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ON COMMUTATIVITY OF QUASI-MINIMAL GROUPS

Slavko Moconja

Abstract. We investigate if every quasi-minimal group is abelian, and give a positive answer for a quasi-minimal pure group having a $\emptyset$-definable partial order with uncountable chains. We also relate two properties of a complete theory in a countable language: the existence of a quasi-minimal model and the existence of a strongly regular type. As a consequence we derive the equivalence of conjectures on commutativity of quasi-minimal groups and commutativity of regular groups.

1. Introduction

In the nineties Zilber initiated the study of model theory of the complex exponential field $(\mathbb{C},+,\cdot,\exp)$ and conjectured that every definable subset of $\mathbb{C}$ is either countable or co-countable (the complement is countable); uncountable first order structures with this property are called quasi-minimal (see [**9**]). This conjecture is still widely open and motivates the study of model theoretical properties of quasi-minimal structures, especially algebraic ones. The interesting infinitary properties are related to $\mathcal{L}_{\omega_1,Q}$-categoricity of the complex exponential field and Schanuel's conjecture of transcendental number theory. A complete review of this topic can be found in the first part of Zilber's recent paper [**10**]. In this article we are interested in elementary first order properties of quasi-minimal structures. This direction was initiated by Itai, Tsuboi and Wakai in [**1**]. In particular, we will investigate quasi-minimal groups. By a group in this article we mean a first-order structure $(G,\cdot,\dots)$ which beside the group structure may have additional operations and relations; groups with no additional structure are called pure groups.

Initially, quasi-minimality was viewed as a generalization of minimality; an infinite first order structure is minimal if any definable subset is either finite or co-finite. Minimal pure groups were classified by Reineke in 1975. In [**6**] he proved that every minimal group is abelian, and then by purely algebraic arguments obtained

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the full list. Concerning minimal fields, it is known that every algebraically closed field is a minimal structure. The converse was conjectured by Podewski in 1973 (see [**5**]).

Conjecture 1.1. *Every minimal field is algebraically closed.*

Podewski's conjecture was confirmed by Wagner for fields of positive characteristic in [**8**]. Partial results in characteristic 0 were obtained by Krupinski et al. in [**3**], where the conjecture is reduced to fields with a proper partial ordering definable in the field language. There are natural conjectures generalizing Podewski's conjecture and Reineke's theorem in the quasi-minimal case.

Conjecture 1.2. *Every quasi-minimal field is algebraically closed.*

Conjecture 1.3. *Every quasi-minimal group is abelian.*

These conjectures were posed in [**4**] and studied recently by Gogacz and Krupinski in [**2**]. Gogacz and Krupinski noted that proofs of the above mentioned results on Podewski's conjecture translate to the context of Conjecture 1.2. Concerning Conjecture 1.3 they proved that in every quasi-minimal non-abelian group all non-central elements are conjugated and that their centralizers are countable. By applying iterated HNN extensions they constructed an uncountable group with those properties, but it is still open if it is quasi-minimal. In this paper we will confirm Conjecture 1.3 in the case when there exists a $\emptyset$-definable partial order with uncountable chains. This assumption is motivated by Theorem 5.1 from [**4**], according to which there exist two kinds of quasi-minimal structures whose generic type is countably based (which holds in the group case): they are either symmetric or there exists a definable, over a finite parameter set, partial order with 'large' chains. So, we prove the following theorem.

Theorem 1.1. *Every quasi-minimal pure group having a $\emptyset$-definable partial order with an uncountable chain is abelian.*

The notion of regular types first arose in Shelah's investigations of stable theories. Recently, Pillay and Tanović in [**4**] introduced the concept of strong regularity for global types in the general first order context (we will further include the definition of this and all other undefined notions used in the introduction). We will investigate the connection between notions of global strong regularity and quasi-minimality and prove the following theorem, indicating that quasi-minimality is closely related to strong regularity.

Theorem 1.2. *Assume that T is a complete theory in a countable language with infinite models.*

(1) *If there exists a global, countably invariant, strongly regular via $x = x$ type, then T has a quasi-minimal model.*

(2) *If $M \models T$ is quasi-minimal and its generic type p is definable, then there exists a global, M-invariant, strongly regular via $x = x$ type extending p.*

Since the generic type of any quasi-minimal or strongly regular group is definable over $\emptyset$ (see Lemma 4.1 below), as an immediate corollary we obtain.

COROLLARY 1.1. (1) *The existence of a non-abelian quasi-minimal group is equivalent with the existence of a strongly regular non-abelian group.*

(2) *The existence of a quasi-minimal, non algebraically closed field is equivalent with the existence of a strongly regular, non algebraically closed field.*

Part (1) of the previous theorem was independently proved in unpublished notes by Haykazyan and by Tanović. Part (2) can be derived from the results in [**4**]. Here we will give direct proofs of both parts.

Throughout the paper we use the standard notions from model theory. Let $\mathcal{L}$ be a first-order language and let M be an $\mathcal{L}$-structure. By $\bar{a}$ we denote a finite tuple of elements from M and by $|\bar{a}|$ its length. We will write $\bar{a} \in A$, where $A \subseteq M$, to denote that $\bar{a}$ is a tuple of elements of A. For a subset $A \subseteq M$, let $\mathcal{L}_A$ be the language obtained by adding to $\mathcal{L}$ constant symbols naming elements of A. For an $\mathcal{L}_A$-formulas we say that it is a *formula with parameters from* A. Usually, when the meaning of the model M is clear from the context, then by a formula we mean a formula with parameters from M. By a solution of $\phi(\bar{x})$ in $A \subset M$ we mean a tuple $\bar{a} \in A^{|\bar{x}|}$ for which $M \models \phi(\bar{a})$ holds; $\phi(A)$ will denote the set of all solutions of $\phi(\bar{x})$ in A. A subset $D \subseteq M$ is *definable over* A, or A-*definable*, if $D = \phi(M)$ holds for some formula $\phi(x)$ with parameters from A; $D \subseteq M$ is *definable* if it is A-definable for some $A \subseteq M$.

A type over $A \subseteq M$ in variables $\bar{x}$ is a set of formulas whose free variables are among $\bar{x}$ and whose parameters are from A, which is consistent with $\mathrm{Th}(M)$. By an n-type we mean a type in n free variables, usually $x_1, \ldots, x_n$. If p is an n-type (over some set of parameters), we say that $\bar{a} \in M$ *realizes* p, and write $\bar{a} \models p$, if $\bar{a}$ is a solution of every formula $\phi(\bar{x}) \in p$. The set of all realizations of p in M is denoted by $p(M)$. We say that p is not realized in M, or that M omits p, if $p(M) = \emptyset$. An n-type over A is *complete* if it contains either $\phi(\bar{x})$ or $\neg\phi(\bar{x})$, for every $\phi(\bar{x})$ with parameters from A. We usually denote complete types by $p, q, r, \ldots$. The set of all complete n-types over A is denoted by $\mathrm{S}_n(A)$. For a tuple $\bar{a} \in M$ and subset $A \subseteq M$, by $\mathrm{tp}(\bar{a}/A)$ we denote the set of all formulas with parameters from A satisfied by $\bar{a}$; it is a complete $|\bar{a}|$-type over A.

