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CONVERGENCE THEOREMS OF A SCHEME FOR *I*-ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPING IN BANACH SPACE

Seyit Temir

Abstract. Let X be a Banach space. Let K be a nonempty subset of X. Let $T : K \to K$ be an I-asymptotically quasi-nonexpansive type mapping and $I : K \to K$ be an asymptotically quasi-nonexpansive type mappings in the Banach space. Our aim is to establish the necessary and sufficient conditions for the convergence of the Ishikawa iterative sequences with errors of an I-asymptotically quasi-nonexpansive type mappping in Banach spaces to a common fixed point of T and I. Also, we study the convergence of the Ishikawa iterative sequences to common fixed point for nonself I-asymptotically quasi-nonexpansive type mapping in Banach spaces.

The results presented in this paper extend and generalize some recent work of Chang and Zhou [**1**], Wang [**19**], Yao and Wang [**20**] and many others.

1. Introduction

Let X be a real Banach space, K be a nonempty subset of Banach space and $T, I : K \to K$. Let $F(T) = \{x \in K : Tx = x\}$ and $F(I) = \{x \in K : Ix = x\}$ denote the set of fixed points of mappings T and I, respectively. Recall some definitions and notations. T is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. The quasi-nonexpansive mappings defined as the following were studied by Diaz and Metcalf [**4**] and Dotson [**5**] in Banach spaces. T is called a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $\|Tx - p\| \leqslant \|x - p\|$ for all $x \in K$ and $p \in F(T)$. The concept of asymptotically nonexpansiveness defined as the following was introduced by Goebel and Kirk [**7**]. T is called asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - p\| \leqslant k_n \|x - p\|$ for all $x \in K$ and $p \in F(T)$ and $n \geqslant 1$. Let X be a Banach space and K be a nonempty subset of the Banach space. Let $T, I : K \to K$ be two mappings. T is called I-nonexpansive if $\|Tx - Ty\| \leqslant \|Ix - Iy\|$ for all $x, y \in K$. T is called I-quasi-nonexpansive if $F(T) \cap F(I) \neq \emptyset$ and $\|Tx - p\| \leqslant \|Ix - p\|$ for all $x \in K$ and $p \in F(T) \cap F(I)$.

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From the above definitions, it follows that if $F(T)\cap F(I)$ is nonempty, an I-nonexpansive mapping must be I quasi-nonexpansive, and linear I quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous I quasi-nonexpansive mappings which are not I-nonexpansive. T is called I-asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that $\|T^nx-p\|\leqslant k_n\|I^nx-p\|$ for all $x\in K$ and $p\in F(T)\cap F(I)$ and $n\geqslant 1$. T is called I-asymptotically nonexpansive type mapping if $\limsup_{n\to\infty}\{\sup\{\|T^nx-T^ny\|-\|I^nx-I^ny\|\}\}\leqslant 0$ for all $x,y\in K$.

T is called I-asymptotically quasi-nonexpansive type if $F(T)\cap F(I)\neq\emptyset$ and

$$\limsup_{n\to\infty}\{\sup\{\|T^nx-p\|-\|I^nx-p\|\}\}\leqslant 0 \tag{1.1}$$

for all $x\in K$ and $p\in F(T)\cap F(I)$.

I is called asymptotically quasi-nonexpansive type if $F(I)\neq\emptyset$ and

$$\limsup_{n\to\infty}\{\sup\{\|I^nx-p\|-\|x-p\|\}\}\leqslant 0 \tag{1.2}$$

for all $x\in K$ and $p\in F(I)$.

From the above definitions, it follows that if $F(I)$ is nonempty, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings all are special cases of asymptotically quasi-nonexpansive type mappings.

Let $\{x_n\}$ be of the Ishikawa iterative scheme [**8**] associated with T, $x_0\in K$,

$$\begin{aligned} y_n &= (1-\beta_n)x_n+\beta_nTx_n\\ x_{n+1} &= (1-\alpha_n)x_n+\alpha_nTy_n \end{aligned}$$

for every $n\in\mathbb{N}$, where $0\leqslant\alpha_n,\beta_n\leqslant 1$.

Let $S,T:K\to K$ be two mappings. In 2006, Lan [**9**] introduced the following iterative scheme with errors. The sequence x_n in K defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n+\beta_nT^nx_n+\psi_n\\ x_{n+1} &= (1-\alpha_n)x_n+\alpha_nS^ny_n+\varphi_n \end{aligned} \tag{1.3}$$

for every $n\in\mathbb{N}$, where $0\leqslant\{\alpha_n\},\{\beta_n\}\leqslant 1$ and $\{\varphi_n\}$, $\{\psi_n\}$ are two sequences in K.

