PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 98(112) (2015), 109–117 DOI: 10.2298/PIM150126018M

INDEPENDENCE COMPLEXES OF COMAXIMAL GRAPHS OF COMMUTATIVE RINGS WITH IDENTITY

Nela Milošević

ABSTRACT. We study topology of the independence complexes of comaximal (hyper)graphs of commutative rings with identity. We show that the independence complex of comaximal hypergraph is contractible or homotopy equivalent to a sphere, and that the independence complex of comaximal graph is almost always contractible.

1. Introduction

To any graph or hypergraph G we can associate its independence complex, which is the abstract simplicial complex formed from all the independent sets of G, and then study topology of its geometric realization. In this paper we investigate independence complexes of the comaximal graph and the comaximal hypergraph associated with a commutative ring with identity and determine their homotopy type. The set of vertices of the comaximal graph are the elements of the ring R, and two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. This graph is denoted by $\Gamma(R)$ and was first defined by Sharma and Bhatwadekar in [**11**], who proved that the chromatic number is finite if and only if the ring itself is finite. The graph was further investigated by Maimani et al. in [**6**] (who actually coined the name comaximal) and by Moconja and Petrović in [**8**] completing the structure of the comaximal graph in case the number of maximal ideals in R is finite. Similarly, we define comaximal hypergraph $H(R)$ as a generalization of the comaximal graph: vertices are the elements of the ring R and hyperedges (or simply edges) are nonempty subsets of R which are generating sets for the ring, that is $\{x_0, x_1, \dots, x_n\}$ is a hyperedge if and only if $Rx_0 + \cdots + Rx_n = R$.

In general, the idea of associating a combinatorial object to a commutative ring with identity has been of great interest to researches, so for other examples of associating a graph to a commutative ring the reader may wish to consult [**1, 2, 3,**

2010 *Mathematics Subject Classification:* 55U10; 57M15; 55P15; 05E45.
Key words and phrases: independence complex; graph; simplicial complex; homotopy type; commutative ring.
Communicated by Zoran Petrović.

10, 13]. For another example of associating simplicial complexes to commutative rings, we refer the reader to [**7**] where the authors associated order complex with a general commutative ring via chains of ideals following a suggestion from Vassiliev in [**12**], and determined homotopy type of that complex.

2. Preliminaries

In this section we collect some definitions mainly concerning simplicial complexes and independence complexes which are needed for the discussion in Section 3 and the exposition of our main results. For more thorough background we refer the reader to [**5, 9**].

Definition 2.1. An abstract simplicial complex K is a set A together with a collection K of finite nonempty subsets of A such that if $X \in K$ and $Y \subseteq X$, then $Y \in K$.

The element $v \in A$ such that $\{v\} \in K$ is called a *vertex* and the set of all vertices is denoted by $V(K)$. The elements of K are called *simplices* and usually denoted by σ. The dimension of a simplex σ is $|\sigma| - 1$, where $|\sigma|$ is the number of elements in the set σ. Any nonempty subset of a simplex is called a *face* of that simplex; faces are simplices themselves. Those simplices that are not faces of any other simplex in K are called *maximal.* For a simplex τ that is contained in only one maximal simplex σ in the complex, we say that τ is a *free face* of σ.

Definition 2.2. Let K and L be two abstract simplicial complexes. A simplicial map from K to L is a set map $f\colon V(K) \to V(L)$ such that if $\{x_0, \dots, x_n\}$ is a simplex of K, then $\{f(x_0), \dots, f(x_n)\}$ is a simplex of L. We simply write $f\colon K \to L$. Furthermore, simplicial map f is an isomorphism of abstract simplicial complexes if the induced map is a bijection and its inverse is a simplicial map as well.

Now we turn to the definitions regarding geometric realization.

Definition 2.3. A geometric n-simplex σ is the convex hull of the set A of $n+1$ affine independent points in $\mathbb{R}^N$ for some $N > n$. The standard n-simplex Δ^n is the convex hull of the set of the endpoints of the standard unit basis in $\mathbb{R}^{n+1}$.

