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ON SOME CLASS OF INTEGRAL OPERATORS RELATED TO THE BERGMAN PROJECTION

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Abstract. We consider the integral operator

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1 - z\bar{\xi})^\alpha}\, dA(\xi), \quad z \in D,$$

where $0 < \alpha < 2$ and D is the unit disc in the complex plane. and investigate boundedness of it on the space $L^p(D, d\lambda)$, $1 < p < \infty$, where $d\lambda$ is the Möbius invariant measure in D. We also consider the spectral properties of C_α when it acts on the Hilbert space $L^2(D, d\lambda)$, i.e., in the case $p = 2$, when C_α maps $L^2(D, d\lambda)$ into the Dirichlet space.

1. Introduction and notation

Throughout the paper let $D = \{z : |z| < 1\}$ be the open unit disc in complex plane $\mathbb{C}$ and let $dA(z) = \frac{1}{\pi}\, dx\, dy$, $z = x+iy$ stands for the normalized area measure in $\mathbb{C}$. For $1 < p < \infty$ we consider the Besov space B_p of D, $1 < p < \infty$, which is defined to be the space of all analytic functions f in D such that

$$\|f\|_{B_p} = \Big(\int_D |f'(z)|^p (1 - |z|^2)^p d\lambda(z) \Big)^{1/p} < \infty,$$

where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the Möbius invariant measure on D. It is known that $\|\cdot\|_{B_p}$ is complete seminorm on B_p. It should be pointed that B_p is a Banach space with norm $\|f\| = |f(0)| + \|f\|_{B_p}$. For $p = 2$ the space B_2 is the classical Dirichlet space, and appropriate semi-inner product is given by the formula

$$\langle f, g\rangle = \int_D f'(z)\, \overline{g'(z)}\, dA(z), \quad f, g \in B_2. \tag{1.1}$$

The weighted Bergman projection P_s, $-1 < s < \infty$ represents a central operator which appears in the research concerning the analytic function spaces. It is given by

$$P_s f(z) = (s+1) \int_D \frac{(1 - |\omega|^2)^s}{(1 - z\bar{\omega})^{2+s}} f(\omega)\, dA(\omega), \quad z \in D.$$

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Particulary, the ordinary Bergman projection $P = P_0$ arises as the orthogonal projection from $L^2(D; dA)$ onto an analytic function subspace. It connects B_p and $L^p(D, d\lambda)$. This relation is expressed in the next theorem.

THEOREM 1.1. *Suppose* $f \in H(D)$ *and* $1 \leqslant p \leqslant \infty$. *Then*

$$f \in B_p \Leftrightarrow f \in P(L^p(D, d\lambda)).$$

The inclusion operator V from B_p into $L^p(D, d\lambda)$ is given by

$$Vf(z) = 3(1 - |z|^2)^2 \int_D \frac{f(\xi)\, dA(\xi)}{(1 - z\bar{\xi})^4}, \quad z \in D.$$

More precisely we have the following lemma (see [**8**]).

LEMMA 1.1. *The operator* V *is an embedding from* B_p *into* $L^p(D, d\lambda)$ *for all* $1 < p < \infty$ *if on* $B_p = P(L^p(D, d\lambda))$ *is given the quotient norm.*

In this paper we consider the class of the operators

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1 - z\bar{\xi})^\alpha}\, d\xi, \quad z \in D,$$

where $0 < \alpha < 2$. For $\alpha = 2$ we have the Bergman projection. The norm of the Bergman projection from $L^p(D, d\lambda)$ onto B_p was estimated in [**7**]. In Theorem 1.3 we prove that C_α is a bounded mapping from $L^p(D, d\lambda)$ into B_p for all $0 < \alpha < 2$ and $1 < p < \infty$. We investigate in the next section some of its spectral properties in the context of the Lebesgue space $L^2(D, d\lambda)$ and the Besov space B_2.

By boundedness of an operator $T : L^p(D, d\lambda) \to B_p$ we mean that there exists a constant $C > 0$ such that $\|Tf\|_{B_p} \leqslant C\|f\|_{L^p(D,d\lambda)}$.

In this section we observe boundedness of C_α defined on $L^p(D, d\lambda)$. We firstly state a technical lemma and a proposition (the Schur test).