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied Ghosh and Debnath [**6**], Goebel and Kirk [**7**], Liu [**10, 11**], Petryshyn and Williamson [**13**] in the settings of Hilbert spaces and uniformly convex Banach spaces. The strong and weak convergences of the sequence of Mann iterates to a fixed point of quasi-nonexpansive maps were studied by Petryshyn and Williamson [**13**]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces were discussed by Ghosh and Debnath [**6**]. The above results and some necessary and sufficient conditions for Ishikawa iterative sequences obtained to converge to a fixed point for asymptotically quasi-nonexpansive mappings were extended by Liu [**10**]. In [**11**], the results of Liu [**10**] were extended and some sufficient and necessary conditions for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings with error member to converge to fixed points were proved. Recently, Temir and Gul [**17**] obtained the weakly almost convergence theorems for I-asymptotically quasi-nonexpansive mapping in a Hilbert space. In [**20**], Yao and Wang established the strong convergence of an iterative scheme with errors involving I-asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. Temir [**18**], studied the convergence to common fixed point of Ishikawa iterative process of generalized I-asymptotically quasi-nonexpansive mappings to common fixed point in Banach space. In [**1**], the convergence theorems for Ishikawa iterative sequences with mixed errors of asymptotically quasi-nonexpansive type mappings in Banach spaces were studied.

2. Preliminaries and notations

We first recall the following definitions. A Banach space X is said to satisfy Opial's condition [**12**] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [**12**] that all l_r spaces for $1 < r < \infty$ have this property. However, the L_r space do not have unless $r = 2$.

In order to prove the main results of this paper, we need the following lemmas.

Lemma 2.1. [**16**] *Let $\{a_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers satisfying the following conditions: $\forall n \geqslant 1$, $a_{n+1} \leqslant a_n + b_n$, where $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2. [**15**] *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $\{\alpha_n\} \subseteq [\epsilon, 1-\epsilon] \subset (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that $\limsup_{n\to\infty} \|x_n\| \leqslant c$, $\limsup_{n\to\infty} \|y_n\| \leqslant c$, and $\limsup_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = c$ for some $c \geqslant 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. [**2**] *Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X and $T : K \to K$ an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$. Then $E - T$ is semi-closed (demi-closed) at zero, i.e., for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $(E-T)\{x_n\}$ converges strongly to 0, then $(E-T)q = 0$.*

3. Convergence theorems for I-asymptotically quasi-nonexpansive type mapping

In this section, X is a Banach space and K is its nonempty subset. Let $T, I : K \to K$ be two mappings, where T is an I-asymptotically quasi-nonexpansive type mapping and $I : K \to K$ is an asymptotically quasi-nonexpansive type mapping. We study the strong and weak convergences of the sequence of Ishikawa iterates with mixed errors to a common fixed point of T and I.

Theorem 3.1. *Let X be a Banach space, K its nonempty subset, and $T, I : K \to K$ two mappings. Let T be an I-asymptotically quasi-nonexpansive type and I be an asymptotically quasi-nonexpansive type in the Banach space satisfying*

$$\|Tx - p\| \leqslant L\|Ix - p\| \tag{3.1}$$

for all $x \in K$ and $p \in F(T) \cap F(I)$, where $L > 0$ is a constant and

$$\|Ix - p\| \leqslant \Gamma \|x - p\| \tag{3.2}$$

for all $x \in K$ and $p \in F(I)$, where $\Gamma > 0$ is a constant. Write $I : K \to K$ instead of $S : K \to K$ in (1.3) and get

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n + \psi_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n I^n y_n + \varphi_n \end{aligned} \tag{3.3}$$

for every $n \in \mathbb{N}$, where $0 \leqslant \{\alpha_n\}, \{\beta_n\} \leqslant 1$ and $\{\varphi_n\}$, $\{\psi_n\}$ be two sequences in K satisfying: (i) $\sum_{n=1}^{\infty} \alpha_n < \infty$; (ii) *$\{\psi_n\}$ is bounded, $\varphi_n = \varphi'_n + \varphi''_n$, $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|\varphi'_n\| < \infty$, $\|\varphi"_n\| = o(\alpha_n)$.*