There is an affine bijection between any geometric n-simplex and the standard geometric n-simplex.

Let us denote by $\mathbb{R}^{\oplus J}$ direct sum of $|J|$ (where J may be infinite; $|J|$ stands here for the cardinality of J) copies of $\mathbb{R}$ (so, it is a subset of $\mathbb{R}^J$ consisting of those $x = (x_j)_{j \in J} \in \mathbb{R}^J$ such that $x_j = 0$, for all but finitely many $j \in J$). A geometric simplicial complex K in $\mathbb{R}^{\oplus J}$ is a collection of geometric simplices in $\mathbb{R}^{\oplus J}$ such that every face of a simplex in K is a simplex of K and the intersection of any two simplices of K is a face of each of them.

For a geometric simplicial complex K, let $|K|$ denote the union of all simplices of K. We can define a topology on $|K|$: every simplex σ of K has the induced topology, and a subset F of $|K|$ is closed if and only if $F \cap \sigma$ is closed for every simplex $\sigma \in K$. Topological space $|K|$, called *geometric realization* of K, is determined up to a homeomorphism. When we refer to the topological properties of the abstract simplicial complex K, we are always actually referring to the topological space $|K|$.

Note that a simplicial map $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$ of topological spaces.

Definition 2.4. A simplicial complex K on a vertex set $V(K)$ is a *cone* with apex $v \in V(K)$ if every $\sigma \in K$ satisfies $\sigma \cup \{v\} \in K$.

If a simplicial complex K is a cone with apex v then its geometric realization can be contracted to a point.

We will also need the following lemma from [**9**].

Lemma 2.1. [**9**, Lemma 2.5] *If K is finite, then $|K|$ is compact. Conversely, if a subset A of $|K|$ is compact, then $A \subset |K_0|$ for some finite subcomplex K_0 of K.*

Note that this lemma is formulated for simplicial complex K that lies in $\mathbb{R}^N$ for some N, which puts limitation on the cardinality of K and on the dimension of the simplices of K. However, the author in [9] removes these restrictions later in the book, and asserts that this result holds in general, which we will use.

There is a large variety of complexes whose description is purely combinatorial, one of which are independence complexes. Given any (hyper)graph G, a set of vertices $S \subseteq V(G)$ is called independent if it contains no (hyper)edge (as its subset).

Definition 2.5. For an arbitrary graph (hyper)G, the independence complex of G, called Ind_G, is the abstract simplicial complex whose set of vertices is $V(G)$ and whose simplices are all finite independent sets of G.

In [**4**] the authors focus on the homotopic properties of a simplicial complex K in terms of its *nonfaces*, that is, the family $\{A \subseteq V(K)\colon A \notin K\}$. A minimal nonface of a simplicial complex is called a *circuit*.

Definition 2.6. A simplicial complex K on the vertex set V is *constrictive* if the complex K is the boundary of the simplex on the vertex set V or there is a vertex v in V belonging to at most one circuit with one of the following properties:

- v belongs to no circuit; or
- v belongs to a unique circuit $B \neq V$ and there is a vertex $u \notin B$ such that contracting the edge $\{u, v\}$ yields a constrictive complex, where an edge $\{u, v\}$ is contractible if and only if no circuit contains $\{u, v\}$.

A constrictive complex K is simple-homotopic to a single vertex or to the boundary complex of a simplex.

3. Independence complex of comaximal graph and comaximal hypergraph

Simplices in the independence complex of the comaximal hypergraph $\mathrm{Ind}_{H(R)}$ are independent sets of $H(R)$, that is nongenerating sets in R, $\{x_0, x_1, \ldots, x_n\} \in$

$\mathrm{Ind}_{H(R)}$ if and only if $Rx_0 + \cdots + Rx_n \neq R$. This naturally forms a simplicial complex since if an $(n+1)$-tuple does not generate the entire ring, then any smaller subset certainly does not generate the entire ring. Simplices in the the independence complex of comaximal graph $\mathrm{Ind}_{\Gamma(R)}$ are the independent sets of $\Gamma(R)$, that is simplices are the sets $\{x_0, x_1, \ldots, x_n\}$ such that $Rx_i + Rx_j \neq R$ for all the pairs of distinct indices $0 \leqslant i, j \leqslant n$.