For an n-type p over A and $B \subseteq A$, we write $p_{\restriction B}$ to denote the restriction of p to the set of parameters B, i.e., the set of all $\mathcal{L}_B$-formulas from p. A formula $\phi(\bar{x})$ (with parameters from M) is *algebraic* if $\phi(M)$ is finite. A type $p \in \mathrm{S}_n(M)$ is *algebraic* if it contains an algebraic formula; otherwise, it is *non-algebraic*.

A complete type $p \in \mathrm{S}_1(B)$ is *finitely satisfiable in* $A \subseteq B$, if every finite subtype $p_0 \subseteq p$ has a realization in A, equivalently if every formula $\phi(x) \in p$ has a solution in A.

REMARK 1.1. If a type $p \in \mathrm{S}_1(A)$ is finitely satisfiable in A, then for every $B \supseteq A$ there exists a type $q \in \mathrm{S}_1(B)$ such that $p \subseteq q$ and q is finitely satisfiable in A. For the proof, consider the set of formulas

$$\Sigma(x) = \{\phi(x, \bar{b}) \mid \phi(x, \bar{y}) \text{ is } \mathcal{L}\text{-formula}, \bar{b} \in B \text{ and } \phi(A, \bar{b}) = A\}.$$

Obviously, $\Sigma(x) \cup p(x)$ is consistent, since p is finitely satisfiable in A. Therefore it has an extension $q \in \mathrm{S}_1(B)$. The type q is finitely satisfiable in A, since otherwise there is a formula $\varphi(x) \in q(x)$ such that $\varphi(A) = \emptyset$. But then $\neg\varphi(A) = A$, hence $\neg\varphi(x) \in \Sigma(x) \subseteq q(x)$. A contradiction.

In fact, one can easily prove that $q \in \mathrm{S}_1(B)$ is finitely satisfiable in A extension of p if and only if it contains $\Sigma(x) \cup p(x)$.

Let $p \in \mathrm{S}_1(M)$ be a type over a model, and let $M \subseteq B$. An extension $q \in \mathrm{S}_1(B)$ of p is an *heir* of p if for every $\mathcal{L}_M$-formula $\phi(x, \bar{y})$ such that $\phi(x, \bar{b}) \in q$ for some $\bar{b} \in B$, there exists $\bar{m} \in M$ such that $\phi(x, \bar{m}) \in p$.

Monster model. Assume that T is a complete theory with infinite models. If we are interested in studying the models of T of cardinality less than κ, then under some additional set-theoretical assumptions we can fix a model of T of cardinality κ, usually denoted by $\mathfrak{C}$, with the following properties:

- every model M of T with $|M| \leqslant \kappa$ is elementary embeddable in $\mathfrak{C}$, hence we may assume that M is an elementary submodel of $\mathfrak{C}$;
- every type with parameters from A, where $A \subseteq \mathfrak{C}$ and $|A| < \kappa$, is realized in $\mathfrak{C}$;
- if $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$, where $A \subseteq \mathfrak{C}$ and $|A| < \kappa$, then there exists an automorphism $f \in \mathrm{Aut}_A(\mathfrak{C})$ such that $f(\bar{a}) = \bar{b}$.

We say that $\mathfrak{C}$ is a *monster model of* T. By a small subset or model we mean a set $A \subseteq \mathfrak{C}$ or a model $M \prec \mathfrak{C}$ such that $|A|, |M| < \kappa$. Also $\models \phi(\bar{a})$ means $\mathfrak{C} \models \phi(\bar{a})$. For a complete type with parameters from $\mathfrak{C}$ we say that it is *global*, and usually denote such type by $\mathfrak{p}$. By $\mathfrak{p}_{\restriction A}$ we denote the restriction of the global type $\mathfrak{p}$ to the set of parameters A.

If $\Sigma(\bar{x})$ is a set of formulas with parameters from A, where A is some small subset of $\mathfrak{C}$, and $\phi(\bar{x})$ is a formula with parameters, then by $\Sigma(\bar{x}) \vdash \phi(\bar{x})$ we mean that whenever $\bar{a} \in \mathfrak{C}$ satisfies all formulas in $\Sigma(\bar{x})$, then $\bar{a}$ also satisfies $\phi(\bar{x})$ (the set of realisation of $\Sigma(\bar{x})$ in $\mathfrak{C}$ is a subset of the set of realisations of $\phi(\bar{x})$ in $\mathfrak{C}$). Now compactness inside $\mathfrak{C}$ can be stated in the following form. If $\Sigma(\bar{x}) \vdash \phi(\bar{x})$, then there exists some finite subset $\Sigma_0(\bar{x}) \subseteq \Sigma(\bar{x})$ such that $\Sigma_0(\bar{x}) \vdash \phi(\bar{x})$. Indeed, by considering $\bar{x}$ as constants, the theory $\Sigma(\bar{x}) \cup \{\neg\phi(\bar{x})\}$ is not consistent with T, since otherwise it would be a type over small set of parameters, hence realized in $\mathfrak{C}$. By compactness, for some finite $\Sigma_0(\bar{x}) \subseteq \Sigma(\bar{x})$, $\Sigma_0(\bar{x}) \cup \{\neg\phi(\bar{x})\}$ is not consistent with T; in particular $\Sigma_0(\bar{x}) \vdash \phi(\bar{x})$.

A global type $\mathfrak{p} \in \mathrm{S}_n(\mathfrak{C})$ is *invariant over* A, or A-*invariant*, if for every automorphism $f \in \mathrm{Aut}_A(\mathfrak{C})$, every formula $\phi(\bar{x}, \bar{y})$ with no parameters and every $\bar{a} \in \mathfrak{C}$: $\phi(\bar{x}, \bar{a}) \in \mathfrak{p}$ iff $\phi(\bar{x}, f(\bar{a})) \in \mathfrak{p}$ holds. The type $\mathfrak{p}$ is *invariant* if it is A-invariant, for some small A, and it is *countably invariant* if it is A-invariant, for some countable A. Note that if $\mathfrak{p}$ is A-invariant, then $\mathfrak{p}$ is B-invariant, for every $B \supseteq A$.

Let $\mathfrak{p} \in \mathrm{S}_n(\mathfrak{C})$ be invariant and let $(I, <)$ be a linear order. The sequence of n-tuples $(\bar{a}_i)_{i \in I}$ is a *Morley sequence in* $\mathfrak{p}$ *over* A if $\bar{a}_i \models \mathfrak{p}_{\restriction A\bar{a}_{<i}}$, for each $i \in I$, where $\bar{a}_{<i}$ denotes the set $\{\bar{a}_j \mid j < i\}$.