LEMMA 1.2. *Suppose* $z \in D$, c *is real,* $t > -1$, *and*

$$I_{c,t}(z) = \int_D \frac{(1 - |\omega|^2)^t}{|1 - z\bar{\omega}|^{2+t+c}}\, dA(\omega).$$

Then we have

(a) *If* $c < 0$, *then* $I_{c,t}(z)$ *is bounded in* z.
(b) *If* $c > 0$, *then* $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$, $|z| \to 1^-$.
(c) *If* $c = 0$, *then* $I_{0,t}(z) \sim \log \frac{1}{1-|z|^2}$, $|z| \to 1^-$.

PROPOSITION 1.1. *Suppose* K *is a nonnegative measurable function on* $X \times X$, *where* (X, μ) *is a measure space. Let* T *be an integral operator induced by* K, *that is* $Tf(x) = \int_X K(x, y)\, f(y)\, d\mu(y)$, *where* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If there exist constants* $C_1, C_2 > 0$ *and a positive measurable function* h *on* X *such that*

$$\int_X K(x, y)\, h(y)^q d\mu(y) \leqslant C_1 h(x)^q \quad \text{for } \mu\text{-almost every } x \in X,$$

$$\int_X K(x, y)\, h(x)^p d\mu(x) \leqslant C_2 h(y)^p \quad \text{for } \mu\text{-almost every } y \in X,$$

then T *is bounded on* $L^p(X, d\mu)$ *with the norm less than or equal to* $C_1^{1/q} C_2^{1/p}$.

In the proof of Theorem 1.3 we will use the Gauss hypergeometric functions and its basic properties. Following [**1**] we recall some facts for the sake of easy reference.

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n \quad \text{for } |z|<1, $$

and by continuation elsewhere. Here $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the shifted factorial, where a is any complex number.

The identity

$$ {}_2F_1(a,b;c;z) = (1-z^2)^{c-a-b}{}_2F_1(c-a,c-b;c;z) \tag{1.2} $$

is known as Euler identity. The following properties of hypergeometric function are also going to be of interest

$$ \begin{gathered} \frac{\partial}{\partial x}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z), \\ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \mathrm{Re}(c-a-b)>0. \end{gathered} \tag{1.3} $$

It is known that ${}_2F_1(a,b;c;z)$ diverges in general for $z=1$ if $\mathrm{Re}(c-a-b) \leqslant 0$. The next theorem, due to Gauss, describes the asymptotic behaviour of the hypergeometric functions as $z \to 1^-$.

Theorem 1.2. *If* $\mathrm{Re}(c-a-b)<0$, *then*

$$ \lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. $$

For $c=a+b$ *we have*

$$ \lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;z)}{\log\left(\frac{1}{1-z}\right)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}. $$

Theorem 1.3. *For* $0<\alpha<2$, C_α *is a bounded mapping from* $L^p(D,d\lambda)$ *into* B_p $(1<p<\infty)$. *The norm may be estimated by* $\|C_\alpha\|_{L^p(D,d\lambda)\to B_p} < \alpha C_1^{1/q}C_2^{1/p}$. *Here*

$$ C_1 = \begin{cases} \frac{p\Gamma(2+\frac{1}{p})\Gamma(\alpha-1-\frac{1}{p})}{(p+1)\Gamma^2(\frac{\alpha+1}{2})}, & \alpha>1+\frac{1}{p}; \\ \frac{pq\Gamma(1+\frac{1}{p})}{e\pi(p+1)}, & \alpha=1+\frac{1}{p}; \\ \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, & \alpha<1+\frac{1}{p}, \end{cases} \quad \textit{and} \quad C_2 = \begin{cases} \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}, & \alpha>\frac{1}{q}; \\ \frac{pq\Gamma(1+\frac{1}{q})}{\pi(p+1)e}, & \alpha=\frac{1}{q}; \\ \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{1}{q}-\alpha)}{\Gamma^2(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2})}, & \alpha<\frac{1}{q}, \end{cases} $$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. First of all, it is easy to see that Cf is an analytic function for every $f \in L^p(D,d\lambda)$. Further, for $f \in L^p(D,d\lambda)$ we have

$$ (Cf)'(z) = \alpha \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}} f(\xi)\,dA(\xi), $$

$$\|Cf\|_{B_p}^p = \alpha^p \int_D \left|(1-|z|^2)^{1-2/p} \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}} f(\xi)\, dA(\xi)\right|^p dA(z).$$