Then $\{x_n\}$ converges strongly to a common fixed point of T and I in K iff

$$\liminf_{n\to\infty} d(x_n, F(T) \cap F(I)) = 0. \tag{3.4}$$

Lemma 3.1. *Suppose all conditions in Theorem 3.1 are satisfied; then for $\varepsilon > 0$, there exists a positive integer n_0 and $M > 0$ such that*

$$\|x_{n+1} - p\| \leqslant \|x_n - p\| + \alpha_n M + \|\varphi'_n\|$$

for all $p \in F(T) \cap F(I)$, $n \geqslant n_0$ and

$$\|x_{n+m} - p\| \leqslant \|x_n - p\| + M \sum_{i=n}^{n+m-1} \alpha_i + \sum_{i=n}^{n+m-1} \|\varphi'_i\|,$$

for all $p \in F(T) \cap F(I), n \geqslant n_0$, $\forall m \geqslant 1$, where $M = \sup_{n\geqslant 0}\{\varepsilon_n + \|\psi_n\|\} + 3\varepsilon \leqslant \infty$, and ε_n is a sequence with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ such that $\|\varphi''_n\| = \varepsilon_n \alpha_n$.

Proof. For $p \in F(T) \cap F(I)$, from (3.3), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(I^n y_n - p) + \varphi_n\| \\ &\leqslant (1 - \alpha_n)\|x_n - p\| + \alpha_n \|I^n y_n - p\| + \|\varphi_n\| \\ &= (1 - \alpha_n)\|x_n - p\| + \alpha_n\{\|I^n y_n - p\| - \|y_n - p\|\} \\ &\quad + \alpha_n \|y_n - p\| + \|\varphi_n\| \end{aligned} \tag{3.5}$$

Now we consider the second term 0n the right-hand side of (3.5). From (1.1) and (1.2), for any given $\varepsilon > 0$, there exists a positive integer n_0 such that $n \geqslant n_0$, so we have

$$\sup_{x\in K, p\in F(T)\cap F(I)} \{\|T^n x - p\| - \|I^n x - p\|\} < \varepsilon,$$
$$\sup_{x\in K, p\in F(I)} \{\|I^n x - p\| - \|x - p\|\} < \varepsilon.$$

Therefore, in particular, we have

$$\{\|T^n x_n - p\| - \|I^n x_n - p\|\} < \varepsilon, \tag{3.6}$$

for all $p \in F(T) \cap F(I)$ and $\forall n \geqslant n_0$.

$$\{\|I^n y_n - p\| - \|y_n - p\|\} < \varepsilon, \tag{3.7}$$

for all $p \in F(I)$ and $\forall n \geqslant n_0$. From (3.7), we have

$$\|x_{n+1} - p\| \leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\varepsilon + \alpha_n\|y_n - p\| + \|\varphi_n\| \tag{3.8}$$

Consider the third term on the right-hand side of (3.8). From (3.6) and (3.7), we get

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p) + \psi_n\| \\ &\leqslant (1-\beta_n)\|x_n - p\| + \beta_n\{\|T^n x_n - p\| - \|I^n x_n - p\|\} \\ &\quad + \beta_n\{\|I^n x_n - p\| - \|x_n - p\|\} + \beta_n\|x_n - p\| + \|\psi_n\| \\ &\leqslant (1-\beta_n)\|x_n - p\| + 2\beta_n\varepsilon + \beta_n\|x_n - p\| + \|\psi_n\| \\ &= \|x_n - p\| + 2\beta_n\varepsilon + \|\psi_n\|\end{aligned} \tag{3.9}$$

Now consider the fourth term on the right side of (3.5); we have $\|\varphi_n\| \leqslant \|\varphi'_n\| + \|\varphi''_n\|$, $\forall n \geqslant 0$. Substituting (3.9) into (3.8), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\varepsilon + \alpha_n\{\|x_n - p\| + 2\beta_n\varepsilon + \|\psi_n\|\} + \|\varphi_n\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\varepsilon + \alpha_n\|x_n - p\| \\ &\quad + 2\alpha_n\beta_n\varepsilon + \alpha_n\|\psi_n\| + \|\varphi'_n\| + \|\varphi''_n\| \\ &= \|x_n - p\| + \alpha_n\varepsilon(1+2\beta_n) + \alpha_n\varepsilon_n + \alpha_n\|\psi_n\| + \|\varphi'_n\|\end{aligned}$$

