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BASS NUMBERS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a Noetherian ring, M a finitely generated R-module and N an arbitrary R-module. We consider the generalized local cohomology modules, with respect to an arbitrary ideal I of R, and prove that, for all non-negative integers r, t and all $\mathfrak{p} \in \operatorname{Spec}(R)$ the Bass number $\mu^r(\mathfrak{p}, H_I^t(M, N))$ is bounded above by $\sum_{j=0}^{t} \mu^r\left(\mathfrak{p}, \operatorname{Ext}_R^{t-j}(M, H_I^j(N))\right)$. A corollary is that $\operatorname{Ass}\left(H_I^t(M,N)\right) \subseteq \bigcup_{j=0}^{t} \operatorname{Ass}\left(\operatorname{Ext}_R^{t-j}(M, H_I^j(N))\right)$. In a slightly different direction, we also present some well known results about generalized local cohomology modules.

1. Introduction

The local cohomology theory has been a significant tool in commutative algebra and algebraic geometry. As a generalization of the ordinary local cohomology modules, Herzog [**8**] introduced the generalized local cohomology modules and these had been studied further by Suzuki [**15**] and Yassemi [**16**] and some other authors. They studied some basic duality theorems, vanishing and other properties of generalized local cohomology modules which also generalize several known facts about Ext and ordinary local cohomology modules.

An important problem in commutative algebra is to determine when the Bass numbers of the i-th local cohomology module is finite. In [**9**] Huneke conjectured that if $(R, \mathfrak{m}, k)$ is a regular local ring, then for any prime ideal $\mathfrak{p}$ of R the Bass numbers $\mu^i(\mathfrak{p}, H_I^j(R)) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), H^j_{IR_\mathfrak{p}}(R_\mathfrak{p}))$ are finite for all i and j. There is evidence that this conjecture is true. It is shown by Huneke and Sharp [**10**] and Lyubeznik [**11**] that the conjecture holds for regular local ring containing a field. This conjecture is also true for unramified regular local rings of mixed characteristic; this is part of the main theorem of [**12**]. On the other hand there is a negative answer to the conjecture (over non-regular ring) that is due to Hartshorne [**7**]. In [**5**] Dibaei and Yassemi studied the relationship between the Bass numbers of a module and its local cohomology modules. We would like to study

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the relationship between the Bass numbers of generalized local cohomology modules $H_I^i(M,N)$ and $\mathrm{Ext}_R^i(M, H_I^j(N))$ whenever R is a Noetherian ring, M is a finitely generated R-module and N is an arbitrary R-module.

2. Main results

Throughout this section R is a Noetherian ring, I is an ideal of R, M is a finitely generated R-module, N is an arbitrary R-module and r,t are non-negative integers. For a prime ideal $\mathfrak{p}$ of R the r-th Bass number of M is denoted by $\mu^r(\mathfrak{p}, M)$.

The following lemma will be used to prove the main result of this paper.

Lemma 2.1.

$$\mu^r(\mathfrak{p}, H_I^1(M,N)) \leqslant \mu^r(\mathfrak{p}, \mathrm{Ext}_R^1(M, \Gamma_I(N))) + \mu^r(\mathfrak{p}, \mathrm{Hom}_R(M, H_I^1(N))).$$

Proof. In view of [**3**, Corollary 11.1.6] and [**6**, Lemma 2.1] $\mu^r(\mathfrak{p}, H_I^t(M,N)) = \mu^r(\mathfrak{p}R_\mathfrak{p}, H_I^t(M_\mathfrak{p}, N_\mathfrak{p}))$; also in view of [**3**, Corollary 4.3.3]

$$\mu^r(\mathfrak{p}, \mathrm{Ext}_R^{t-j}(M, H_I^j(N))) = \mu^r(\mathfrak{p}R_\mathfrak{p}, \mathrm{Ext}_{R_\mathfrak{p}}^{t-j}(M_\mathfrak{p}, H_{IR_\mathfrak{p}}^j(N_\mathfrak{p}))).$$