Thus

$$\begin{aligned}\|Cf\|_{B_p}^p &= \alpha^p \int_D \left|(1-|z|^2)^{1-2/p} \int_D \frac{\bar{\xi}}{(1-z\bar{\xi})^{\alpha+1}} f(\xi)\, dA(\xi)\right|^p dA(z)\\ &= \alpha^p \int_D \left|(1-|z|^2) \int_D \frac{\bar{\xi}(1-|\xi|^2)^2}{(1-z\bar{\xi})^{\alpha+1}} f(\xi)\, d\lambda(\xi)\right|^p d\lambda(z)\end{aligned}$$

Therefore, we should consider the operator

$$Tf(z) = (1-|z|^2) \int_D \frac{f(\xi)(1-|\xi|^2)^2}{|1-z\bar{\xi}|^{\alpha+1}}\, d\lambda(\xi)$$

on $L^p(D, d\lambda)$. We will prove that it is bounded there and we will estimate its norm. Using the obvious relation $\|C_\alpha\|_{L^p(D,d\lambda)\to B_p} \leqslant \alpha\|T\|_{L^p(D,d\lambda)\to L^p(D,d\lambda)}$ we can estimate the norm of C_α.

We use Proposition 1.1 and test function $h(z) = (1-|z|^2)^{1/pq}$ for the kernel $K(z,\xi) = \frac{(1-|z|^2)(1-|\xi|^2)^2}{|1-z\bar{\xi}|^{\alpha+1}}$. We have to prove existence of the constants C_1, C_2 such that

$$\begin{aligned}\int_D K(z,\xi)\, h^q(\xi)\, d\lambda(\xi) &\leqslant C_1 h^q(z), \quad z \in D,\\ \int_D K(z,\xi)\, h^p(z)\, d\lambda(z) &\leqslant C_2 h^p(\xi), \quad \xi \in D,\end{aligned}$$

which is equivalent with

$$\begin{aligned}(1-|z|^2)^{1/q} \int_D \frac{(1-|\xi|^2)^{1/p}}{|1-z\bar{\xi}|^{\alpha+1}}\, dA(\xi) &\leqslant C_1, \quad z \in D,\\ (1-|\xi|^2)^{1+1/p} \int_D \frac{(1-|z|^2)^{1/q-1}}{|1-z\bar{\xi}|^{\alpha+1}}\, dA(z) &\leqslant C_2, \quad \xi \in D.\end{aligned} \tag{1.4}$$

From Lemma 1.2 we can easily check that both functions on the left-hand side in (1.4) are bounded and consequently relations (1.4) are true for some constants C_1 and C_2. In the sequel we will determine the upper bounds for the constants C_1 and C_2.

By using the uniform convergence and orthogonality we have

$$\begin{aligned}(1-|z|^2)^{1/q} \int_D \frac{(1-|\xi|^2)^{1/p}}{|1-z\bar{\xi}|^{\alpha+1}} dA(\xi) &= (1-|z|^2)^{1/q} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{p}+1)\Gamma^2(\frac{\alpha+1}{2}+n)}{\Gamma(n+\frac{1}{p}+2)\Gamma^2(\frac{\alpha+1}{2})n!}|z|^{2n}\\ &= \frac{p}{1+p}(1-|z|^2)^{1/q}{}_2F_1\Big(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}; 2+\frac{1}{p}, |z|^2\Big).\end{aligned}$$

In a similar way we obtain that

$$(1-|\xi|^2)^{1+1/p} \int_D \frac{(1-|z|^2)^{1/q-1}}{|1-z\bar{\xi}|^{\alpha+1}} dA(z)$$

$$= q(1-|\xi|^2)^{1+1/p}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Let us denote

$$C_1 = \frac{p}{1+p}\sup_{|z|<1}(1-|z|^2)^{1/q}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};2+\frac{1}{p},|z|^2\Big),$$
$$C_2 = q\sup_{|\xi|<1}(1-|\xi|^2)^{1+1/p}{}_2F_1\Big(\frac{\alpha+1}{2},\frac{\alpha+1}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Then the Schur test implies $\|C_\alpha\|_{L^p(d\lambda)\to B_p} \leqslant \alpha C_1^{1/q}C_2^{1/p}$. By using the Euler transformation (1.2) for the hypergeometric functions, we obtain

$$C_1 = \frac{p}{1+p}\sup_{|z|<1}(1-|z|^2)^{2-\alpha}{}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},|z|^2\Big),$$
$$C_2 = q\sup_{|\xi|<1}(1-|\xi|^2)^{2-\alpha}{}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},|\xi|^2\Big).$$

Both functions

$${}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},|z|^2\Big),$$
$${}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},|\xi|^2\Big)$$

are increasing in $|z|$ and $|\xi|$, respectively (see (1.3)).