Taking $M = \sup_{n\geqslant 0}\{\varepsilon_n + \|\psi_n\|\} + 3\varepsilon$ we obtain

$$\|x_{n+1} - p\| \leqslant \|x_n - p\| + \alpha_n M + \|\varphi'_n\| \tag{3.10}$$

for all $p \in F(T)\cap F(I)$, $n \geqslant n_0$. Writing $n+m-1$ instead of n in inequality (3.10), for $m \geqslant 1$, we get

$$\begin{aligned}\|x_{n+m} - p\| &\leqslant \|x_{n+m-1} - p\| + \alpha_{n+m-1}M + \|\varphi'_{n+m-1}\| \\ &\leqslant \|x_{n+m-2} - p\| + (\alpha_{n+m-1} + \alpha_{n+m-2})M + \|\varphi'_{n+m-2}\| + \|\varphi'_{n+m-1}\| \\ &\ \ \vdots \\ &\leqslant \|x_n - p\| + M\sum_{i=n}^{n+m-1}\alpha_i + \sum_{i=n}^{n+m-1}\|\varphi'_i\|\end{aligned}$$

for all $p \in F(T)\cap F(I)$, $n \geqslant n_0$. Thus Lemma 3.1 is proved. $\square$

Since $\{\psi_n\}$ is bounded, $\varphi_n = \varphi'_n + \varphi''_n$, $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty}\|\varphi'_n\| < \infty$, $\|\varphi''_n\| = o(\alpha_n)$, then we have $\sum_{n=0}^{\infty}(M\alpha_n + \|\varphi'_n\|) < \infty$. From Lemma 2.1, we take $\{a_n\} = \{x_n - p\}$ and $\{b_n\} = M\alpha_n + \|\varphi'_n\|$. This implies that $\lim_{n\to\infty}\|x_n - p\|$ exists.

Proof of Theorem 3.1. We only prove the sufficiency of Theorem 3.1. Suppose that (3.4) is satisfied; then $\lim_{n\to\infty} d(x_n, F(T)\cap F(I)) = 0$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. For $\varepsilon > 0$ and $n \geqslant n_1$ there exists $n_1 \geqslant n_0$ such that $d(x_n, F(T)\cap F(I)) < \varepsilon$, $\sum_{n=n_1}^{\infty}\alpha_n < \frac{\varepsilon}{M}$, $\sum_{n=n_1}^{\infty}\|\varphi'_n\| < \varepsilon$. By the definition of infimum and $d(x_n, F(T)\cap F(I)) < \varepsilon$ there exists $p_0 \in F(T)\cap$

$F(I)$ such that $d(x_{n_1}, p) < 2\varepsilon$. Furthermore, for $n \geqslant n_1 \geqslant n_0$ and $\forall m \geqslant 1$

$$\begin{aligned}\|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - p_0\| + \|x_n - p_0\| \\ &\leqslant \|x_{n_1} - p_0\| + M \sum_{i=n_1}^{n+m-1} \alpha_i + \sum_{i=n_1}^{n+m-1} \|\varphi_i'\| \\ &\quad + \|x_{n_1} - p_0\| + M \sum_{i=n_1}^{n-1} \alpha_i + \sum_{i=n_1}^{n-1} \|\varphi_i'\|.\end{aligned}$$

Then for $n \geqslant n_1 \geqslant n_0$ and $\forall m \geqslant 1$ we have $\|x_{n+m} - x_n\| \leqslant 8\varepsilon$. Since ε is arbitrary, then $\{x_n\}$ is a Cauchy sequence in K. Since X is a Banach space, let $\{x_n\} \to p^*$ as $n \to \infty$. We prove that $p^* \in F(T) \cap F(I)$. We have $\{x_n\} \to p^*$ as $n \to \infty$ and $\lim_{n\to\infty} d(x_n, F(T) \cap F(I)) = 0$, for $\varepsilon > 0$, there exists a positive integer $n_2 \geqslant n_1 \geqslant n_0$ and $n \geqslant n_2$ we have $\|x_n - p^*\| < \varepsilon$, $d(x_n, F(T) \cap F(I)) < \varepsilon$. Then there exists $q \in F(T) \cap F(I)$ such that $d(x_{n_2}, q) < 2\varepsilon$. Furthermore, for $n \geqslant n_2$

$$\begin{aligned}\|T^n p^* - p^*\| &\leqslant \{\|T^n p^* - q\| - \|p^* - q\|\} + 2\|p^* - q\| \\ &\leqslant \{\|T^n p^* - q\| - \|I^n p^* - q\|\} + \{\|I^n p^* - q\| - \|p^* - q\|\} + 3\|p^* - q\| \\ &< 2\varepsilon + 3\{3\varepsilon\} = 11\varepsilon\end{aligned}$$

Since T is I-asymptotically quasi nonexpansive type and I is asymptotically quasi nonexpansive type, this implies that $\{T^n p^*\} \to p^*$ as $n \to \infty$. Furthermore,

$$\|T^n p^* - Tp^*\| \leqslant \{\|T^n p^* - q\| - \|p^* - q\|\} + \|p^* - q\| + \|Tp^* - q\|.$$