Whenever $Rx_0 + \cdots + Rx_n \neq R$, then $\{x_i, x_j\}$ is not an edge in $\Gamma(R)$ for any $0 \leqslant i, j \leqslant n$ and $i \neq j$, so $\{x_0, x_1, \ldots, x_n\}$ is an independent set in $\Gamma(R)$ and hence a simplex in $\mathrm{Ind}_{\Gamma(R)}$. Therefore the independence complex of comaximal hypergraph is a subcomplex of the independence complex of comaximal graph. We will then first look at the independence complex of comaximal hypergraph and determine homotopy type of its geometric realization and then use these results and extend them to the investigation of independence complex of comaximal graph.

Note that in both complexes units are not adjacent to any vertex and elements of $J(R)$ are adjacent to every nonunit. Therefore, it will be more interesting to exclude units and elements of Jacobson radical from the set of vertices in these complexes; if we were to include elements of $J(R)$, then we would always have contractible complexes whenever $J(R) \neq 0$. Thus, we will be looking at the independence complexes of the subgraph of the comaximal hypergraph $H'(R)$ and the subgraph of the comaximal graph $\Gamma_2'(R)$ whose vertex set is $V(H'(R)) = V(\Gamma_2'(R)) = R \smallsetminus (U(R) \cup J(R))$ (we use same notation for subgraph $\Gamma_2'(R)$ as in [**6**]). We denote these complexes by $\mathrm{Ind}_{H'(R)}$ and $\mathrm{Ind}_{\Gamma_2'(R)}$ respectively, and determine their homotopy type.

Remark 3.1. Studying topology of independence complexes has been of interest to researchers and in [**4**] Ehrenborg and Hetyei studied topology of a certain class of independence complexes called *constrictive* complexes. They showed that constrictive complexes are either contractible or homotopy equivalent to a sphere. Conclusions concerning the independence complexes of the comaximal graph and comaximal hypergraph are similar to those regarding constrictive complexes, so it is natural to question whether the results can be derived by proving that the complexes we are studying are constrictive. It turns out that the independence complexes we are studying are not always constrictive, which we show by providing a counterexample. Let $R = \mathbb{Z}/p_1^2 p_2 \mathbb{Z}$ where p_1 and p_2 are distinct prime numbers, and consider the complexes $\mathrm{Ind}_{H'(R)}$ and $\mathrm{Ind}_{\Gamma_2'(R)}$ which are equal for this ring. Let a be any vertex in this complex. Since elements of $J(R)$ are not in the vertex set, a is divisible by either p_1 or p_2, but not by both. The geometric realization of this complex consists of two disconnected cones with vertices being elements of each maximal ideal that are not in $J(R)$. Clearly this complex is not a boundary of a simplex, and since each cone has at least two vertices, then every vertex of the complex belongs to more than one circuit. Therefore this complex is not constrictive.

We will determine homotopy type of the independence complex of comaximal hypergraph $\mathrm{Ind}_{H'(R)}$ and the independence complex of comaximal graph $\mathrm{Ind}_{\Gamma_2'(R)}$

using a direct approach by considering three cases: (1) local rings, (2) rings with infinitely many maximal ideals, and (3) semilocal rings.

Observe that in the case when the ring R is local with the maximal ideal M, the resulting complexes $\mathrm{Ind}_{H'(R)}$ and $\mathrm{Ind}_{\Gamma'_2(R)}$ are empty since every nonunit in R is contained in M (we have $J(R) = M$). Also, note that the equivalent definition of $\mathrm{Ind}_{H'(R)}$ is $\{x_0, x_1, \dots, x_n\} \in \mathrm{Ind}_{H'(R)}$ if and only if there exists a maximal ideal M in R such that $x_0, x_1, \dots, x_n \in M$.