We distinguish the following five cases:

1) If $\alpha > 1+\frac{1}{p}$, then

$$C_1 < \frac{p}{p+1}{}_2F_1\Big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},1\Big) = \frac{p\Gamma(2+\frac{1}{p})\Gamma(\alpha-1-\frac{1}{p})}{(p+1)\Gamma^2(\frac{\alpha+1}{2})},$$
$$C_2 < q\,{}_2F_1\Big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},1\Big) = \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}.$$

2) If $\alpha < \frac{1}{q}$, then according to Theorem 1.2 we have

$$C_1 \leqslant \frac{p}{p+1}\limsup_{|z|\to 1^-}(1-|z|^2)^{2-\alpha}\frac{{}_2F_1\big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2},\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2};2+\frac{1}{p},1\big)}{(1-|z|^2)^{\alpha-1-\frac{1}{p}}}$$
$$< \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})},$$
$$C_2 \leqslant q\limsup_{|\xi|\to 1^-}(1-|\xi|^2)^{2-\alpha}\frac{{}_2F_1\big(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2},\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2};1+\frac{1}{q},1\big)}{(1-|\xi|^2)^{\alpha-1/q}}$$
$$< \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{1}{q}-\alpha)}{\Gamma^2(\frac{1}{2}+\frac{1}{q}-\frac{\alpha}{2})}.$$

3) If $\alpha = 1 + \frac{1}{p}$, then

$$C_1 \leqslant \frac{p}{p+1} \limsup_{|z|\to 1^-} (1-|z|^2)^{1/q} \log \frac{1}{1-|z|^2} \frac{{}_2F_1\big(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2}, \frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2}; 2+\frac{1}{p}, 1\big)}{\log \frac{1}{1-|z|^2}}$$

$$< \frac{pq}{e(p+1)} \frac{\Gamma(3+\frac{2}{p}-\alpha)}{\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})} = \frac{pq\Gamma(1+\frac{1}{p})}{e\pi(p+1)}, \qquad C_2 < \frac{q\Gamma(1+\frac{1}{q})\Gamma(\frac{2}{p})}{\Gamma^2(\frac{\alpha+1}{2})},$$

since the maximal value of the function $\phi(x) = (1-x)^{1/q} \log \frac{1}{1-x}$, $x \in (0,1)$ is $\frac{q}{e}$.

4) If $\alpha = \frac{1}{q}$, then

$$C_1 < \frac{p\Gamma(2+\frac{1}{p})\Gamma(\frac{2}{p})}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, \quad C_2 < \frac{pq\Gamma(1+\frac{1}{q})}{\pi(p+1)e}.$$

5) If $\frac{1}{q} < \alpha < 1 + \frac{1}{p}$, then

$$C_1 < \frac{p\Gamma(2+\frac{1}{p})\Gamma(1+\frac{1}{p}-\alpha)}{(p+1)\Gamma^2(\frac{3}{2}+\frac{1}{p}-\frac{\alpha}{2})}, \quad C_2 < \frac{q\Gamma(1+\frac{1}{q})\Gamma(\alpha-\frac{1}{q})}{\Gamma^2(\frac{\alpha+1}{2})}. \qquad \square$$

2. Hilbert case and spectral properties

Folowing [**6**] let us recall some basic facts from spectral-operator theory. Let us firstly recall that for the bounded measurable function $A(z,\xi)$ the operator

$$Af(z) = \int_D \frac{A(x,\xi)f(\xi)}{|z-\xi|^\alpha}\, dA(\xi), \quad z \in D$$

is compact on $L^2(D, dA)$, where $0 < \alpha < 2$.

For a compact operator T defined on a separable Hilbert space H, let $s_n(T)$, $n \geqslant 1$ denote the eigenvalues of the operator $(T^*T)^{1/2}$ arranged in nondecreasing order [**4**]. In general, if T is a compact operator on a separable Hilbert space H, then there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in H such that

$$Tx = \sum_n \lambda_n \langle x, e_n\rangle \sigma_n, \quad x \in H,$$

where λ_n is n-th singular value of T.

