal* if there exists a nonzero ideal J of R such that $IJ = 0$. We use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. By the *annihilating-ideal graph* of R, $\mathbb{AG}(R)$, we mean the graph with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ such that two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-ideal graph was first introduced in [**6**] and some interesting properties of this graph have been studied. In this article, we study the domination number of the annihilating-ideal graphs. For reduced rings with finitely many minimal primes and Artinian rings, the domination number of the annihilating-ideal graphs is given. Also, some relations between the domination numbers of annihilating-ideal graphs and zero-divisor graphs are studied.

2. Main results

We start with the following remark which completely characterizes all rings for which either $\gamma(\mathbb{AG}(R)) = 1$ or $\gamma(\Gamma(R)) = 1$.

Remark 2.1. Let R be a ring. By [**6**, Theorem 2.2], there is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex if and only if either $R \cong F \times D$, where F is a field and D is an integral domain, or $Z(R)$ is an annihilator ideal. Also, by [**3**, Theorem 2.5], there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R)$ is an annihilator ideal. Now, let R be a reduced ring. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if $R \cong F \times D$, where F is a field and D is an integral domain. Moreover, $\gamma(\Gamma(R)) = 1$ if and only if $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain.

Now, we can state the following proposition.

Proposition 2.1. *Let R be a ring. Then we have the following:*

(i) *If $\gamma(\mathbb{AG}(R)) = 1$, then $\gamma(\Gamma(R)) \in \{1, 2\}$.*
(ii) *If $\gamma(\Gamma(R)) = 1$, then $\gamma(\mathbb{AG}(R)) = 1$.*

Proof. The result follows from Remark 2.1 and [**10**, Proposition 8]. □

The following result describes the relation between $\gamma_t(\mathbb{AG}(R))$ ($\gamma_t(\Gamma(R))$) and $\gamma(\mathbb{AG}(R))$ ($\gamma(\Gamma(R))$).

Theorem 2.1. *Let R be a ring. Then*

(i) *$\gamma_t(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R))$ or $\gamma_t(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R)) + 1$.*

(ii) $\gamma_t(\Gamma(R)) = \gamma(\Gamma(R))$ *or* $\gamma_t(\Gamma(R)) = \gamma(\Gamma(R)) + 1$.

Proof. (i) Let $\gamma_t(\mathbb{AG}(R)) \neq \gamma(\mathbb{AG}(R))$ and D be a γ-set of $\mathbb{AG}(R)$. If $\gamma(\mathbb{AG}(R)) = 1$, then it is clear that $\gamma_t(\mathbb{AG}(R)) = 2$. So let $\gamma(\mathbb{AG}(R)) > 1$ and put $k = \mathrm{Max}\{n \mid \exists\, I_1, \dots, I_n \in D \textit{ s.t. } \prod_{i=1}^{n} I_i \neq 0\}$. Since $\gamma_t(\mathbb{AG}(R)) \neq \gamma(\mathbb{AG}(R))$, we have $k \geqslant 2$. Let $I_1, \dots, I_k \in D$ be such that $\prod_{i=1}^{k} I_i \neq 0$. Then $S = \{\prod_{i=1}^{k} I_i, \operatorname{ann} I_1, \dots, \operatorname{ann} I_k\} \cup D \setminus \{I_1, \dots, I_k\}$ is a γ_t-set. Hence $\gamma_t(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R)) + 1$.

(ii) It is clear by the same argument of part (i). □

In the following result we find the total domination number of $\mathbb{AG}(R)$.

Theorem 2.2. *Let M be the set of all maximal elements of the set $\mathbb{A}(R)$. If $|M| > 1$, then $\gamma_t(\mathbb{AG}(R)) = |M|$.*