Definable types. Let M be a first order structure and $p \in S_n(M)$. We say that the type p is *definable over* $A \subseteq M$ if for every formula $\phi(\bar{x}, \bar{y})$ with no parameters, there exists formula $\mathrm{d}_p\,\phi(\bar{y})$ with parameters from A, such that

$$\phi(\bar{x},\bar{m}) \in p \quad \text{if and only if} \quad M \models \mathrm{d}_p\,\phi(\bar{m})$$

holds for every $\bar{m} \in M$.

We say that formula $\mathrm{d}_p\,\phi(\bar{y})$ is a *definition over* A of formula $\phi(\bar{x},\bar{y})$, and the correspondence d_p is a *defining schema* of p. A type $p \in \mathrm{S}_n(M)$ is *definable* if it is definable over M.

Remark 1.2. If we work in a countable language $\mathcal{L}$, then every definable type is definable over some countable set. Indeed, there are only countably many formulas with no parameters, hence if $p \in \mathrm{S}_n(M)$ is a definable type, then the definition of each formula uses only finitely many parameters from M, hence p is definable over some countable $A \subseteq M$.

2. Closure operations associated to types

We start with the definition of a closure operation on a set.

Definition 2.1. Suppose that S is a non-empty set and let $\mathrm{cl} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$. We say that cl is *closure operation* on S if it satisfies (for all $X, Y \subseteq S$):

- $X \subseteq Y$ implies $X \subseteq \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$; (Monotonicity)
- $\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(X_0) \mid X_0 \subseteq X,\ X_0 \text{ is finite}\}$; (Finite character)
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$. (Transitivity)

We say that cl is pregeometry operation if in addition it satisfies (for all $a, b \in S$ and $X \subseteq S$):

- $b \in \mathrm{cl}(X \cup \{a\}) \smallsetminus \mathrm{cl}(X)$ implies $a \in \mathrm{cl}(X \cup \{b\})$. (Exchange property)

If $\mathrm{cl} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ is an operation on S, then for any $A \subseteq S$ one can define another operation $\mathrm{cl}^A : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ on S with

$$\mathrm{cl}^A(X) = \mathrm{cl}(A \cup X).$$

We call the operation cl^A *the relativization of* cl *in* A. It is easy to see that any relativization keeps monotonicity, finite character, transitivity and exchange property.

An important notion related to a closure operation cl, which we will use, is the notion of a cl-free sequence.

Definition 2.2. Suppose that cl is a closure operation on S, and $(I, \leqslant)$ is a linear order. We say that the sequence $(a_i)_{i\in I}$ of elements of S, is cl-*free over* $A \subseteq S$ if

$$a_i \notin \mathrm{cl}(A \cup \{a_j \mid j < i\})$$

holds for every $i \in I$. In particular, (a, b) is a cl-free sequence over $A \subseteq S$ if $a \notin \mathrm{cl}(A)$ and $b \notin \mathrm{cl}(A \cup \{a\})$.

Assume that $p \in \mathrm{S}_1(N)$ is a non-algebraic type, where N is a first order structure (possibly $N = \mathfrak{C}$). We associate to p an operation cl_p on $\mathcal{P}(N)$ defined by

$$\mathrm{cl}_p(X) = \{a \in N \mid a \not\models p_{\restriction X}\}.$$

Equivalently

$$\mathrm{cl}_p(X) = \bigcup\{\phi(N,\bar{a}) \mid \phi(x,\bar{y}) \text{ is } \mathcal{L}\text{-formula, } \bar{a} \in X \text{ and } \phi(x,\bar{a}) \notin p\}.$$

It is not common, but we will say that a formula is p-large if it belongs to p; otherwise it is p-small. If the type p is clear from the context, we just say that a formula is large (small), instead of p-large (p-small). So, keeping in mind this convention, we see that $\mathrm{cl}_p(X)$ is the union of all sets definable by a p-small formula with parameters from X.

Remark 2.1. Let $p \in \mathrm{S}_1(N)$.

(1) Operation cl_p always satisfies monotonicity and finite character. Since p is non-algebraic, $X \subseteq \mathrm{cl}_p(X)$ immediately follows. If $X \subseteq Y$, then $\mathrm{cl}_p(X) \subseteq \mathrm{cl}_p(Y)$ since every formula with parameters from X is also a formula with parameters from Y. For finite character, if $a \in \mathrm{cl}_p(X)$, then a is a solution of some small formula with parameters from X. Since this formula uses only finitely many parameters X_0 from X, it follows that $a \in \mathrm{cl}_p(X_0)$.

(2) As we have remarked earlier, if p is a non-algebraic type, then every relativization cl_p^A satisfies Monotonicity and Finite character.

Definition 2.3. Assume that $\mathfrak{p} \in \mathrm{S}_1(\mathfrak{C})$ is a global non-algebraic type. The type $\mathfrak{p}$ is *strongly regular* if there exists some small $A \subseteq \mathfrak{C}$ such that:

- $\mathfrak{p}$ is A-invariant;
- for every $B \supseteq A$ and every $a \not\models \mathfrak{p}_{\restriction B}$: $\quad \mathfrak{p}_{\restriction B} \vdash \mathfrak{p}_{\restriction Ba}$ holds.

In that case we say that A witnesses the strong regularity of $\mathfrak{p}$.

Remark 2.2. (1) Pillay and Tanović in [**4**] define that a global, nonalgebraic type $\mathfrak{p} \in \mathrm{S}_1(\mathfrak{C})$ is *strongly regular via formula* $\phi(x)$ if there exists some small $A \subseteq \mathfrak{C}$ such that:

- $\phi(x) \in \mathfrak{p}_{\restriction A}$ and $\mathfrak{p}$ is A-invariant;
- for every $B \supseteq A$ and every $a \in \phi(\mathfrak{C}) \smallsetminus \mathfrak{p}_{\restriction B}(\mathfrak{C})$: $\mathfrak{p}_{\restriction B} \vdash \mathfrak{p}_{\restriction Ba}$ holds.

Note that our definition of a strongly regular type corresponds to the definition of a strongly regular type via $x = x$ in the sense of Pillay and Tanović.

(2) An equivalent characterization of a strongly regular type is the following: A global, non-algebraic type $\mathfrak{p}$ is strongly regular, witnessed by small A, if:

- $\mathfrak{p}$ is A-invariant;
- $\mathrm{cl}_{\mathfrak{p}}^A$ is a closure operation on $\mathfrak{C}$.

The proof of this characterisation can be found in [**4**].

(3) It is clear from the definition that if A witnesses strong regularity of $\mathfrak{p}$, then any small $B \supseteq A$ witnesses strong regularity of $\mathfrak{p}$. Moreover, if $\mathfrak{p}$ is strongly regular, then every small subset B, such that $\mathfrak{p}$ is B-invariant, witnesses strong regularity of $\mathfrak{p}$. For the proof of this fact see [**7**].

Let us fix a strongly regular type $\mathfrak{p} \in \mathrm{S}_1(\mathfrak{C})$, witnessed by A. We consider $\mathrm{cl}_{\mathfrak{p}}^A$-free sequences. We will freely use the following fact.