So, we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. We denote $\mu^r(\mathfrak{p}, M)$ by $\mu^r(M)$ and we have to show that

$$\mu^r(H_I^1(M,N)) \leqslant \mu^r(\mathrm{Ext}_R^1(M, \Gamma_I(N))) + \mu^r(\mathrm{Hom}_R(M, H_I^1(N))).$$

By Theorem 11.38 of [**14**] there is a Grothendieck spectral sequence

$$E_2^{p,q} := \mathrm{Ext}_R^p(M, H_I^q(N)) \Rightarrow_p H_I^{p+q}(M,N) = E^{p+q}$$

and there exists a finite filtration $0 = \phi^2 E^1 \subseteq \phi^1 E^1 \subseteq \phi^0 E^1 = H_I^1(M,N)$ such that $E_\infty^{0,1} = E^1/\phi^1 E^1$ and $E_\infty^{1,0} \cong \phi^1 E^1$. Thus $\mu^r(E^1) \leqslant \mu^r(E_\infty^{0,1}) + \mu^r(E_\infty^{1,0})$. Now, by the sequence $0 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}$ we have $E_\infty^{0,1} \cong E_3^{0,1} \cong \ker d_2^{0,1}$. Also $E_\infty^{1,0} \cong E_2^{1,0}$. Hence, $\mu^r(E^1) \leqslant \mu^r(\ker d_2^{0,1}) + \mu^r(E_2^{1,0}) \leqslant \mu^r(E_2^{0,1}) + \mu^r(E_2^{1,0})$ from which the result follows. $\square$

Theorem 2.1. *We have* $\mu^r(\mathfrak{p}, H_I^t(M,N)) \leqslant \sum_{j=0}^{t} \mu^r(\mathfrak{p}, \mathrm{Ext}_R^{t-j}(M, H_I^j(N)))$.

Proof. The proof which we include for the reader's convenience, is based on [**5**, Theorem 2.1]. By the same argument as Lemma 2.1 we may assume that R is a local ring with maximal ideal $\mathfrak{p}$. We have to show that

$$\mu^r(H_I^t(M,N)) \leqslant \sum_{j=0}^{t} \mu^r(\mathrm{Ext}_R^{t-j}(M, H_I^j(N))).$$

We use induction on t. In the case $t = 0$, we have $H_I^0(M,N) = \mathrm{Hom}_R(M, \Gamma_I(N))$ so that there is nothing to prove. In the case when $t = 1$, the claim follows from Lemma 2.1. We therefore assume, inductively, that $t > 1$ and the result has been proved for smaller values of t. Then the exact sequence $0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0$ induces a long exact sequence

$$\cdots \longrightarrow H_I^t(M, \Gamma_I(N)) \longrightarrow H_I^t(M,N) \longrightarrow H_I^t(M, N/\Gamma_I(N)) \longrightarrow \cdots$$

which implies that

$$\mu^r(H_I^t(M,N)) \leqslant \mu^r(H_I^t(M,\Gamma_I(N))) + \mu^r(H_I^t(M,N/\Gamma_I(N))).$$

Let E be an injective hull of $\Gamma_I(N)$ and let $L = E/\Gamma_I(N)$. Then by the sequence $0 \longrightarrow \Gamma_I(N) \longrightarrow E \longrightarrow L \longrightarrow 0$ it follows that $H_I^{i-1}(M,L)) \cong H_I^i(M,\Gamma_I(N))$, for all $i \geqslant 2$. Thus by induction hypothesis for $t-1$, we have

$$\mu^r(H_I^{t-1}(M,L)) \leqslant \sum_{j=0}^{t-1} \mu^r(\operatorname{Ext}_R^{t-1-j}(M,H_I^j(L))) = \mu^r(\operatorname{Ext}_R^t(M,\Gamma_I(N)))$$