For $0 < p < \infty$, we define the Schattene p-class of H denoted by $S_p(H)$, or simply S_p, to be the space of all compact operators T on H with singular value sequence $\{\lambda_n\}$ belonging to l^p (p-summable sequence space). The Schattene class S_p is a Banach space for the range $1 \leqslant p < \infty$, and appropriate norm of the operator $T \in S_p$ is given by $\|T\|_p = \big(\sum_n |\lambda_n|^p\big)^{1/p}$.

Theorem 2.1. *Let A be a compact operator. Then A has the norm convergent expansion*

$$A = \sum_{n=1}^{N} \mu_n(A) \langle \phi_n, \cdot\rangle \psi_n \tag{2.1}$$

(*where N is a finite non-negative integer or infinity*), *each* $\mu_n(A) > 0$, $\mu_1(A) \geqslant \mu_2(A) \geqslant \cdots$, *and* (ϕ_n) *and* (ψ_n) *are (not necessarily complete) orthonomal sets. Moreover,* $\mu_n(A)$ *are uniquely determined and* ϕ*'s and* ψ*'s are essentially uniquely determined.*

Here $\mu_n(A)$ are singular values of A and formula (2.1) is called canonical expansion for A.

Now we state a known result related to minimax properties of eigenvalues for compact nonnegative operators [**4**].

THEOREM 2.2. *Let* A $(\neq 0)$ *be a nonnegative compact operator and let* φ_j $(j = 1, 2, \ldots)$ *be an orthonomal system of its eigenvalues which is complete in the range of* A, *so that* $A\varphi_j = \lambda_j(A)\varphi_j$, $j = 1, 2, \ldots$ *where* $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots$. *Then its eigenvalues have the following minimax properties*

$$\lambda_1(A) = \max_{\varphi\in H} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle} \tag{2.2}$$

where the maximum in (2.2) *is attained only for those eigenvalues of* A *that correspond to* $\lambda_1(A)$.

$$\lambda_{j+1}(A) = \min_{L\in N_j} \max_{\varphi\in L^T} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}, \quad j = 1, 2, \ldots \tag{2.3}$$

where N_j *is the set of all* j*-dimensional lineals of* H, *and the minimum in* (2.3) *is attained when* L *coincides with the linear hull* L_j *of the eigenvectors* $\varphi_1, \varphi_2, \ldots, \varphi_j, \ldots$ *so that*

$$\lambda_{j+1}(A) = \max_{\varphi\in L_j^T} \frac{\langle A\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}.$$

We note that

$$s_1(A) = \|A\|. \tag{2.4}$$

Dostanić [**3**] investigated the singular values of the operator $S : L^2(D) \to L^2(D)$ defined by

$$Sf(z) = \frac{1}{\pi}\int_D \frac{\bar{\xi}}{1 - z\bar{\xi}} m(\xi)\, f(\xi)\, dA(\xi),$$

where $m \in C(\bar{D})$. He obtained that $s_n(S) \sim \frac{1}{2n\pi}\int_0^{2\pi} |m(e^{i\theta})|\, d\theta$.

The next theorem is our second main result and is related to finding singular numbers of the operator VC_α.

THEOREM 2.3. *The operator* $VC_\alpha : L^2(D, d\lambda) \to L^2(D, d\lambda)$ *is compact for* $0 < \alpha < 2$. *The following asymptotic formula holds*

$$s_n(C_\alpha|_{L^2(D,d\lambda)}) = s_n(VC_\alpha) = \frac{\Gamma(n+\alpha)}{12\Gamma(\alpha)\,\Gamma(n+2)} \sim \frac{1}{n^{2-\alpha}}, \quad n \to \infty.$$

In the proof we will need the following inequalities for Γ function (see [**2**]).

PROPOSITION 2.1. *Let* m, p *and* k *be real numbers with* $m, p > 0$ *and* $p > k > -m$. *If* $k(p - m - k) \geqslant 0$ $(\leqslant 0)$, *then* $\Gamma(p)\Gamma(m) \geqslant$ $(\leqslant)\Gamma(p-k)\,\Gamma(m+k)$.