Fact 2.1. *If $(I,<)$ is a linear order, and $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are two $\mathrm{cl}_{\mathfrak{p}}^A$-free sequences over some small $B \supseteq A$, then* $\mathrm{tp}((a_i)_{i\in I}/B) = \mathrm{tp}((b_i)_{i\in I}/B)$.

Note that for a strongly regular type $\mathfrak{p}$, the $\mathrm{cl}_{\mathfrak{p}}^A$-free sequences over A are precisely the Morley sequences in $\mathfrak{p}$ over A. Indeed, if $(I,<)$ is a linear order, $(a_i)_{i\in I}$ is a $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence over A iff $a_i \notin \mathrm{cl}_{\mathfrak{p}}^A(\bar{a}_{<i})$ iff $a_i \models \mathfrak{p}_{\restriction A\bar{a}_{<i}}$ iff $(a_i)_{i\in I}$ is a Morley sequence in $\mathfrak{p}$ over A, where $\bar{a}_{<i}$ denotes the set $\{a_j \mid j<i\}$.

According to the dichotomy theorem for regular types from [**4**] we have two kinds of regular types: symmetric and asymmetric.

Symmetric kind: If $\mathrm{cl}_{\mathfrak{p}}^A$ is a pregeometry operation on $\mathfrak{C}$.

In this case every $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence over A is invariant under permutations, i.e., if $(a_1,a_2,\dots,a_n)$ is a $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence over A and π is a permutation of $\{1,2\dots,n\}$, then $(a_{\pi(1)},a_{\pi(2)},\dots,a_{\pi(n)})$ is also a $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence over A. In particular, by Fact 2.1 every $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence is totally indiscernible over A.

Asymmetric kind: If $\mathrm{cl}_{\mathfrak{p}}^A$ is not a pregeometry operation on $\mathfrak{C}$.

In this case there exist a finite extension A_0 of A and a A_0-definable partial order on $\mathfrak{C}$ such that every $\mathrm{cl}_{\mathfrak{p}}^A$-free sequence over A_0 is strictly increasing.

Now, we turn to the notion of a quasi-minimal structure.

Definition 2.4. A first order structure M in a countable language is said to be *quasi-minimal* if it is uncountable and every M-definable subset of M is either countable or co-countable (its complement in M is countable).

Naturally, we say that a definable subset is *small* if it is countable, and that it is *large* if it is co-countable. Accordingly, we say that a formula is small (large) if it defines small (large) subset.

For a quasi-minimal structure M, denote by p the set of all large formulas with parameters from M; then $p\in \mathrm{S}_1(M)$. Indeed, p is closed for finite conjunctions, since the intersection of finitely many co-countable subsets is co-countable. For completeness note that by quasi-minimality every formula defines either a countable or a co-countable subset of M, hence either a formula or its negation belongs to p. We say that p is the *generic type* of M.

Remark 2.3. Let M be a quasi-minimal structure and let p be its generic type.

(1) The set $\mathrm{cl}_p(X)$ is countable for every at most countable subset $X\subseteq M$. Indeed, there are only countably many formulas with parameters from X, hence $\mathrm{cl}_p(X)$ is a countable union of countable sets.

(2) Since $\mathrm{cl}_p(X)$ is defined as the union of all countable X-definable subsets, it follows that $f(\mathrm{cl}_p(X))=\mathrm{cl}_p(f(X))$ for every automorphism $f\in \mathrm{Aut}(M)$.

(3) The operation cl_p doesn't need to be a closure operation on M. An example can be found in [**4**].

Lemma 2.1. *Assume that M is a quasi-minimal structure whose generic type p is definable over countable $A\subseteq M$. Then cl_p^A is a closure operation on M.*

Proof. As we remarked earlier, cl_p^A satisfies Monotonicity and Finite character. It remains to prove Transitivity. We have to prove that $\mathrm{cl}_p^A(\mathrm{cl}_p^A(X))\subseteq \mathrm{cl}_p^A(X)$, since the other inclusion holds by Monotonicity.

Let $c \in \mathrm{cl}_p^A(\mathrm{cl}_p^A(X))$. Then $M \models \phi(c, \bar{b})$, where $\phi(x, \bar{y})$ is a formula with parameters from A, $\bar{b} = b_1 b_2 \dots b_n \in \mathrm{cl}_p^A(X)$ and $\phi(x, \bar{b}) \notin p$. For each i, $1 \leqslant i \leqslant n$, $b_i \in \mathrm{cl}_p^A(X)$ implies that $M \models \psi_i(b_i, \bar{a})$, where $\psi_i(y_i, \bar{z})$ is a formula with parameters from A, $\bar{a} \in X$ and $\psi_i(y_i, \bar{a}) \notin p$. By definability of p, $\phi(x, \bar{b}) \notin p$ and $\psi_i(y_i, \bar{a}) \notin p$ imply $M \models \neg\, \mathrm{d}_p\, \phi(\bar{b})$ and $M \models \neg\, \mathrm{d}_p\, \psi_i(\bar{a})$, where $\mathrm{d}_p\phi(\bar{y})$ and $\mathrm{d}_p\psi_i(\bar{z})$ are the definitions over A of formulas $\phi(x, \bar{y})$ and $\psi_i(y_i, \bar{z})$. So c satisfies the following formula with parameters from $A \cup X$

$$\theta(x, \bar{a}) := \exists \bar{y}\, (\phi(x, \bar{y}) \wedge \neg \mathrm{d}_p \phi(\bar{y}) \wedge \bigwedge_{i=1}^{n} \psi_i(y_i, \bar{a})).$$

It suffices to prove that $\theta(x, \bar{a}) \notin p$; $c \in \mathrm{cl}_p^A(X)$ follows. Note that:

$$\theta(M, \bar{a}) = \bigcup \{\phi(M, \bar{d}) \mid \bar{d} \in M,\ M \models \neg \mathrm{d}_p \phi(\bar{d}),\ M \models \psi_i(d_i, \bar{a}),\ \text{for all } 1 \leqslant i \leqslant n\}.$$

Since $M \models \neg\, \mathrm{d}_p\, \psi_i(\bar{a})$, we conclude that there are only countably many choices for $\bar{d}$ such that $M \models \psi_i(d_i, \bar{a})$ for all $1 \leqslant i \leqslant n$. Therefore, the former set is countable. For any $\bar{d} \in M$ such that $M \models \neg \mathrm{d}_p \phi(\bar{d})$ and $M \models \psi_i(d_i, \bar{a})$, for all $1 \leqslant i \leqslant n$, $M \models \neg \mathrm{d}_p \phi(\bar{d})$ implies that $\phi(x, \bar{d}) \notin p$, hence $\phi(M, \bar{d})$ is countable. So, $\theta(M, \bar{a})$ is a countable union of countable sets, hence is countable. Therefore, $\theta(x, \bar{a}) \notin p$. $\square$

3. Strong regularity and quasi-minimality

Throughout this section we assume that T is a complete theory in a countable language, with $\mathfrak{C}$ being its monster model. In this section we prove Theorem 1.2. Part (1) is proven in Proposition 3.1 and part (2) in Proposition 3.2. We need few lemmas.