3.1. Independence complexes for rings with infinitely many maximal ideals. Consider the case when the ring R has infinitely many maximal ideals. To determine the homotopy type of the independence complex $\mathrm{Ind}_{H'(R)}$ in this case, first we need the following lemma.

LEMMA 3.1. *Suppose that R is such that the set of maximal ideals $\mathrm{Max}(R)$ is infinite. If K_0 is a finite subcomplex of $\mathrm{Ind}_{H'(R)}$, then there is a subcomplex K_1 of $\mathrm{Ind}_{H'(R)}$ such that K_0 is a subcomplex of K_1 and $|K_1|$ is contractible.*

PROOF. Let $\sigma_1, \dots, \sigma_m$ be all the maximal simplices in K_0. For every simplex $\sigma_i = \{x_{i0}, \dots, x_{in_i}\} \in K_0$ there exists a maximal ideal M_i containing elements $x_{i0}, \dots, x_{in_i}$, so we have a finite set of maximal ideals $\{M_1, \dots, M_m\}$ (not necessarily distinct) containing elements which are vertices of appropriate maximal simplices. We will show that there exists an element $y \in R \setminus (U(R) \cup J(R))$ such that $\{x_{i0}, \dots, x_{in_i}, y\} \in \mathrm{Ind}_{H'(R)}$ for all $1 \leqslant i \leqslant m$. This will show that $|\mathrm{Ind}_{H'(R)}|$ is connected.

Let y be an element of $\bigcap_{i=1}^{m} M_i$, such that $y \notin J(R)$. Such element exists because the intersection of finitely many maximal ideals is not equal to $J(R)$. Namely if we were to have $\bigcap_{i=1}^{m} M_i = J(R)$, then for some other maximal ideal $M_{m+1} \notin \{M_1, \dots, M_m\}$ we would have $\bigcap_{i=1}^{m} M_i \subset M_{m+1}$, which would mean that $M_i \subseteq M_{m+1}$ for some $i \in \{1, \dots, m\}$ which is a contradiction.

Consider K_1 as a subcomplex of $\mathrm{Ind}_{H'(R)}$ spanned by all of the vertices of K_0 together with the vertex y (note that y might already be a vertex of K_0, in that case $K_1 = K_0$). For every simplex $\sigma_i = \{x_{i0}, \dots, x_{in_i}\} \in K_0$, we have $x_{i0}, \dots, x_{in_i}$, $y \in M_i$, hence σ_i is a face of $\{x_{i0}, \dots, x_{in_i}, y\} \in K_1$ so $|K_1|$ is a cone with the apex y, hence contractible. $\square$

We have a similar lemma for the independence complex of comaximal graph $\mathrm{Ind}_{\Gamma'_2(R)}$.

LEMMA 3.2. *Suppose that R is such that the set of maximal ideals $\mathrm{Max}(R)$ is infinite. If K_0 is a finite subcomplex of $\mathrm{Ind}_{\Gamma'_2(R)}$, then there is a subcomplex K_1 of $\mathrm{Ind}_{\Gamma'_2(R)}$ such that K_0 is a subcomplex of K_1 and $|K_1|$ is contractible.*

PROOF. Let $\sigma_1, \dots, \sigma_m$ be all the maximal simplices in K_0. For every simplex $\sigma_i = \{x_{i0}, \dots, x_{in_i}\} \in K_0$ and every pair $\{x_{ij}, x_{ik}\}$, $j \neq k$, that is a face of σ_i, there exists a maximal ideal M containing elements x_{ij}, x_{ik}, so we have some finite set of maximal ideals $\{M_1, \dots, M_t\}$ containing pairs of elements which are vertices of appropriate maximal simplices. We will show that there exists an element $y \in R \setminus (U(R) \cup J(R))$ such that $\{x_{i0}, \dots, x_{in_i}, y\} \in \mathrm{Ind}_{\Gamma'_2(R)}$ for all $1 \leqslant i \leqslant m$.