since $H_I^i(L) \cong H_I^{i+1}(\Gamma_I(N)) = 0$, for all $i \geqslant 1$. Also, the exact sequence $0 \longrightarrow \Gamma_I(N) \longrightarrow \Gamma_I(E) \longrightarrow \Gamma_I(L) \longrightarrow 0$ shows $\operatorname{Ext}_R^{t-1}(M,H_I^0(L)) \cong \operatorname{Ext}_R^t(M,\Gamma_I(N)))$ because $\Gamma_I(E)$ is an injective R-module. Let E' be an injective hull of $N/\Gamma_I(N)$ and let $K = E'/N/\Gamma_I(N)$. Then by the sequence $0 \longrightarrow N/\Gamma_I(N) \longrightarrow E' \longrightarrow K \longrightarrow 0$ it follows that $H_I^{i-1}(M,K)) \cong H_I^i(M,N/\Gamma_I(N))$, for all $i \geqslant 2$. Thus by induction hypothesis for $t-1$, we have

$$\begin{aligned}
\mu^r(H_I^{t-1}(M,K)) &\leqslant \sum_{j=0}^{t-1} \mu^r(\operatorname{Ext}_R^{t-1-j}(M,H_I^j(K))) \\
&= \sum_{j=1}^{t-1} \mu^r(\operatorname{Ext}_R^{t-1-j}(M,H_I^j(K))) + \mu^r(\operatorname{Ext}_R^{t-1}(M,H_I^0(K))) \\
&= \sum_{j=1}^{t-1} \mu^r(\operatorname{Ext}_R^{t-1-j}(M,H_I^{j+1}(N/\Gamma_I(N)))) + \mu^r(\operatorname{Ext}_R^{t-1}(M,H_I^0(K))) \\
&= \sum_{j=1}^{t-1} \mu^r(\operatorname{Ext}_R^{t-1-j}(M,H_I^{j+1}(N))) + \mu^r(\operatorname{Ext}_R^{t-1}(M,H_I^0(K))) \\
&= \sum_{j=2}^{t} \mu^r(\operatorname{Ext}_R^{t-j}(M,H_I^j(N))) + \mu^r(\operatorname{Ext}_R^{t-1}(M,H_I^0(K))).
\end{aligned}$$

Now, the exact sequence $0 \longrightarrow \Gamma_I(E') \longrightarrow \Gamma_I(K) \longrightarrow H_I^1(N/\Gamma_I(N)) \longrightarrow 0$ shows that $\operatorname{Ext}_R^{t-1}(M,H_I^0(K)) \cong \operatorname{Ext}_R^{t-1}(M,H_I^1(N/\Gamma_I(N))) \cong \operatorname{Ext}_R^{t-1}(M,H_I^1(N))$. Thus

$$\begin{aligned}
\mu^r(H_I^{t-1}(M,K)) &\leqslant \sum_{j=2}^{t} \mu^r(\operatorname{Ext}_R^{t-j}(M,H_I^j(N))) + \mu^r(\operatorname{Ext}_R^{t-1}(M,H_I^1(N))) \\
&= \sum_{j=1}^{t} \mu^r(\operatorname{Ext}_R^{t-j}(M,H_I^j(N))).
\end{aligned}$$

Hence,

$$\begin{aligned}\mu^r(H_I^t(M,N)) &\leqslant \mu^r(H_I^t(M,\Gamma_I(N))) + \mu^r(H_I^t(M,N/\Gamma_I(N)))\\ &= \mu^r(H_I^{t-1}(M,L)) + \mu^r(H_I^{t-1}(M,K))\\ &\leqslant \mu^r(\operatorname{Ext}_R^t(M,\Gamma_I(N))) + \sum_{j=1}^{t}\mu^r(\operatorname{Ext}_R^{t-j}(M,H_I^j(N)))\end{aligned}$$

which is claimed. □

Corollary 2.1. [**13**, Theorem 1.1] *We have*

$$\operatorname{Ass}(H_I^t(M,N)) \subseteq \bigcup_{j=0}^{t}\operatorname{Ass}(\operatorname{Ext}_R^{t-j}(M,H_I^j(N))).$$