Lemma 3.1. *Assume that M is a countable model of T and $\bar{a} \in \mathfrak{C}$. There exists a countable model N of T such that:*

- *$M \subseteq N$, $\bar{a} \in N$;*
- *for every $b \in N \smallsetminus M$, $\mathrm{tp}(b/M\bar{a})$ is not finitely satisfiable in M.*

Proof. Consider the following set of formulas with parameters from $M\bar{a}$:

$$\Sigma(x) = \{\phi(x) \mid \phi(x) \text{ is } \mathcal{L}_{M\bar{a}}\text{–formula and } \phi(M) = M\} \cup \{x \neq m \mid m \in M\}.$$

The set $\Sigma(x)$ is obviously (incomplete) type over $M\bar{a}$.

We *claim* that the following conditions are equivalent for every $q \in \mathrm{S}_1(M\bar{a})$:

(1) q is finitely satisfiable in M and q is not realized in M;

(2) $\Sigma(x) \subseteq q(x)$.

(1)⇒(2): Assume that q is finitely satisfiable in M, but not realized in M. Obviously $\{x \neq m \mid m \in M\} \subseteq q(x)$, since q is complete and not realized in M. Assume that $\phi(x) \in \Sigma(x)$ is such that $\phi(M) = M$. Since q is complete, either $\phi(x)$ or $\neg\phi(x)$ belongs to $q(x)$. But since $\phi(M) = M$, we get that $\neg\phi(x)$ has no solution in M, so $\neg\phi(x) \notin q(x)$ since q is finitely satisfiable in M. Therefore, $\phi(x) \in q(x)$.

(2)⇒(1): Assume now that $\Sigma(x) \subseteq q(x)$. Since $\{x \neq m \mid m \in M\} \subseteq q(x)$, q is not realized in M. If q is not finitely satisfiable in M, then there exists a formula

$\phi(x) \in q(x)$ that has no solution in M, i.e. $\phi(M) = \emptyset$. Then $\neg\phi(M) = M$, so $\neg\phi(x) \in \Sigma(x)$. Therefore, $\neg\phi(x) \in \Sigma(x) \smallsetminus q(x)$. A contradiction.

By the claim, if a countable model N of T, such that $M \subseteq N$ and $\bar{a} \in N$, omits $\Sigma(x)$, then for every $b \in N \smallsetminus M$, $\mathrm{tp}(b/M\bar{a})$ is not finitely satisfiable in M. (Note that $\mathrm{tp}(b/M\bar{a})$ is not realized in M, since this implies $b \in M$.) Thus, by Omitting Types Theorem, it suffices to show that for no consistent formula $\theta(x)$ with parameters from $M\bar{a}$, $\theta(x) \vdash \Sigma(x)$ holds.

Toward a contradiction, assume that $\theta(x)$ is a consistent formula with parameters from $M\bar{a}$ such that $\theta(x) \vdash \Sigma(x)$. The formula $\theta(x)$ has a solution in $\mathfrak{C}$, so choose $b \in \mathfrak{C}$ such that $\models \theta(b)$. Since $\theta(x) \vdash \Sigma(x)$, we get that $\Sigma(x) \subseteq \mathrm{tp}(b/M\bar{a})$, and by the claim above, $\mathrm{tp}(b/M\bar{a})$ is finitely satisfiable in M. In particular, $\theta(x) \in \mathrm{tp}(b/M\bar{a})$ has a solution in M. But this is not possible since $\theta(x) \vdash \{x \neq m \mid m \in M\}$. A contradiction. $\square$

LEMMA 3.2. *Assume that $\mathfrak{p} \in S_1(\mathfrak{C})$ is a strongly regular type, witnessed by A. Let M be a small model containing A and let $a \models \mathfrak{p}_{\restriction M}$. If $\mathrm{tp}(b/Ma)$ is not finitely satisfiable in M, then $b \models \mathfrak{p}_{\restriction M}$.*

PROOF. Assume that $\mathrm{tp}(b/M)$ is finitely satisfiable in M, then by Remark 1.1 it has a finitely satisfiable in M extension $q \in \mathrm{S}_1(Ma)$. Let $b' \models q$. Then $\mathrm{tp}(b'/M) = \mathrm{tp}(b/M)$.

Toward a contradiction, assume that $b \not\models \mathfrak{p}_{\restriction M}$. Then $b' \not\models \mathfrak{p}_{\restriction M}$ as well. By strong regularity of $\mathfrak{p}$, we get that $\mathfrak{p}_{\restriction M} \vdash \mathfrak{p}_{\restriction Mb}$ and $\mathfrak{p}_{\restriction M} \vdash \mathfrak{p}_{\restriction Mb'}$, so $a \models \mathfrak{p}_{\restriction Mb}$ and $a \models \mathfrak{p}_{\restriction Mb'}$, since $a \models \mathfrak{p}_{\restriction M}$. But this implies that $\mathrm{tp}(b/Ma) = \mathrm{tp}(b'/Ma)$, which is a contradiction since $\mathrm{tp}(b'/Ma)$ is finitely satisfiable in M, and $\mathrm{tp}(b/Ma)$ is not. $\square$

COROLLARY 3.1. *Assume that $\mathfrak{p} \in S_1(\mathfrak{C})$ is a strongly regular type, witnessed by A. Let M be a countable model of T containing A and $a \models \mathfrak{p}_{\restriction M}$. There exists a countable model N of T such that: $M \subseteq N$, $a \in N$ and $\mathfrak{p}_{\restriction M}(N) = N \smallsetminus M$.*

PROOF. By Lemma 3.1 there exists a countable model N of T such that $M \subseteq N$, $a \in N$ and for every $b \in N \smallsetminus M$, $\mathrm{tp}(b/Ma)$ is not finitely satisfiable in M. Then for every $b \in N \smallsetminus M$, $b \models \mathfrak{p}_{\restriction M}$ follows by Lemma 3.2. Therefore, $N \smallsetminus M \subseteq \mathfrak{p}_{\restriction M}(N)$. The other inclusion is obvious. $\square$

Now we are ready to prove Theorem 1.2 (1).

PROPOSITION 3.1. *Assume that $\mathfrak{p} \in S_1(\mathfrak{C})$ is a countably invariant, strongly regular type. Then there exists a quasi-minimal model N of T.*

PROOF. Let A be a countable set such that $\mathfrak{p}$ is A-invariant. By Remark 2.2 (3), A witnesses strong regularity of $\mathfrak{p}$.

We build a sequence $(M_\alpha, a_\alpha)_{\alpha<\omega_1}$ such that:

- M_α is a countable model of T;
- $M_\alpha \subseteq M_{\alpha+1}$, $a_\alpha \in M_{\alpha+1} \smallsetminus M_\alpha$ and $a_\alpha \models \mathfrak{p}_{\restriction M_\alpha}$;
- $\mathfrak{p}_{\restriction M_\alpha}(M_{\alpha+1}) = M_{\alpha+1} \smallsetminus M_\alpha$.