Proof. Let M be the set of all maximal elements of the set $\mathbb{A}(R)$, $I \in M$ and $|M| > 1$. First we show that $I = \operatorname{ann}(\operatorname{ann} I)$ and there exists $x \in R$ such that $I = \operatorname{ann}(x)$. Let $I \in M$. Then $\operatorname{ann} I \neq 0$ and so there exists $0 \neq x \in \operatorname{ann} I$. Hence $I \subseteq \operatorname{ann}(\operatorname{ann} I) \subseteq \operatorname{ann}(x)$. Thus by the maximality of I, we have $I = \operatorname{ann}(\operatorname{ann} I) = \operatorname{ann}(x)$. By Zorn's Lemma it is clear that if $\mathbb{A}(R) \neq \emptyset$, then $M \neq \emptyset$. For any $I \in M$ choose $x_I \in R$ such that $I = \operatorname{ann}(x_I)$. We assert that $D = \{Rx_I \mid I \in M\}$ is a total dominating set of $\mathbb{AG}(R)$. Since for every $J \in \mathbb{A}(R)$ there exists $I \in M$ such that $J \subseteq I = \operatorname{ann}(x_I)$, J and Rx_I are adjacent. Also for each pair $I, I' \in M$, we have $Rx_I Rx_{I'} = 0$. Namely, if there exists $x \in Rx_I Rx_{I'} \setminus \{0\}$, then $I = I' = \operatorname{ann}(x)$. Thus $\gamma_t(\mathbb{AG}(R)) \leqslant |M|$. To complete the proof, we show that each element of an arbitrary γ_t-set of $\mathbb{AG}(R)$ is adjacent to exactly one element of M. Assume to the contrary, that a vertex K of a γ_t-set of $\mathbb{AG}(R)$ is adjacent to I and I', for $I, I' \in M$. Thus $I = I' = \operatorname{ann} K$, which is impossible. Therefore $\gamma_t(\mathbb{AG}(R)) = |M|$. □

Theorem 2.3. *Let R be a ring. Then $\gamma_t(\Gamma(R)) \leqslant \gamma_t(\mathbb{AG}(R))$.*

Proof. Let $\gamma_t(\mathbb{AG}(R)) = k$. In the light of the proof of Theorem 2.2, there exists $x_i \in R$ such that $\operatorname{ann}(x_i)$ is a maximal ideal of $\mathbb{A}(R)$, for $i = 1, \dots, k$, and $D = \{Rx_1, \dots, Rx_k\}$ is a minimum total dominating set of $\mathbb{AG}(R)$. It is easy to check that $D' = \{x_1, \dots, x_k\}$ is a total dominating set of $\Gamma(R)$. Thus $\gamma_t(\Gamma(R)) \leqslant \gamma_t(\mathbb{AG}(R))$. □

It is interesting to find some rings for which $\gamma_t(\Gamma(R)) = \gamma(\Gamma(R)) = \gamma_t(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R))$. In the next result, we study the domination number of the annihilating-ideal graphs of reduced rings with finitely many minimal primes.

Theorem 2.4. *Let R be a reduced ring and $|\operatorname{Min}(R)| < \infty$. If $\gamma(\mathbb{AG}(R)) > 1$, then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\operatorname{Min}(R)|$.*

Proof. Since R is reduced and $\gamma(\mathbb{AG}(R)) > 1$, we have $|\operatorname{Min}(R)| > 1$. Suppose that $\operatorname{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. If $n = 2$, the result follows from [**7**, Corollary 2.5]. Therefore, suppose that $n \geqslant 3$. Define $\widehat{\mathfrak{p}}_i = \mathfrak{p}_1 \dots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \dots \mathfrak{p}_n$, for every $i = 1, \dots, n$. Clearly, $\widehat{\mathfrak{p}}_i \neq 0$, for every $i = 1, \dots, n$. Since R is reduced, we deduce that $\widehat{\mathfrak{p}}_i \mathfrak{p}_i = 0$. Therefore, every $\mathfrak{p}_i$ is a vertex of $\mathbb{AG}(R)$. If I is a vertex of