This will show that $|\operatorname{Ind}_{\Gamma_2'(R)}|$ is connected. Let y be an element of $\bigcap_{i=1}^{t} M_i$, such that $y \notin J(R)$. As in the previous lemma, such element always exists. Consider K_1 as a subcomplex of $\operatorname{Ind}_{\Gamma_2'(R)}$ spanned by all of the vertices of K_0 together with the vertex y (note that y might already be a vertex of K_0, in that case $K_1 = K_0$). For every simplex $\sigma_i = \{x_{i0}, \dots, x_{in_i}\} \in K_0$, we have $\{x_{ij}, y\}$ in some maximal ideal for every $0 \leqslant j \leqslant n_i$, hence $\{x_{ij}, y\}$ is not an edge in $\Gamma_2'(R)$. Therefore $\{x_{i0}, \dots, x_{in_i}, y\}$ is an independent set and therefore a simplex in the complex $\operatorname{Ind}_{\Gamma_2'(R)}$. We have that each σ_i is a face of $\{x_{i0}, \dots, x_{in_i}, y\} \in K_1$ so $|K_1|$ is a cone with the apex y, hence contractible. $\square$

Theorem 3.1. *If* $\operatorname{Max}(R)$ *is infinite, then* $|\operatorname{Ind}_{H'(R)}|$ *and* $|\operatorname{Ind}_{\Gamma_2'(R)}|$ *are contractible.*

Proof. The proof is completely analogous to the proof in [**7**] which we give here for the sake of completeness. It applies for both complexes so let Ind stand for both $\operatorname{Ind}_{H'(R)}$ and $\operatorname{Ind}_{\Gamma_2'(R)}$.

Since $|\operatorname{Ind}|$ has the homotopy type of a CW complex, we may use the Whitehead theorem. We only need to show that all homotopy groups of $|\operatorname{Ind}|$ are trivial. Suppose that $n \geqslant 1$ and that $g\colon S^n \to |\operatorname{Ind}|$ is a continuous map. Since the image $g[S^n]$ is compact, by Lemma 2.1 there is a finite subcomplex K_0 such that $g[S^n] \subseteq |K_0|$. By Lemma 3.1, there is a subcomplex K_1 such that $K_0 \subset K_1$ and $|K_1|$ is contractible. So, the map g may be factored through the contractible space $|K_1|$ and it is homotopically trivial. We conclude that $\pi_n(|\operatorname{Ind}|, *)$ is trivial. Since this holds for all n, by Whitehead's theorem we get that $|\operatorname{Ind}|$ is contractible. $\square$

3.2. Independence complex of comaximal hypergraph when R is a semilocal ring. Let R be a semilocal ring with m maximal ideals $M_1, M_2, \dots, M_m$. We can easily observe that the maximal simplices in $\operatorname{Ind}_{H'(R)}$ are formed from the set of elements in each maximal ideal. Therefore it will be useful to first look at a related complex whose vertices are proper ideals of the ring R (naturally excluding $J(R)$), namely the nongenerating ideal complex.

Definition 3.1. Let R be a commutative ring and let $I^*(R)$ be the set of all proper ideals of R properly containing $J(R)$. The nongenerating ideal complex $\mathscr{C}(R)$ is defined by:

$$V(\mathscr{C}(R)) = I^*(R), \text{ and } \{I_0, \dots, I_n\} \in \mathscr{C}(R) \text{ if and only if } I_0 + \cdots + I_n \neq R$$

Again, we have an equivalent definition $\{I_0, \dots, I_n\} \in \mathscr{C}(R)$ if and only if $I_0, \dots, I_n \subseteq M$ for some maximal ideal M. Consider the subcomplex $\widetilde{K}$ of $\mathscr{C}(R)$ whose vertices are all the maximal ideals and their intersections (except the intersection of all of them), that is, vertices are $\bigcap_{i\in S} M_i$ where S is any proper nonempty subset of the set $\{1, 2, \dots, m\}$. Geometric realization of this subcomplex for the case $m = 3$ is given below in Figure 1.