We proceed by induction. For $\alpha = 0$ take any countable model M_0 of T containing A, and take any $a_0 \models \mathfrak{p}_{\restriction M_0}$; note that $a_0 \notin M_0$.

If α is a limit ordinal, take $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ and take any $a_\alpha \models \mathfrak{p}_{\restriction M_\alpha}$; note that M_α is a countable model of T and $a_\alpha \notin M_\alpha$.

Given (M_α, a_α) such that M_α is a countable model of T containing A and $a_\alpha \models \mathfrak{p}_{\restriction M_\alpha}$, by Corollary 3.1 there is a countable model $M_{\alpha+1}$ such that $M_\alpha \subseteq M_{\alpha+1}$, $a_\alpha \in M_{\alpha+1}$ and $\mathfrak{p}_{\restriction M_\alpha}(M_{\alpha+1}) = M_{\alpha+1} \smallsetminus M_\alpha$. Take any $a_{\alpha+1} \models \mathfrak{p}_{\restriction M_{\alpha+1}}$; note that $a_{\alpha+1} \notin M_{\alpha+1}$.

It is clear that the defined sequence $(M_\alpha, a_\alpha)_{\alpha<\omega_1}$ satisfies the desired conditions.

Take $N = \bigcup_{\alpha<\omega_1} M_\alpha$. Obviously, N is an uncountable (since it contains $(a_\alpha)_{\alpha<\omega_1}$) model of T. Also note that $\mathfrak{p}_{\restriction M_\alpha}(N) = N \smallsetminus M_\alpha$, for every $\alpha < \omega_1$. Indeed, if $b \in N \smallsetminus M_\alpha$, then $b \in M_{\beta+1}$, for some $\beta \geqslant \alpha$. By construction, $b \models \mathfrak{p}_{\restriction M_\beta}$, hence $b \models \mathfrak{p}_{\restriction M_\alpha}$, since $M_\alpha \subseteq M_\beta$. Therefore $\mathfrak{p}_{\restriction M_\alpha}(N) \supseteq N \smallsetminus M_\alpha$. The other inclusion is obvious.

We claim that N is quasi-minimal. Let $\phi(x)$ be a formula with parameters from N. Then $\phi(x)$ is a formula with parameters from M_α, for some $\alpha < \omega_1$. If $\phi(x) \notin \mathfrak{p}_{\restriction M_\alpha}$, then no element of $\mathfrak{p}_{\restriction M_\alpha}(N) = N \smallsetminus M_\alpha$ satisfies $\phi(x)$. Thus $\phi(N) \subseteq M_\alpha$, so $\phi(N)$ is countable. If $\phi(x) \in \mathfrak{p}_{\restriction M_\alpha}$, then $\neg\phi(x) \notin \mathfrak{p}_{\restriction M_\alpha}$, hence $\neg\phi(N)$ is countable and $\phi(N)$ is co-countable. Therefore, N is indeed a quasi-minimal model of T. □

Corollary 3.2. *Assume that $\mathfrak{p} \in S_1(\mathfrak{C})$ is a definable, strongly regular type. Then there exists a quasi-minimal model N of T.*

Proof. Since $\mathfrak{p}$ is definable, by Remark 1.2 it is definable over some countable A. Now it is clear that $\mathfrak{p}$ is A-invariant: if $\phi(x,\bar{y})$ is an $\mathcal{L}$-formula, $\bar{a} \in \mathfrak{C}$ and $f \in \mathrm{Aut}_A(\mathfrak{C})$, then

$$\phi(x,\bar{a}) \in \mathfrak{p} \quad \text{iff} \quad \models \mathrm{d}_{\mathfrak{p}}\, \phi(\bar{a}) \quad \text{iff} \quad \models \mathrm{d}_{\mathfrak{p}}\, \phi(f(\bar{a})) \quad \text{iff} \quad \phi(x, f(\bar{a})) \in \mathfrak{p},$$

where $\mathrm{d}_{\mathfrak{p}}$ is the defining schema of $\mathfrak{p}$ over A.

Therefore, $\mathfrak{p}$ is countably invariant, hence by Proposition 3.1, T has a quasi-minimal model. □

In the following proposition we prove Theorem 1.2 (2).

Proposition 3.2. *Assume that M is a quasi-minimal model whose generic type p is definable. Then the global heir $\mathfrak{p}$ of p is strongly regular.*

Proof. Since p is definable, by Remark 1.2 it is definable over some countable $A \subseteq M$. Let $\mathfrak{C}$ be a monster model of a theory $T = \mathrm{Th}(M)$, and let d_p be the defining schema of the type p over A. The global heir of p is defined with:

$$\mathfrak{p} = \{\phi(x,\bar{a}) \mid \phi(x,\bar{y}) \text{ is } \mathcal{L}\text{-formula, } \bar{a} \in \mathfrak{C} \text{ and } \models \mathrm{d}_p\, \phi(\bar{a})\}.$$

We claim that A witnesses the regularity of $\mathfrak{p}$. First, note that $\mathfrak{p}$ is A-invariant. Assume that $\phi(x,\bar{a}) \in \mathfrak{p}$ and $f \in \mathrm{Aut}_A(\mathfrak{C})$. Then $\models \mathrm{d}_p\, \phi(\bar{a})$ holds, hence also $\models \mathrm{d}_p\, \phi(f(\bar{a}))$ holds, since $\mathrm{d}_p\, \phi(\bar{y})$ is a formula with parameters from A. Therefore $\phi(x, f(\bar{a})) \in \mathfrak{p}$. It remains to prove that $\mathrm{cl}^A_{\mathfrak{p}}$ is a closure operation on $\mathfrak{C}$.

Note that the corresponding $\mathrm{cl}_{\mathfrak{p}}^A$ operation on $\mathfrak{C}$ is defined by

$$\begin{aligned}\mathrm{cl}_{\mathfrak{p}}^A(X) &= \bigcup\{\phi(\mathfrak{C},\bar{a}) \mid \phi(x,\bar{y}) \text{ is } \mathcal{L}\text{-formula, } \bar{a}\in A\cup X \text{ and } \phi(x,\bar{a})\notin\mathfrak{p}\}\\ &= \bigcup\{\phi(\mathfrak{C},\bar{a}) \mid \phi(x,\bar{y}) \text{ is } \mathcal{L}\text{-formula, } \bar{a}\in A\cup X \text{ and } \models\neg\,\mathrm{d}_p\,\phi(\bar{a})\}.\end{aligned}$$