$\mathbb{AG}(R)$, then by [**8**, Corollary 2.4], $I \subseteq Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. It follows from the Prime Avoidance Theorem (see [**12**, Theorem 3.61]) that $I \subseteq \mathfrak{p}_i$, for some i, $1 \leqslant i \leqslant n$. Thus $\mathfrak{p}_i$ is a maximal element of $\mathbb{A}(R)$, for every $i = 1, \ldots, n$. From Theorem 2.2, $\gamma_t(\mathbb{AG}(R)) = |\operatorname{Min}(R)|$. Now, we show that $\gamma(\mathbb{AG}(R)) = n$. Assume to the contrary, that $B = \{J_1, \ldots, J_{n-1}\}$ is a dominating set for $\mathbb{AG}(R)$. Since $n \geqslant 3$, the ideals $\mathfrak{p}_i$ and $\mathfrak{p}_j$, for $i \neq j$ are not adjacent (from $\mathfrak{p}_i\mathfrak{p}_j = 0 \subseteq \mathfrak{p}_k$ it would follow that $\mathfrak{p}_i \subseteq \mathfrak{p}_k$, or $\mathfrak{p}_j \subseteq \mathfrak{p}_k$ which is not true). Because of that, we may assume that for some $k < n-1$, $J_i = \mathfrak{p}_i$ for $i = \overline{1,k}$, but none of the other of ideals from B are equal to some $\mathfrak{p}_s$ (if $B = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}\}$ then $\mathfrak{p}_n$ would be adjacent to some $\mathfrak{p}_i$, for $i \neq n$). So, every ideal in $\{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\}$ is adjacent to an ideal in $\{J_{k+1}, \ldots, J_{n-1}\}$. It follows that for some $s \neq t$ there is an l such that $\mathfrak{p}_s J_l = 0 = \mathfrak{p}_t J_l$. Since $\mathfrak{p}_s \nsubseteq \mathfrak{p}_t$, it follows that $J_l \subset \mathfrak{p}_t$, so $J_l^2 = 0$, which is impossible, since the ring R is reduced. So $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\operatorname{Min}(R)|$. □

Theorem 2.4 leads to the following corollary.

Corollary 2.1. *Let R be a reduced ring. If $\gamma(\mathbb{AG}(R)) > 1$, then the following are equivalent:*

(i) $\gamma(\mathbb{AG}(R)) = 2$.
(ii) $\mathbb{AG}(R)$ *is a bipartite graph with two nonempty parts.*
(iii) $\mathbb{AG}(R)$ *is a complete bipartite graph with two nonempty parts.*
(iv) R *has exactly two minimal primes.*

Proof. The result follows from Theorem 2.4 and [**7**, Corollary 2.5]. □

In Theorem 15 of [**10**], it is proved that if R is a finite reduced ring such that $\gamma(\Gamma(R)) \neq 1$, then $\gamma(\Gamma(R)) = |\operatorname{Min}(R)|$. In the next theorem, we prove this result, where R is not necessarily finite.

Theorem 2.5. *Let R be a reduced ring and $|\operatorname{Min}(R)| < \infty$. If $\gamma(\Gamma(R)) > 1$, then $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |\operatorname{Min}(R)|$.*

Proof. Using the notations in the proof of Theorem 2.4, set $A = \{\widehat{x}_i \mid 1 \leqslant i \leqslant n\}$, where, for every i, $\widehat{x}_i$ is an element of $\widehat{\mathfrak{p}}_i$. We show that A is a dominating set in $\Gamma(R)$. Since R is reduced, it is easily seen that $\widehat{x}_i$ is a vertex of $\Gamma(R)$, for $i = 1, \ldots, n$. Assume that $x \notin A$ is a vertex of $\Gamma(R)$. Then $x \in \mathfrak{p}_i$, for some i. The equality $\widehat{x}_i\mathfrak{p}_i = 0$ implies that $x\widehat{x}_i = 0$. In the sequel, we prove that $\gamma(\Gamma(R)) = n$. If $n = 2$, then [**2**, Theorem 2.4] completes the proof. Thus assume that $n \geqslant 3$. Assume to the contrary, the set $B = \{y_1, \ldots, y_{n-1}\}$ is a dominating set for $\Gamma(R)$. By the Prime Avoidance Theorem, there exists $x_i \in \mathfrak{p}_i \setminus \bigcup_{j=1, j\neq i}^{n} \mathfrak{p}_j$. Thus there exists k, $1 \leqslant k \leqslant n-1$, such that $y_k x_i = y_k x_j = 0$, for some different i, j, $1 \leqslant i, j \leqslant n$. Since $x_i \notin \mathfrak{p}_j$ and $x_j \notin \mathfrak{p}_i$, we have $y_k \in \mathfrak{p}_i \cap \mathfrak{p}_j$. As R is a reduced ring, we conclude that $y_k \notin \mathfrak{p}_l$, for some l, $1 \leqslant l \leqslant n$. Now, $y_k x_i = 0 \in \mathfrak{p}_l$ implies that either $y_k \in \mathfrak{p}_l$ or $x_i \in \mathfrak{p}_l$, a contradiction. Clearly, $\widehat{x}_i\widehat{x}_j = 0$, where $i \neq j$. Therefore, $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |\operatorname{Min}(R)|$. □