Note that this complex is connected unless $|\operatorname{Max}(R)| = 2$.

It will be useful to denote $[m] = \{1, 2, \dots, m\}$, while for the intersection (union) of all maximal ideals whose indices belong to a set $S \subset [m]$ we use the same notation as in [**8**], that is $M_S = \bigcap_{i\in S} M_i$ and $M^S = \bigcup_{i\in S} M_i$.

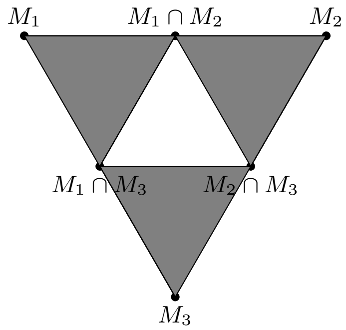

FIGURE 1.

LEMMA 3.3. *Let R be a semilocal ring. If $|\operatorname{Max}(R)| = m > 1$, then $|\widetilde{K}| \simeq \dot{\Delta}^{m-1}$.*

PROOF. Note that there are exactly m maximal simplices in $\widetilde{K}$, namely simplices whose vertices are ideals contained in M_i for each $i = 1, \dots, m$. For each such maximal simplex σ_i, consider its face τ_i whose vertices are ideals $M_{[m]\smallsetminus\{j\}}$ for each $j = 1, \dots, m$ and $j \neq i$. Let K_0 be the subcomplex with m vertices $M_{[m]\smallsetminus\{j\}}$ for each $j = 1, \dots, m$. Then K_0 has exactly m maximal simplices $\tau_1, \dots, \tau_m$, and its geometric realization is the boundary of an $(m-1)$-simplex. We define a continuous function $f\colon |\widetilde{K}| \to |K_0|$ by mapping each vertex M_S in $|\widetilde{K}|$ to the barycenter of the simplex whose vertices are $M_{[m]\smallsetminus\{j\}}$ for each $j = 1, \dots, m$ and $j \notin S$ (this is exactly the simplex $\bigcap_{i\in S} \tau_i$), and any point $x = \sum_{i=0}^{n} a_i M_{S_i}$ to $f(x) = \sum_{i=0}^{n} a_i f(M_{S_i})$. In this manner, each maximal simplex σ_i is projected onto its face τ_i, as any point of $|\widetilde{K}|$ which is inside σ_i is projected onto appropriate point of τ_i along the line that is inside the simplex σ_i. Then f is a strong deformation retraction, so $|\widetilde{K}| \simeq |K_0|$, that is, $|\widetilde{K}| \simeq \dot{\Delta}^{m-1}$. □

For example, in case $|\operatorname{Max}(R)| = 3$ (see Figure 1), there are three maximal simplices (in the geometric realization they are represented by three full triangles), which we are projecting onto the appropriate edge of the hollow triangle in the center. The function f maps vertex M_1 to the barycenter of the simplex $\{M_1 \cap M_2, M_1 \cap M_3\}$, M_2 to the barycenter of the simplex $\{M_1 \cap M_2, M_2 \cap M_3\}$, and M_3 to the barycenter of the simplex $\{M_1 \cap M_3, M_2 \cap M_3\}$.

Now we can finally determine the homotopy type of the independence complex $\operatorname{Ind}_{H'(R)}$ when R is a semilocal ring.

THEOREM 3.2. *Let R be a semilocal ring with $|\operatorname{Max}(R)| = m > 1$; then $|\operatorname{Ind}_{H'(R)}| \simeq \dot{\Delta}^{m-1}$.*

PROOF. In order to prove this, we show that $|\operatorname{Ind}_{H'(R)}| \simeq |\widetilde{K}|$ and use this together with the above lemma.