Assume on the contrary that $\mathrm{cl}_{\mathfrak{p}}^A$ is not a closure operation on $\mathfrak{C}$. Then Transitivity condition fails, so we can choose $X\subseteq\mathfrak{C}$ and $c\in\mathfrak{C}$ such that $c\in\mathrm{cl}_{\mathfrak{p}}^A(\mathrm{cl}_{\mathfrak{p}}^A(X))\smallsetminus\mathrm{cl}_{\mathfrak{p}}^A(X)$. Since $c\in\mathrm{cl}_{\mathfrak{p}}^A(\mathrm{cl}_{\mathfrak{p}}^A(X))$, there exists a formula $\phi(x,\bar{y})$ with parameters from A and $\bar{b}=b_1b_2\dots b_n\in\mathrm{cl}_{\mathfrak{p}}^A(X)$ such that $\phi(x,\bar{b})\notin\mathfrak{p}$ and $\models\phi(c,\bar{b})$. For each i, $1\leqslant i\leqslant n$, $b_i\in\mathrm{cl}_{\mathfrak{p}}^A(X)$ implies that there exists a formula $\psi_i(y_i,\bar{z})$ with parameters from A and $\bar{a}\in X$ such that $\psi_i(y_i,\bar{a})\notin\mathfrak{p}$ and $\models\psi_i(b_i,\bar{a})$. By the definition of $\mathfrak{p}$, $\phi(x,\bar{b})\notin\mathfrak{p}$ and $\psi_i(y_i,\bar{a})\notin\mathfrak{p}$ imply $\models\neg\,\mathrm{d}_p\,\phi(\bar{b})$ and $\models\neg\,\mathrm{d}_p\,\psi_i(\bar{a})$.

Therefore, $\bar{b}$ witnesses the existential quantifier that c satisfies

$$\theta(x,\bar{a}):=\exists\bar{y}\,[\phi(x,\bar{y})\wedge\neg\,\mathrm{d}_p\,\phi(\bar{y})\wedge\bigwedge_{i=1}^{n}\psi_i(y_i,\bar{a})\wedge\bigwedge_{i=1}^{n}\neg\,\mathrm{d}_p\,\psi_i(\bar{a})],$$

where $\theta(x,\bar{z})$ is a formula with parameters from A. Since $c\notin\mathrm{cl}_{\mathfrak{p}}^A(X)$ and $\bar{a}\in X$ we conclude that $\theta(x,\bar{a})\in\mathfrak{p}$, i.e., $\models\mathrm{d}_p\,\theta(\bar{a})$, and accordingly: $\models\exists\bar{z}\,\mathrm{d}_p\,\theta(\bar{z})$ holds.

Since $M\prec\mathfrak{C}$ and $\exists\bar{z}\,\mathrm{d}_p\,\theta(\bar{z})$ is a formula with parameters from $A\subseteq M$, $M\models\exists\bar{z}\,\mathrm{d}_p\,\theta(\bar{z})$ also holds, i.e., there exists $\bar{a}'\in M$ such that $M\models\mathrm{d}_p\,\theta(\bar{a}')$. This implies that $\theta(x,\bar{a}')\in p$, i.e. $\theta(M,\bar{a}')$ is co-countable. Since $\mathrm{cl}_p^A(\bar{a}')$ is countable, we can choose $c'\in\theta(M,\bar{a}')\smallsetminus\mathrm{cl}_p^A(\bar{a}')$. Now $M\models\theta(c',\bar{a}')$ implies that there exists $\bar{b}'\in M$ such that

$$M\models\phi(c',\bar{b}')\wedge\neg\,\mathrm{d}_p\,\phi(\bar{b}')\wedge\bigwedge_{i=1}^{n}\psi_i(b_i',\bar{a}')\wedge\bigwedge_{i=1}^{n}\neg\,\mathrm{d}_p\,\psi_i(\bar{a}').$$

The last two conjuncts imply that $\bar{b}'\in\mathrm{cl}_p^A(\bar{a}')$, and then the first two say that $c'\in\mathrm{cl}_p^A(\mathrm{cl}_p^A(\bar{a}'))$. Therefore $c'\in\mathrm{cl}_p^A(\mathrm{cl}_p^A(\bar{a}'))\smallsetminus\mathrm{cl}_p^A(\bar{a}')$. This is a contradiction, since by Lemma 2.1, cl_p^A is a closure operation on M. □

4. Around quasi-minimal group

In this section we prove Theorem 1.1.

According to the general definition of a quasi-minimal structure, a group G in a countable language is quasi-minimal if it is uncountable and every G-definable subset of G is either countable or co-countable (its complement in G is countable).

Let p be the generic type of G, i.e. the type consisting of all large formulas with parameters from G. It turns out that the type p is definable over $\emptyset$. This fact is well known (see [**2, 3, 4**]), but for the sake of completeness we formulate and prove it in the following lemma.

LEMMA 4.1. *The generic type p of a quasi-minimal group G is definable over $\emptyset$.*

PROOF. First we prove that for any co-countable subset $S\subseteq G$ there exists $a\in G$ such that $S\cup aS=G$. Since S is co-countable, then also S^{-1} is co-countable, as well as gS^{-1}, for each $g \in G$. Consider the family $\{gS^{-1} \mid g \notin S\}$. Since it is a countable family of co-countable subsets, it has co-countable intersection, so choose a in its intersection. But then for each $g \notin S$ we have $a \in gS^{-1}$, hence $g \in aS$. Therefore $G = S \cup aS$.

Also note that if $G = S \cup aS$ holds for a subset $S \subseteq G$, then S is uncountable (since G is uncountable). Therefore, if in addition S is definable, then by quasi-minimality S is co-countable.

Suppose now that $\phi(x, \bar{y})$ is a formula with no parameters. Note that for any $\bar{g} \in G$, $\phi(x, \bar{g}) \in p$ iff $\phi(G, \bar{g})$ is co-countable iff $G = \phi(G, \bar{g}) \cup a\,\phi(G, \bar{g})$ for some $a \in G$. Therefore, $\phi(x, \bar{g}) \in p$ iff $G \models \mathrm{d}_p\,\phi(\bar{g})$, where $\mathrm{d}_p\,\phi(\bar{y})$ is the formula

$$\exists z\, \forall x\, [\phi(x, \bar{y}) \vee \exists x'\, (\phi(x', \bar{y}) \wedge x = zx')].$$

Note that $\mathrm{d}_p\,\phi(\bar{y})$ has no parameters, hence it is the definition of $\phi(x, \bar{y})$ over $\emptyset$. □

By Lemma 2.1 we have an immediate corollary.

Corollary 4.1. *Let G be a quasi-minimal group whose generic type is p. Then $\mathrm{cl}_p = \mathrm{cl}_p^{\emptyset}$ is a closure operation on G.*

In order to prove Theorem 1.1 we first consider some properties that hold if a non-abelian quasi-minimal group exists. The same properties hold in the minimal case and some of them are proven in [**6**]. The proof of Lemma 4.2 in the quasi-minimal case can also be found in [**2**]. In Lemma 4.2 and Corollaries 4.2 and 4.3 we deal under the following assumption.

Assumption 4.1. Assume that G is a non-abelian quasi-minimal group.