Proof. From Theorem 1.3 and properties of the operator V we have that VC_α maps $L^2(D,d\lambda)$ into itself. Let $H(\cdot,\cdot)$ be appropriate kernel of VC_α, i.e.,

$$VC_\alpha f(z) = \int_D H(z,\xi)\, f(\xi)\, dA(\xi).$$

The the kernel $H(\cdot,\cdot)$ is given by

$$H(z,\xi) = (1-|z|^2)^2 \int_D \frac{dA(t)}{(1-z\bar{t})^4(1-\bar{\xi}t)^\alpha}.$$

On the other hand

$$\begin{aligned} H(z,\xi) &= (1-|z|^2)^2 \int_D \frac{dA(t)}{(1-z\bar{t})^4(1-\bar{\xi}t)^\alpha} \\ &= (1-|z|^2)^2 \int_D \sum_{n=0}^{\infty} \frac{\Gamma(n+4)}{\Gamma(4)n!}(z\bar{t})^n \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!}(\bar{\xi}t)^k dA(t) \\ &= (1-|z|^2)^2 \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!} z^n\bar{\xi}^n. \end{aligned}$$

So,

$$\begin{aligned} VC_\alpha f(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!}(1-|z|^2)^2 z^n \int_D f(\xi)\bar{\xi}^n dA(\xi) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+4)\Gamma(n+\alpha)}{12\Gamma(\alpha)n!(n+1)!}(1-|z|^2)^2 z^n \int_D f(\xi)(1-|\xi|^2)^2\bar{\xi}^n d\lambda(\xi). \end{aligned}$$

Let us note that $e_n(z) = \sqrt{\frac{1}{2}(n+3)(n+2)(n+1)}(1-|z|^2)^2 z^n,\ n = 0,1,2,\dots$ represents orthonomal set in $L^2(D,d\lambda)$, which implies that

$$VC_\alpha f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)} e_n(z)\langle f, e_n\rangle, \quad z \in D.$$

Since, Stirling's formula implies $s_n(VC_\alpha) \sim \frac{1}{n^{2-\alpha}}$, as $n \to \infty$, and by using the fact

$$\|VC_\alpha f\|_2^2 = \sum_{n=0}^{\infty} \Big(\frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)}\Big)^2 |\langle f, e_n\rangle|^2, \quad f \in L^2(D,d\lambda),$$

we conclude that VC_α is compact for $0 < \alpha < 2$. The sequence $\big(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}\big)$ is decreasing in n, and this is a consequence of Proposition 2.1 with $p = \alpha+n+1$, $m = n+2$, $k = 1$. Then by Theorem 2.1 we get $s_n(VC_\alpha) = \frac{\Gamma(n+\alpha)}{6\Gamma(\alpha)\Gamma(n+2)}$. □

The next corollary is a direct consequence of the previous theorem.

Corollary 2.1. *For $0 < \alpha < 2$, $VC_\alpha \in S_p$ holds, where $p > \frac{1}{2-\alpha}$ and*

$$\|VC_\alpha\|_p = \frac{1}{12\Gamma(\alpha)}\bigg(\sum_{n=0}^{\infty}\Big(\frac{\Gamma(n+\alpha)}{\Gamma(n+2)}\Big)^p\bigg)^{1/p}.$$

According to (2.4) and Theorem 2.3 we easily obtain the following result.

THEOREM 2.4. *If* $VC_\alpha : L^2(D, d\lambda) \to L^2(D, d\lambda)$, $0 < \alpha < 2$, *then*

$$\|VC_\alpha\|_{L^2(d\lambda)\to L^2(d\lambda)} = \frac{\Gamma(1+\alpha)}{12\Gamma(\alpha)}.$$

In the next theorem we will consider the operator C_α defined on the Dirichlet space B_2.