Then for $n \geqslant n_2$ by (3.1), (3.2), (3.6) and (3.7) we have

$$\begin{aligned}\|T^n p^* - Tp^*\| &\leqslant \{\|T^n p^* - q\| - \|I^n p^* - q\|\} + \{\|I^n p^* - q\| - \|p^* - q\|\} \\ &\quad + 2\|p^* - q\| + L\|Ip^* - q\| \\ &\leqslant 2\varepsilon + 2\|p^* - q\| + L\Gamma\|p^* - q\| \\ &\leqslant 2\varepsilon + (2 + L\Gamma)\{\|x_{n_2} - p^*\| + \|x_{n_2} - q\|\} \\ &< 2\varepsilon + (2 + L\Gamma)3\varepsilon < \varepsilon(8 + 3L\Gamma)\end{aligned}$$

Since ε is arbitrary, $\{T^n p^*\} \to Tp^*$ as $n \to \infty$, implying $Tp^* = p^* \in F(T) \cap F(I)$.

Further we apply for $I : K \to K$ asymptotically quasi nonexpansive type mapping. Then for $n \geqslant n_2$ we have

$$\begin{aligned}\|I^n p^* - p^*\| &\leqslant \{\|I^n p^* - q\| - \|q - p^*\|\} + 2\|p^* - q\| \\ &\leqslant \varepsilon + 2\{\|x_{n_2} - p^*\| + \|x_{n_2} - q\|\} < \varepsilon + 2\{\varepsilon + 2\varepsilon\} = 7\varepsilon\end{aligned}$$

This implies that $\{I^n p^*\} \to p^*$ as $n \to \infty$. Furthermore,

$$\|I^n p^* - Ip^*\| \leqslant \{\|I^n p^* - q\| - \|p^* - q\|\} + \|p^* - q\| + \|Ip^* - q\|.$$

Then for $n \geqslant n_2$ by (3.2) and (3.7) we have

$$\begin{aligned}\|I^n p^* - Ip^*\| &\leqslant \{\|I^n p^* - q\| - \|p^* - q\|\} + \|p^* - q\| + \Gamma\|Ip^* - q\| \\ &\leqslant \varepsilon + \|p^* - q\| + \Gamma\|p^* - q\| \\ &\leqslant \varepsilon + (1+\Gamma)\{\|x_{n_2} - p^*\| + \|x_{n_2} - q\|\} \\ &< \varepsilon + (1+\Gamma)3\varepsilon < \varepsilon(4 + 3\Gamma).\end{aligned}$$

Since ε is arbitrary, $\{I^n p^*\} \to p^*$ as $n \to \infty$. Also

$$\|I^n p^* - Ip^*\| \leqslant \|I^n p^* - q\| + \|Ip^* - q\| < 2\varepsilon$$

Since ε is arbitrary, $\{I^n p^*\} \to Ip^*$ as $n \to \infty$. This shows that $Ip^* = p^* \in F(I)$. From this we obtain $p^* \in F(T) \cap F(I)$.

Thus $\{x_n\}$ converges strongly to a common fixed point of T and I in K, subset of X Banach space. □

Now we establish the weak convergence theorem for Ishikawa iterates of I-asymptotically quasi-nonexpansive type mappings in Banach spaces. First, we prove the following lemma.

Lemma 3.2. *Let X be a uniformly convex Banach space and K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as in Lemma* 3.1*. If $F = F(T) \cap F(I) \neq \emptyset$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ix_n - x_n\| = 0$.*

Proof. By Lemma 3.1, for any $p \in F(T) \cap F(I)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$. If $c = 0$, then the proof is completed.

Now suppose $c > 0$. From (3.9), we have $\|y_n - p\| \leqslant \|x_n - p\| + 2\beta_n\varepsilon + \|\psi_n\|$. Taking lim sup on both sides in the above inequality,

$$\limsup_{n\to\infty} \|y_n - p\| \leqslant c. \tag{3.11}$$

Since I is asymptotically nonexpansive type self-mappings on K, from (3.7), which is on taking $\limsup_{n\to\infty}$ and using (3.11), then we get $\limsup_{n\to\infty} \|I^n y_n - p\| \leqslant c$. Further, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\lim_{n\to\infty} \|\alpha_n I^n y_n + (1-\alpha_n)x_n - p\| = c$$

$$\lim_{n\to\infty} (1-\alpha_n)\|x_n - p\| + \alpha_n\|I^n y_n - p\| = c.$$

It follows from Lemma 2.2

$$\lim_{n\to\infty} \|I^n y_n - x_n\| = 0. \tag{3.12}$$

Further,

$$\lim_{n\to\infty} \|\alpha_n(T^n x_n - p) + (1-\alpha_n)(x_n - p)\| = \lim_{n\to\infty} \|y_n - p\| = c.$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\lim_{n\to\infty} \|I^n x_n - x_n\| = 0. \tag{3.14}$$