For each proper nonempty subset $S \subset [m]$, the set $M_S \smallsetminus M^{S^c}$ is nonempty by the Prime avoidance lemma. Choose an element $a_S \in M_S \smallsetminus M^{S^c}$, and consider subcomplex $\overline{\mathrm{Ind}}_{H'(R)}$ of the independence complex whose vertices are such elements. Consider a simplicial map between $\overline{\mathrm{Ind}}_{H'(R)}$ and the complex $\widetilde{K}$ discussed above, given by the vertex map $f : a_S \mapsto M_S$. This map is well defined and a bijection, since $a_S = a_T$ if and only if $S = T$, that is, if and only if $M_S = M_T$. Furthermore this is an isomorphism because $\{a_{S_0}, \ldots, a_{S_n}\} \in \overline{\mathrm{Ind}}_{H'(R)}$ if and only if there exists $i \in S_0 \cap \cdots \cap S_n$ if and only if $M_{S_0}, \ldots, M_{S_n} \subsetneq M_i$ if and only if $\{M_{S_0}, \ldots, M_{S_n}\} \in \widetilde{K}$. Therefore the complexes $\overline{\mathrm{Ind}}_{H'(R)}$ and $\widetilde{K}$ are isomorphic, hence $|\overline{\mathrm{Ind}}_{H'(R)}| \simeq |\widetilde{K}|$.

Let $g : |\,\mathrm{Ind}_{H'(R)}\,| \to |\overline{\mathrm{Ind}}_{H'(R)}|$ be the simplicial map such that a vertex x is mapped to a_S where S is the set of indices of all maximal ideals containing x, that is, we are mapping each element of $R \setminus (U(R) \cup J(R))$ to appropriate representative of the intersection of the maximal ideals containing that element. For any simplex $\{x_0, \ldots, x_n\}$ in $\mathrm{Ind}_{H'(R)}$ there is a maximal ideal containing elements $x_0, \ldots, x_n$, which then also contains $g(x_0), \ldots, g(x_n)$. Hence, simplex $\{g(x_0), \ldots, g(x_n)\}$ in $\mathrm{Ind}_{H'(R)}$ is just a face of the simplex $\{x_0, \ldots, x_n, g(x_0), \ldots, g(x_n)\}$ and our map is the projection of that larger simplex onto its face. This is clearly a strong deformation retraction since every point in $|\mathrm{Ind}_{H'(R)}|$ is mapped onto its image along a line inside the appropriate simplex. Therefore $|\mathrm{Ind}_{\Gamma'_2(R)}| \simeq |\overline{\mathrm{Ind}}_{H'(R)}| \simeq |\widetilde{K}|$, so $|\mathrm{Ind}_{H'(R)}| \simeq \dot{\Delta}^{m-1}$. □

3.3. Independence complex of comaximal graph when R is a semilocal ring. Unlike the independence complex of comaximal hypergraph, we will show that the independence complex of comaximal graph is contractible for semilocal rings with more than two maximal ideals. To show this we use a similar idea as before.

THEOREM 3.3. *Let R be a semilocal ring with m maximal ideals $M_1, \ldots, M_m$. If $m = 2$, then $|\,\mathrm{Ind}_{\Gamma'_2(R)}\,|$ is homotopy equivalent to a pair of points with the discrete topology, and if $m > 2$ the complex $|\,\mathrm{Ind}_{\Gamma'_2(R)}\,|$ is contractible.*