THEOREM 2.5. *The operator* $C_\alpha : B_2 \to B_2$ *is compact for* $0 < \alpha < 2$ *and*

$$s_n(C|_{B_2}) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!} \sim \frac{1}{n^{2-\alpha}}, \quad n \to \infty.$$

PROOF. Let us note that the sequence $e_n(z) = \frac{z^n}{\sqrt{n}}$, $n \geqslant 1$ is orthonomal in B_2 according to the invariant integral pairing defined in (1.1). Then, for the function $f \in B_2, f(z) = \sum_{n=0}^{\infty} a_n z^n$, where we can add the condition $a_0 = 0$, we have

$$\langle f, e_n\rangle = \sqrt{n}\int_D \sum_{k=1}^{\infty} k a_k z^{k-1}\bar{z}^{n-1} dA(z) = \sqrt{n} a_n,$$

$$C_\alpha f(z) = \int_D \frac{f(\xi)}{(1-z\bar{\xi})^\alpha} dA(\xi) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)a_n}{\Gamma(\alpha)(n+1)!} z^n = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)e_n(z)}{\Gamma(\alpha)(n+1)!}\langle f, e_n\rangle.$$

By Stirling's formula we obtain $\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+2)} \sim \frac{1}{n^{2-\alpha}}$, $n \to \infty$. On the other hand,

$$\|C_\alpha f\|_{B_2}^2 = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+2)}\right)^2 |\langle f, e_n\rangle|^2$$

So, we conclude that C_α is a compact operator on B_2 and Theorem 2.1 implies $s_n(C_\alpha|_{B_2}) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}$. ☐

COROLLARY 2.2. *For the operator* $C_\alpha : B_2 \to B_2$, $(0 < \alpha < 2)$ *holds* $C_\alpha \in S_p$, *where* $p > \frac{1}{2-\alpha}$, *and*

$$\|C_\alpha\|_p = \frac{1}{\Gamma(\alpha)}\left(\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+\alpha)}{\Gamma(n+2)}\right)^p\right)^{1/p}.$$

A direct consequence of Theorem 2.5 is that $\|C_\alpha\|_{B_2\to B_2} = s_1(C_\alpha|_{B_2})$, i.e.,

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

However, we present here a direct way for finding the norm of C_α on B_2 without using singular numbers.

THEOREM 2.6. *The operator* $C_\alpha : B_2 \to B_2$, $0 < \alpha < 2$ *is bounded and*

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

Proof. Every function in B_2 can be approximated in norm by a sequence of polynomials. It is enough to find the norm of C_α on the set of polynomials $p_m(z) = \sum_{k=0}^{m} a_k z^k$, where m is a nonnegative integer. From the proof of Theorem 2.5 we get

$$C_\alpha f(z) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)e_n(z)}{\Gamma(\alpha)(n+1)!}\langle f, e_n\rangle.$$

Thus,

$$C_\alpha p_m(z) = \sum_{k=1}^{m} \frac{\Gamma(\alpha+n)a_n z^n}{\Gamma(\alpha)(n+1)!},$$

$$\|C_\alpha p_m\|_{B_2}^2 = \sum_{n=1}^{m} \frac{n|\Gamma(\alpha+n)|^2|a_n|^2}{|\Gamma(\alpha)(n+1)!|^2}, \quad \|p_m\|_{B_2}^2 = \sum_{n=1}^{m} n|a_n|^2.$$

We want to find the minimal constant A such that

$$\sum_{n=1}^{m} \frac{n|\Gamma(\alpha+n)|^2|a_n|^2}{|\Gamma(\alpha)(n+1)!|^2} \leqslant A^2 \sum_{n=1}^{m} n|a_n|^2, \tag{2.5}$$

for every polynomial p_m. In the above inequality we can treat the sequences $(n|a_n|^2)$ and $\big(\frac{|\Gamma(\alpha+n)|^2}{|\Gamma(\alpha)(n+1)!|^2}\big)$ as elements of l^1 and l^∞, respectively, so (2.5) can be rewritten as

$$\left\langle (n|a_n|^2), \left(\frac{|\Gamma(\alpha+n)|^2}{|\Gamma(\alpha)(n+1)!|^2}\right)\right\rangle \leqslant A^2\|(n|a_n|^2)\|_{l^1},$$

where $\langle(\xi_n),(\eta_n)\rangle = \sum_{n=1}^{\infty} \xi_n\eta_n$. By using the duality argument $(l^1)^* = l^\infty$ we obtain

$$A^2 = \sup_{n\geqslant 1} \frac{\Gamma^2(\alpha+n)}{\Gamma^2(\alpha)((n+1)!)^2} = \frac{\Gamma^2(\alpha+1)}{4\Gamma(\alpha)^2}, \tag{2.6}$$

i.e.,

$$\|C_\alpha\|_{B_2\to B_2} = \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha)}.$$

In (2.6) we used again the fact that the sequence $\big(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)(n+1)!}\big)$ is decreasing in n. $\square$

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