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(H_I^t(M,N))$. Then $\mu^0(\mathfrak{p}, H_I^t(M,N)) \neq 0$. So, by Theorem 2.1 we have $\sum_{j=1}^{t}\mu^0(\mathfrak{p},\operatorname{Ext}_R^{t-j}(M,H_I^j(N))) \neq 0$. Hence, there exists $0 \leqslant j \leqslant t$ such that $\mu^0(\mathfrak{p},\operatorname{Ext}_R^{t-j}(M,H_I^j(N))) \neq 0$. Therefore, $\mathfrak{p} \in \bigcup_{j=0}^{t}\operatorname{Ass}(\operatorname{Ext}_R^{t-j}(M,H_I^j(N)))$. □

Definition 2.1. An R-module X is said to be I-cofinite, whenever $\operatorname{Supp}(X) \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I,X)$ is finitely generated R-module, for all $i \geqslant 0$.

Corollary 2.2. *If* $\operatorname{Supp}(M) \subseteq V(I)$ *and* $H_I^j(N)$ *is* I*-cofinite, for all* $0 \leqslant j \leqslant t$*, then* $\mu^r(\mathfrak{p}, H_I^i(M,N)) < \infty$*, for all* $0 \leqslant i \leqslant t$*. In particular, if* I *is an ideal of* R *with* $\operatorname{Spec}(R) = V(I)$ *and* $H_I^j(N)$ *is* I*-cofinite, for all* $0 \leqslant j \leqslant t$*, then* $\mu^r(\mathfrak{p}, H_I^i(N)) < \infty$*, for all* $0 \leqslant i \leqslant t$*.*

Proof. In view of [**4**, Proposition 1] $\operatorname{Ext}_R^{t-j}(M,H_I^j(N))$ is a finitely generated R-module, for all $0 \leqslant j \leqslant t$. Now, the claim is obvious by Theorem 2.1. □

Corollary 2.3. *If* $\operatorname{Supp}(M) \subseteq V(I)$ *and* N *is a finitely generated* R*-module for which* $H_I^j(N)$ *is finitely generated, for all* $0 \leqslant j < t$*, then* $\mu^r(\mathfrak{p}, H_I^i(M,N)) < \infty$*, for all* $0 \leqslant i \leqslant t$*. In particular, if* I *is an ideal of* R *with* $\operatorname{Spec}(R) = V(I)$ *and* N *is a finitely generated* R*-module for which* $H_I^j(N)$ *is finitely generated, for all* $0 \leqslant j < t$*, then* $\mu^r(\mathfrak{p}, H_I^i(N)) < \infty$*, for all* $0 \leqslant i \leqslant t$*.*

Proof. In view of [**1**, Theorem 2.5] and [**4**, Proposition 1] $\operatorname{Ext}_R^{t-j}(M,H_I^j(N))$ is a finitely generated R-module, for all $0 \leqslant j \leqslant t$. Now, the claim follows by Theorem 2.1. □

Corollary 2.4. [**2**, Proposition 5.2] *Let* $\operatorname{pd} M < \infty$ *and let* $\dim N < \infty$*, then* $H_I^t(M,N) = 0$*, for all* $t > \operatorname{pd} M + \dim N$*. In particular,*

$$\mu^r(\mathfrak{p}, H_I^{\operatorname{pd} M + \dim N}(M,N)) \leqslant \mu^r(\mathfrak{p}, \operatorname{Ext}_R^{\operatorname{pd} M}(M, H_I^{\dim N}(N))).$$

Proof. By Theorem 2.1 it follows that $\mu^0(\mathfrak{p}, H_I^t(M,N)) = 0$, for any prime ideal $\mathfrak{p}$ of R and for all $t > \operatorname{pd} M + \dim N$. Thus $H_I^t(M,N)) = 0$. The second assertion also follows by Theorem 2.1. □