Using (3.1), (3.2), (3.3), (3.13) and (3.14), it is easy to show that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0 \tag{3.15}$$

$$\lim_{n\to\infty} \|Ix_n - x_n\| = 0. \tag{3.16}$$

Then the proof is completed. □

THEOREM 3.2. *Let X be a uniformly convex Banach space which satisfies Opial's condition, K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as in Lemma* 3.1. *If $F(T) \cap F(I) \neq \emptyset$, the mappings $E - T$ and $E - I$ are semi-closed at zero, then $\{x_n\}$ converges weakly to a common fixed point of T and I.*

PROOF. By assumption, $F(T) \cap F(I)$ is nonempty. Take $p \in F(T) \cap F(I)$. It follows from Lemma 3.1 that the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Therefore, $\{x_n - p\}$ is a bounded sequence in X. Since X is a uniformly convex Banach space and K is a nonempty closed convex subset of X, then K is weakly compact. This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $p \in w(\{x_n\})$, where $w(\{x_n\})$ denotes the weak limit set of $\{x_n\}$, which shows that $w(\{x_n\})$ is nonempty. For any $p \in w(\{x_n\})$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to p$ weakly. Hence, it follows from (3.15) and (3.16) in Lemma 3.2 that $Tp = p$ and $Ip = p$. By Opial's condition, $\{x_n\}$ has only one weak limit point, i.e., $\{x_n\}$ converges weakly to a common fixed point of T and I. □

4. Convergence for nonself I-asymptotically quasi-nonexpansive type mappings

In this section, the convergence of the Ishikawa iterative sequences to common fixed point for nonself I-asymptotically quasi-nonexpansive type mapppings is obtained in Banach spaces.

A subset K of X is called a *retract* of X if there exists a continuous map $P : X \to K$ such that $Px = x$ for all $x \in K$. A map $P : X \to K$ is called a retraction if $P^2 = P$. In particular, a subset K is called a *nonexpansive retract* of X if there exists a *nonexpansive retraction* $P : X \to K$ such that $Px = x$ for all $x \in K$.

Next, we introduce the following concepts for nonself mappings. Let X be a real Banach space. A subset K of X be nonempty nonexpansive retraction of X and P be nonexpansive retraction from X onto K. A nonself mapping $T : K \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{\upsilon_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \upsilon_n = 1$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant \upsilon_n \|x - y\|$$

for all $x, y \in K$ and $n \geqslant 1$. T is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant L\|x - y\|$$

for all $x, y \in K$ and $n \geqslant 1$. From the above definition, it is obvious that nonself asymptotically nonexpansive mappings is uniformly L-Lipschitzian.

Let $I : K \to X$ be a nonself asymptotically quasi-nonexpansive type mappings and $T : K \to X$ be a nonself I-asymptotically quasi-nonexpansive type mappings with $F(T) \cap F(I) = \{x \in K : Tx = x = Ix\} \neq \emptyset$. A mapping $T : K \to X$ is called Λ-*Lipschitzian* if there exists constant $\Lambda > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leqslant \Lambda\|I(PI)^{n-1}x - I(PI)^{n-1}y\|$$

for all $x, y \in K$ and $n \geqslant 1$.

Iterative techniques for converging fixed points of nonexpansive non-self mappings have been studied by many authors (see, for example, [**3, 19, 14**]). The concept of nonself asymptotically nonexpansive mappings was introduced in [**3**] as a generalization of asymptotically nonexpansive self-mappings and some strong and weak convergence theorems for such mappings were obtained. The sequence $\{x_n\}_{n\geqslant 1}$ generated as follows: $x_1 \in K$,

$$\begin{aligned} y_n &= P(\alpha_n T(PT)^{n-1}x_n + \beta_n x_n),\\ x_{n+1} &= P(\alpha'_n T(PT)^{n-1}y_n + \beta'_n x_n), \quad \forall n \geqslant 1, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\alpha'_n\}, \{\beta'_n\} \in (0, 1)$.

Let $T : K \to X$ be a nonself I-asymptotically quasi-nonexpansive type mapping and $I : K \to X$ be a nonself asymptotically quasi-nonexpansive type mapping.