Lemma 4.2.
- (i) *Every definable proper subgroup of G is countable.*
- (ii) *The center $Z(G)$ is countable.*
- (iii) *For every $a \notin Z(G)$, the centralizer $C_G(a)$ is countable and the conjugation class a^G is co-countable.*
- (iv) *For every $a \notin Z(G)$, $G = Z(G) \cup a^G$ holds.*

Proof. Assume that H is a definable proper subgroup of G. Assume that H is not countable, hence by quasi-minimality it is co-countable. Choose $a \notin H$. Then H and aH are disjoint co-countable sets of G. This contradiction proves (i). For (ii) note that $Z(G)$ is definable proper subgroup of G, since G is non-abelian. Therefore, (ii) follows from (i). Similarly, for (iii) note that $C_G(a)$ is definable proper subgroup of G, since $a \notin Z(G)$. Hence, the countability of $C_G(a)$ follows from (i) as well. Since $|a^G| = |G : C_G(a)|$ holds by the orbit-stabilizer theorem, we get that a^G is uncountable, hence by quasi-minimality it is co-countable.

(iv) Take $a, b \notin Z(G)$. By (iii) a^G and b^G are co-countable, hence they meet. Therefore $a^G = b^G$, and we get that $G = Z(G) \cup a^G$. □

Corollary 4.2. *For every $a, b \notin Z(G)$, the set $\{x \in G \mid a^x = b\}$ is countable.*

Proof. By Lemma 4.2 (iv) there exists $g \in G$ such that $b = a^g$. Then $b = a^x$ iff $a^g = a^x$ iff $a^{xg^{-1}} = a$ iff $xg^{-1} \in C_G(a)$ iff $x \in C_G(a)g$. Therefore, there exists a bijection between $\{x \in G \mid a^x = b\}$ and $C_G(a)$, which is countable by Lemma 4.2 (iii). $\square$

If in addition we assume that G is a pure group, then we get the following corollary.

Corollary 4.3. *We have* $\mathrm{cl}_p(\emptyset) = Z(G)$.

Proof. Obviously, $Z(G) \subseteq \mathrm{cl}_p(\emptyset)$, since by Lemma 4.2(ii), $Z(G)$ is countable and also $\emptyset$-definable. Toward a contradiction, assume that $Z(G) \subsetneq \mathrm{cl}_p(\emptyset)$ and choose $a \in \mathrm{cl}_p(\emptyset) \setminus Z(G)$. Then there exists $\emptyset$-definable countable subset $D \subseteq G$ containing a. But then for every $g \in G$, $a^g \in D^g$, and $D^g = D$ since D is $\emptyset$-definable and conjugation is an automorphism of a pure group. Therefore, $a^G \subseteq D$, which is a contradiction since by Lemma 4.2(iii), a^G is co-countable. $\square$

Now, till the end of the section we deal under a stronger assumption.

Assumption 4.2. Assume that G is non-abelian quasi-minimal pure group with $\emptyset$-definable partial order $\leqslant$ with an uncountable chain.

Lemma 4.3. *Exactly one of the following holds:*
(1) $a < x$ defines a co-countable subset of G, for every $a \notin \mathrm{cl}_p(\emptyset)$;
(2) $x < a$ defines a co-countable subset of G, for every $a \notin \mathrm{cl}_p(\emptyset)$.

Proof. Since $\mathrm{cl}_p(\emptyset)$ is countable and there exists an uncountable chain $\mathcal{C}$, we can choose an element $a_0 \in \mathcal{C} \setminus \mathrm{cl}_p(\emptyset)$. Then at least one of $a_0 < x$ and $x < a_0$ defines an uncountable subset, hence by quasi-minimality exactly one of $a_0 < x$ and $x < a_0$ defines a co-countable subset of G.

Assume that $a_0 < x$ defines a co-countable subset of G and take any $a \notin \mathrm{cl}_p(\emptyset)$. By Corollary 4.3, $\mathrm{cl}_p(\emptyset) = Z(G)$, so by Lemma 4.2(iv) we can choose an element $g \in G$ such that $a_0^g = a$. Since $\leqslant$ is 0-definable and the conjugation is an automorphism of a pure group, we get that $a < x$ defines a co-countable subset of G as well.

In a similar way we can prove that $x < a$ defines a co-countable subset of G, for every $a \notin \mathrm{cl}_p(\emptyset)$, assuming that $x < a_0$ defines a co-countable subset of G. $\square$

Lemma 4.4. *For every $a, b \notin \mathrm{cl}_p(\emptyset)$, $\mathrm{cl}_p(a) \subseteq \mathrm{cl}_p(b)$ or $\mathrm{cl}_p(b) \subseteq \mathrm{cl}_p(a)$.*

Proof. By Lemma 4.3, without loss of generality, assume that $c < x$ defines a co-countable subset of G, for every $c \notin \mathrm{cl}_p(\emptyset)$. Then $x < c$ and $c \not< x$ define countable subsets of G.

If $a < b$, since $x < b$ defines a countable set, we get that $a \in \mathrm{cl}_p(b)$, and $\mathrm{cl}_p(a) \subseteq \mathrm{cl}_p(b)$ follows by Transitivity. If $a \not< b$, since $a \not< x$ defines a countable set, we get $b \in \mathrm{cl}_p(a)$, and $\mathrm{cl}_p(b) \subseteq \mathrm{cl}_p(a)$ follows by Transitivity. $\square$

Proof of Theorem 1.1. Let (a, b) be a cl_p-free sequence over $\emptyset$. Since $b \notin \mathrm{cl}_p(a)$, by Lemma 4.4, $\mathrm{cl}_p(a) \subsetneq \mathrm{cl}_p(b)$ holds, so $\mathrm{cl}_p(a, b) = \mathrm{cl}_p(b)$. By Lemma 4.2(iv) choose $g \in G$ such that $b = a^g$. Since $g \in \{x \in G \mid a^x = b\}$, and since this set is countable by Corollary 4.2, we get that $g \in \mathrm{cl}_p(a, b) = \mathrm{cl}_p(b)$, hence $\mathrm{cl}_p(g) \subseteq \mathrm{cl}_p(b)$. On the other hand, obviously $b \in \mathrm{cl}_p(a, g)$, and since $b \notin \mathrm{cl}_p(a)$, we conclude that

$g \notin \mathrm{cl}_p(a)$. Hence, by Lemma 4.4, $\mathrm{cl}_p(a) \subseteq \mathrm{cl}_p(g)$, and finally $b \in \mathrm{cl}_p(a, g) = \mathrm{cl}_p(g)$. Therefore, $\mathrm{cl}_p(b) = \mathrm{cl}_p(g)$.

Since $\mathrm{cl}_p(a) \subsetneq \mathrm{cl}_p(b) = \mathrm{cl}_p(g)$ and conjugation is an automorphism of a pure group, by Remark 2.3 (2) we have that $\mathrm{cl}_p(a^g) \subsetneq \mathrm{cl}_p(g^g)$, i.e., $\mathrm{cl}_p(b) \subsetneq \mathrm{cl}_p(g)$. A contradiction. $\square$

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Faculty of Mathematics
University of Belgrade
Belgrade
Serbia
slavko@matf.bg.ac.rs

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