Corollary 2.2. *Let R be a reduced ring and $|\operatorname{Min}(R)| < \infty$. If $\gamma(\mathbb{AG}(R)) \neq 1$, then $\gamma(\mathbb{AG}(R)) = \gamma(\Gamma(R))$.*

PROOF. The result follows from Part (2) of Proposition 2.1 and Theorems 2.4 and 2.5. □

In the following theorem the domination number of bipartite annihilating-ideal graphs is given.

THEOREM 2.6. *If $\mathbb{AG}(R)$ is a bipartite graph, then $\gamma(\mathbb{AG}(R)) \leqslant 2$.*

PROOF. If $\mathbb{AG}(R)$ is bipartite, it follows from [**1**, Theorem 27] that one of the following cases occurs: (a): $\mathbb{AG}(R)$ is a star graph; (b): $\mathbb{AG}(R)$ is the path of order 4; (c): $\mathrm{Nil}(R) = \mathrm{Soc}(R)$. If (a) or (b) happen, then we are done. Suppose that $\mathrm{Nil}(R) = \mathrm{Soc}(R)$. If R is reduced, then the result follows from Corollary 2.1. If R is nonreduced, then [**1**, Theorem 20] completes the proof. □

The next theorem is on the domination number of the annihilating-ideal graphs of Artinian rings.

THEOREM 2.7. *Let R be an Artinian ring and $R \ncong F_1 \times F_2$, where F_1 and F_2 are two fields. Then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\,\mathrm{Min}(R)|$.*

PROOF. Since R is Artinian, we deduce that each ideal of R is an annihilating-ideal. So, the set of maximal elements of $\mathbb{A}(R)$ and $\mathrm{Max}(R)$ are equal. By [**4**, Theorem 8.7], $R \cong R_1 \times \cdots \times R_k$, where $(R_i, \mathfrak{m}_i)$ is an Artinian local ring, for $i = 1, \ldots, k$. Let $\mathrm{Max}(R) = \{\mathfrak{n}_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_k \mid 1 \leqslant i \leqslant k\}$. By Theorem 2.2, $\gamma_t(\mathbb{AG}(R)) = |\,\mathrm{Max}(R)|$. In the sequel, we prove that $\gamma(\mathbb{AG}(R)) = k$. Assume to the contrary, the set $\{J_1, \ldots, J_{k-1}\}$ is a dominating set for $\mathbb{AG}(R)$. Since $R \ncong F_1 \times F_2$, where F_1 and F_2 are two fields, we find that $J_i\mathfrak{n}_s = J_i\mathfrak{n}_t = 0$, for some i, t, s, where $1 \leqslant i \leqslant k-1$ and $1 \leqslant t, s \leqslant k$. This means that $J_i = 0$, a contradiction. □

The condition of R to be an Artinian ring in the previous theorem is necessary; see the next example.

EXAMPLE 2.1. Let $R = \frac{k[x,y,z]}{(xy,xz,yz)}$, where k is a field and x, y and z are indeterminates. Then $\gamma(\mathbb{AG}(R)) = 2$ but $|\,\mathrm{Min}(R)| > 2$.

Theorem 2.7 gives the following immediate corollary.

COROLLARY 2.3. *Let R be an Artinian ring and $R \ncong F_1 \times F_2$, where F_1 and F_2 are two fields. Then $\gamma(\mathbb{AG}(R)) = \gamma(\Gamma(R)) = |\,\mathrm{Min}(R)|$.*

PROOF. The result follows from Theorem 2.7 and [**10**, Theorem 11]. □

EXAMPLE 2.2. Let n be a natural number and $n = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}$, where p_i's are distinct primes and n_i's are natural numbers. Then one of the following holds:

(i) $\gamma(\mathbb{AG}(\mathbb{Z}_n)) = 1$ if and only if either $n = p_1p_2$ or $n = p_1^{n_1}$, where $n_1 > 1$.
(ii) $\gamma(\mathbb{AG}(\mathbb{Z}_n)) = 2$ if and only if $m = 2$ and either $n_1 > 1$ or $n_2 > 1$.
(iii) If $m \geqslant 3$, then $\gamma(\mathbb{AG}(\mathbb{Z}_n)) = m$.