Now we define an $\{x_n\}_{n\geqslant 1}$ sequence as follows:

$$\begin{aligned} y_n &= P(\alpha_n T(PT)^{n-1}x_n + \beta_n x_n + \gamma_n\psi_n),\\ x_{n+1} &= P(\alpha'_n I(PI)^{n-1}y_n + \beta'_n x_n + \gamma'_n\varphi_n), \quad \forall n \geqslant 1, \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $(0, 1)$ with $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{\psi_n\}, \{\varphi_n\}$ are bounded sequences in K.

$$\limsup_{n\to\infty}\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\|\}\Big) \leqslant 0.$$

Observe that

$$\begin{aligned} &\limsup_{n\to\infty}\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\|\}\Big)\\ &\qquad\times\limsup_{n\to\infty}\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\| + \|I(PI)^{n-1}x - p\|\}\Big)\\ &= \limsup_{n\to\infty}\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\|^2 - \|I(PI)^{n-1}x - p\|^2\}\Big) \leqslant 0. \end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty}\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\|\}\Big) \leqslant 0.$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \geqslant n_0$ we have

$$\Big(\sup_{x\in X, p\in F(T)\cap F(I)}\{\|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\|\}\Big) \leqslant 0.$$

THEOREM 4.1. *Let X be a Banach space and K be a nonempty subset of the Banach space. Let $T, I : K \to X$ be two nonself mappings. Let T be a nonself I-asymptotically quasi-nonexpansive type and I be a nonself asymptotically quasi-nonexpansive type in Banach space with $F(T) \cap F(I) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (4.1) and for every $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, $\sum_{i=1}^{\infty} \gamma_n < \infty$, $\sum_{i=1}^{\infty} \gamma'_n < \infty$, and $\{\psi_n\}, \{\varphi_n\}$ are bounded sequences in K.*

Then $\{x_n\}$ converges strongly to a common fixed point of T and I in K iff

$$\liminf_{n\to\infty} d(x_n, F(T) \cap F(I)) = 0. \tag{4.2}$$

PROOF. The necessity of condition (4.2) is obvious. Next we prove the sufficiency of condition (4.2). Let the sequence $\{x_n\}$ be defined by (4.1). Let $p \in F(T) \cap F(I)$, by boundedness of the sequences $\{\psi_n\}, \{\varphi_n\}$, so we can put

$$M = \max\{\sup_{n\geqslant 1} \|\psi_n - p\|, \sup_{n\geqslant 1} \|\varphi_n - p\|\}.$$

For any given $\varepsilon > 0$, there exists a positive integer n_0 such that $n \geqslant n_0$

$$\sup_{x\in K, p\in F(T)\cap F(I)} \{\|T(PT)^{n-1}x - p\| - \|I(PI)^{n-1}x - p\|\} < \varepsilon.$$
$$\sup_{x\in K, p\in F(I)} \{\|I(PI)^{n-1}x - p\| - \|x - p\|\} < \varepsilon.$$

Therefore, in particular, we have

$$\{\|T(PT)^{n-1}x_n - p\| - \|I(PI)^{n-1}x_n - p\|\} < \varepsilon, \tag{4.3}$$

for all $p \in F(T) \cap F(I)$ and $\forall n \geqslant n_0$.

$$\{\|I(PI)^{n-1}y_n - p\| - \|y_n - p\|\} < \varepsilon, \tag{4.4}$$

for all $p \in F(I)$ and $\forall n \geqslant n_0$. Thus for each $n \geqslant 1$ and for any $p \in F(T) \cap F(I)$, using (4.1), (4.3) and (4.4), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|P(\alpha'_n x_n + \beta'_n I(PI)^{n-1} y_n + \gamma'_n \varphi_n - p)\| \\ &\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|I(PI)^{n-1} y_n - p\| + \gamma'_n \|\varphi_n - p\| \\ &= \alpha'_n \|x_n - p\| + \beta'_n \{\|I(PI)^{n-1} y_n - p\| - \|y_n - p\|\} \\ &\quad + \beta'_n \|y_n - p\| + \gamma'_n \|\varphi_n - p\| \\ &\leqslant \alpha'_n \|x_n - p\| + \beta'_n \{\varepsilon\} + \beta'_n \|y_n - p\| + \gamma'_n M \end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
\|y_n - p\| &= \|P(\alpha_n x_n + \beta_n T(PT)^{n-1}x_n + \gamma_n \psi_n - p)\| \\
&\leqslant \alpha_n\|x_n - p\| + \beta_n\|T(PT)^{n-1}x_n - p\| + \gamma_n\|\psi_n - p\| \\
&\leqslant \alpha_n\|x_n - p\| + \beta_n\{\|T(PT)^{n-1}x_n - p\| - \|I(PI)^{n-1}x_n - p\|\} \\
&\quad + \beta_n\{\|I(PI)^{n-1}x_n - p\| - \|x_n - p\|\} + \beta_n\|x_n - p\| + \gamma_n M \\
&\leqslant \alpha_n\|x_n - p\| + 2\beta_n\{\varepsilon\} + \beta_n\|x_n - p\| + \gamma_n M \\
&\leqslant (\alpha_n + \beta_n)\|x_n - p\| + 2\beta_n\varepsilon + \gamma_n M \\
&\leqslant (1 - \gamma_n)\|x_n - p\| + 2\beta_n\varepsilon + \gamma_n M \\
&\leqslant \|x_n - p\| + D_n
\end{aligned} \tag{4.6}
$$