PROOF. Elements of maximal ideals form simplices in the complex (because no pair of elements generates entire ring), so again for each proper nonempty subset $S \subset [m]$ we choose an element $a_S \in M_S \setminus M^{S^c}$, and consider subcomplex $\overline{\mathrm{Ind}}_{\Gamma'_2(R)}$ of the independence complex whose vertices are such elements. Let $g\colon |\,\mathrm{Ind}_{\Gamma'_2(R)}\,| \to |\overline{\mathrm{Ind}}_{\Gamma'_2(R)}|$ be the simplicial map such that a vertex x is mapped to a_S where S is the set of indices of all maximal ideals containing x. Then for any simplex $\{x_0, \ldots, x_n\}$ in $\mathrm{Ind}_{\Gamma'_2(R)}$ we also have $\{x_0, \ldots, x_n, g(x_0), \ldots, g(x_n)\}$ in $\mathrm{Ind}_{\Gamma'_2(R)}$ because every pair is in some maximal ideal hence not generating entire ring; namely $\{x_i, g(x_j)\}$ (i and j need not be distinct) is in the same maximal ideal as is $\{x_i, x_j\}$, and $\{g(x_i), g(x_j)\}$ is in the same maximal ideal as $\{x_i, x_j\}$. Now, same as in the proof of Theorem 3.2, we conclude that g is a strong deformation retraction, so $|\,\mathrm{Ind}_{\Gamma'_2(R)}\,| \simeq |\overline{\mathrm{Ind}}_{\Gamma'_2(R)}|$. Now consider the map $f\colon |\overline{\mathrm{Ind}}_{\Gamma'_2(R)}| \to |\overline{\mathrm{Ind}}_{\Gamma'_2(R)}|$ that sends a_S to the barycenter of the simplex whose vertices are $a_{[m] \setminus \{j\}}$ for each $j = 1, \ldots, m$ and $j \notin S$. Every point of each simplex σ is projected onto appropriate point of its face along the line that is inside σ so this is a strong deformation retraction. Therefore the complex $|\overline{\mathrm{Ind}}_{\Gamma_2'(R)}|$ is reduced to its subcomplex consisting of the vertices $a_{[m]\smallsetminus\{1\}}, \ldots, a_{[m]\smallsetminus\{m\}}$, and for $m > 2$ these vertices form a simplex because every pair of elements is in some maximal ideal. Therefore for $m > 2$ the independence complex $\mathrm{Ind}_{\Gamma_2'(R)}$ is contractible while for $m = 2$ it is homotopy equivalent to two disjoint points. □

References

1. S. Akbari, R. Nikadish, M. J. Nikmehr, *Some results on the intersection graphs of ideals of rings*, J. Algebra Appl. **12**(4) (2013), 1250200.
2. D. F. Anderson, P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.
3. I. Chakrabarty, S. Ghosh, T. K. Mukherjee, M. K. Sen, *Intersection graphs of ideals of rings*, Discrete Math. **309**(17) (2009), 5381–5392.
4. R. Ehrenborg, G. Hetyei, *The topology of the independence complex*, Eur. J. Comb.**27**(6) (2006), 906–923.
5. D. Kozlov, *Combinatorial Algebraic Topology*, Algorithms Comput Math. **21**, Springer, Berlin, 2008.
6. H. R. Maimani, M. Salimi, A. Sattari, S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319**(4) (2008), 1801–1808.
7. N. Milošević, Z. Z. Petrović, *Order complex of ideals in a commutative ring with identity*, Czech. Math. J., to appear.
8. S. Moconja, Z. Z. Petrović, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc.**83**(1) (2011), 11–21.
9. J. R. Munkres, *Elements of Algebraic Topology*, Benjamin/Cummings, Menlo Park, California, 1984.
10. Z. Z. Petrović, Z. S. Pucanović, *Toroidality of intersection graphs of ideals of commutative rings*, Graphs Comb. **30**(3) (2014), 707–716.
11. P. K. Sharma, S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra **176**(1) (1995), 124–127.
12. V. A. Vassiliev, *Topology of discriminants and their complements*; in: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994)*, 209–226, Birkhäuser, Basel, 1995.
13. M. Ye, T. Wu, *Co-maximal ideal graphs of commutative rings*, J. Algebra Appl.**11**(6) (2012), 1250114, 14 pp.

Faculty for Information Systems and Technologies
University of Donja Gorica
Podgorica
Montenegro
nela.milosevic@udg.edu.me

(Received 23 10 2014)
(Revised 26 01 2015 and 06 08 2015)