The following theorem provides an upper bound for the domination number of the annihilating-ideal graph of a Noetherian ring.

THEOREM 2.8. *If R is a Notherian ring, then $\gamma(\mathbb{AG}(R)) \leqslant |\operatorname{Ass}(R)| < \infty$.*

PROOF. By [**12**, Remark 9.33], $|\operatorname{Ass}(R)| < \infty$. Let $\operatorname{Ass}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\mathfrak{p}_i = \operatorname{ann}(x_i)$, for some $x_i \in R$ and for every $i = 1, \ldots, n$. Set $A = \{Rx_i \,|\, 1 \leqslant i \leqslant n\}$. We show that A is a dominating set of $\mathbb{AG}(R)$. Clearly, every Rx_i is a vertex of $\mathbb{AG}(R)$, for $i = 1, \ldots, n$. If I is a vertex of $\mathbb{AG}(R)$, then [**12**, Corollary 9.36] implies that $I \subseteq Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. It follows from the Prime Avoidance Theorem that $I \subseteq \mathfrak{p}_i$, for some i, $1 \leqslant i \leqslant n$. Thus $IRx_i = 0$, as desired. □

The inequality in the above theorem may be strict; see the next example.

EXAMPLE 2.3. Let $R = \frac{k[x,y]}{(x^2,xy)}$, where k is a field and x, y are indeterminate. Then $\operatorname{Ass}(R) = \{(x), (x, y)\}$ but $\gamma(\mathbb{AG}(R)) = 1$ (Indeed the vertex (x) is adjacent to all other vertices).

REMARK 2.2. If R is Noetherian, then by a similar argument to that of the proof of Theorem 2.8, one can show that $\gamma(\Gamma(R)) \leqslant |\operatorname{Ass}(R)|$.

We end the paper with the following result about the domination number of the annihilating-ideal graph of a finite direct product of rings.

THEOREM 2.9. *For a ring R, which is a product of two (nonzero) rings, one of the following holds:*

(i) *If $R \cong F \times D$, where F is a field and D is an integral domain, then $\gamma(\mathbb{AG}(R)) = 1$.*

(ii) *If $R \cong D_1 \times D_2$, where D_1 and D_2 are integral domains which are not fields, then $\gamma(\mathbb{AG}(R)) = 2$.*

(iii) *If $R \cong R_1 \times D$, where R_1 is a ring which is not integral domain and D is an integral domain, then $\gamma(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R_1)) + 1$.*

(iv) *If $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings which are not integral domains, then $\gamma(\mathbb{AG}(R)) = \gamma(\mathbb{AG}(R_1)) + \gamma(\mathbb{AG}(R_2))$.*

PROOF. Parts (i) and (ii) are clear.

(iii) With no loss of generality, one can assume that $\gamma(\mathbb{AG}(R_1)) < \infty$. Suppose that $\gamma(\mathbb{AG}(R_1)) = n$ and $\{I_1, \ldots, I_n\}$ is a minimal dominating set of $\mathbb{AG}(R_1)$. It is not hard to see that $\{I_1 \times 0, \ldots, I_n \times 0, 0 \times D\}$ is the smallest dominating set of $\mathbb{AG}(R)$.

(iv) We may assume that $\gamma(\mathbb{AG}(R_1)) = m$ and $\gamma(\mathbb{AG}(R_2)) = n$, for some positive integers m and n. Let $\{I_1, \ldots, I_m\}$ and $\{J_1, \ldots, J_n\}$ be two minimal dominating sets in $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_1)$, respectively. It is easily seen that $\{I_1 \times 0, \ldots, I_m \times 0, 0 \times J_1 \ldots 0 \times J_n\}$ is the smallest dominating set in $\mathbb{AG}(R)$. □

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