Corollary 2.5. [**16**, Theorem 2.5] *Let* $\operatorname{pd} M < \infty$ *and let* $\operatorname{ara}(I) < \infty$*; then* $H_I^t(M, N) = 0$*, for all* $t > \operatorname{pd} M + \operatorname{ara}(I)$*, where* $\operatorname{ara}(I)$*, the arithmetic rank of the ideal* I*, is the least number of elements of* R *required to generate an ideal which has the same radical as* I*. In particular,*

$$\mu^r(\mathfrak{p}, H_I^{\operatorname{pd} M+\operatorname{ara}(I)}(M, N)) \leqslant \mu^r(\mathfrak{p}, \operatorname{Ext}_R^{\operatorname{pd} M}(M, H_I^{\operatorname{ara}(I)}(N))).$$

Proof. It follows by the same argument as that of Corollary 2.4. □

Theorem 2.2. *We have*

$$\mu^r(\mathfrak{p}, \operatorname{Ext}_R^t(M, H_I^0(N))) \leqslant \sum_{j=2}^{t} \mu^r(\mathfrak{p}, \operatorname{Ext}_R^{t-j}(M, H_I^{j-1}(N))) + \mu^r(\mathfrak{p}, H_I^t(M, N)).$$

Proof. We may assume that R is a local ring with maximal ideal $\mathfrak{p}$. So we have to show that

$$\mu^r(\operatorname{Ext}_R^t(M, H_I^0(N))) \leqslant \sum_{j=2}^{t} \mu^r(\operatorname{Ext}_R^{t-j}(M, H_I^{j-1}(N))) + \mu^r(H_I^t(M, N)).$$

Theorem 11.38 of [**14**] shows that there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}_R^p(M, H_I^q(N)) \Rightarrow_p H_I^{p+q}(M, N) = E^{p+q}.$$

Now, the exact sequence

$$\cdots \longrightarrow E_2^{t-2,1} \xrightarrow{d_2^{t-2,1}} E_2^{t,0} \longrightarrow 0$$

and $E_3^{t,0} = E_2^{t,0}/\operatorname{Im} d_2^{t-2,0}$ show that $\mu^r(E_2^{t,0}) \leqslant \mu^r(E_2^{t-2,1}) + \mu^r(E_3^{t,0})$. Also, the exact sequence

$$\cdots \longrightarrow E_3^{t-3,2} \xrightarrow{d_3^{t-3,2}} E_3^{t,0} \longrightarrow 0$$

and $E_4^{t,0} = E_3^{t,0}/\operatorname{Im} d_3^{t-3,2}$ show that $\mu^r(E_3^{t,0}) \leqslant \mu^r(E_4^{t,0}) + \mu^r(E_3^{t-3,2})$. Hence,

$$\begin{aligned}
\mu^r(E_2^{t,0}) &\leqslant \mu^r(E_2^{t-2,1}) + \mu^r(E_3^{t-3,2}) + \mu^r(E_4^{t,0}) \\
&\leqslant \mu^r(E_2^{t-2,1}) + \mu^r(E_2^{t-3,2}) + \mu^r(E_4^{t,0}) \\
&\leqslant \cdots = \sum_{j=2}^{t} \mu^r(E_2^{t-j,j-1}) + \mu^r(E_{t+1}^{t,0}) \\
&= \sum_{j=2}^{t} \mu^r(E_2^{t-j,j-1}) + \mu^r(E^t)
\end{aligned}$$

since $E_k^{t-j,j-1}$ is a subquotient of $E_2^{t-j,j-1}$, for all $3 \leqslant k \leqslant t$, and $E_{t+1}^{t,0}$ is a subquotient of E^t, where $E_\infty^{t,0} \cong \phi^t E^t/\phi^{t+1}E^t \cong \phi^t E^t \subset E^t$ and

$$0 = \phi^t E^t \subseteq \cdots \subseteq \phi^1 E^t \subseteq \phi^0 E^t = E^t$$

is a finite filtration. This completes the proof. □

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