where $D_n = 2\beta_n\varepsilon + \gamma_n M$. Then $\sum_{n=1}^{\infty} D_n < \infty$ since $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Substituting (4.6) into (4.5), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\leqslant \alpha'_n\|x_n - p\| + \beta'_n\varepsilon + \beta'_n(\|x_n - p\| + D_n) + \gamma'_n M \\
&\leqslant (\alpha'_n + \beta'_n)\|x_n - p\| + \beta'_n(\varepsilon + D_n) + \gamma'_n M \\
&\leqslant (1 - \gamma'_n)\|x_n - p\| + G_n \\
&\leqslant \|x_n - p\| + G_n
\end{aligned} \tag{4.7}
$$

where $G_n = \beta'_n(\varepsilon + D_n) + \gamma'_n M$. Then $\sum_{n=1}^{\infty} G_n < \infty$ since $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} D_n < \infty$.

It follows from (4.7) that $d(x_{n+1}, F(T) \cap F(I)) \leqslant d(x_n, F(T) \cap F(I)) + G_n$.

By Lemma 2.1, we can get that $\lim_{n\to\infty} d(x_n, F(T)\cap F(I))$ exists. By condition $\liminf_{n\to\infty} d(x_n, F(T) \cap F(I)) = 0$, we have

$$
\lim_{n\to\infty} d(x_n, F(T) \cap F(I)) = 0. \tag{4.8}
$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in X. In fact, for any $n \geqslant n_0$, any $m \geqslant n_1$ and any $p \in F(T) \cap F(I)$ we have

$$
\begin{aligned}
\|x_{n+m} - p\| &\leqslant \|x_{n+m-1} - p\| + G_{n+m-1} \\
&\leqslant \|x_{n+m-2} - p\| + G_{n+m-1} + G_{n+m-2} \\
&\leqslant \ldots \leqslant \|x_n - p\| + \sum_{k=n}^{\infty} G_k.
\end{aligned} \tag{4.9}
$$

So by (4.9), we have

$$
\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - p\| + \|x_n - p\| \leqslant 2\|x_n - p\| + \sum_{k=n}^{\infty} G_k. \tag{4.10}
$$

By the arbitrariness of $p \in F(T) \cap F(I)$ and (4.10), we have

$$
\|x_{n+m} - p\| \leqslant 2d(x_n, F(T) \cap F(I)) + \sum_{k=n}^{\infty} G_k \qquad \forall n \geqslant n_0.
$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geqslant n_0$, such that for any $n \geqslant n_1$, $d(x_n, F(T) \cap F(I)) < \frac{\varepsilon}{4}$ and $\sum_{k=n}^{\infty} G_k < \frac{\varepsilon}{2}$, we have $\|x_{n+m} - x_n\| < \varepsilon$, and so for any $m \geqslant 1$

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a $p^* \in X$ such that $x_n \to p^*$ as $n \to \infty$.

Finally, by the routine method, we have to prove that $p^* \in F(T) \cap F(I)$. By contradiction, we assume that p^* is not in $F(T) \cap F(I)$. Since $F(T) \cap F(I)$ is a closed set, $d(p^*, F(T) \cap F(I)) > 0$. Hence for all $p \in F(T) \cap F(I)$, we have

$$\|p^* - p\| \leqslant \|x_n - p^*\| + \|x_n - p\|.$$

This implies that

$$d(p^*, F(T) \cap F(I)) \leqslant \|x_n - p^*\| + d(x_n, F(T) \cap F(I)). \tag{4.11}$$

Letting $n \to \infty$ in (4.11) and noting (4.8), we have $d(p^*, F(T) \cap F(I)) \leqslant 0$. This is a contradiction. Hence $p^* \in F(T) \cap F(I)$. This completes the proof of Theorem 4.1. $\square$

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Department of Mathematics
Art and Science Faculty
Harran University
Sanliurfa
Turkey
temirseyit@harran.edu.tr

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