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A NOTE ON DIVIDED DIFFERENCES

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Abstract. We obtain a new recurrence formula for sequences of divided differences. In a particular case, the recurrence formula simplifies the classical Newton–Girard identities relating power sums and elementary symmetric polynomials.

1. Introduction and preliminary results

Divided differences are a basic tool in numerical analysis often used both in theory and in practice. In the last two decades the classical topic of interpolation has been strongly reinvigorated. A new insight was provided into the study of efficient numerical algorithms for the calculations of divided difference, chain rules, mean value theorems and representations of the divided differences of ordinary and implicit functions. In this context, far from being exhaustive, we briefly mention the papers: [**1**–**17**, **20**] and the references therein.

Throughout the paper n and m denote non-negative integers. Let $z_0, \dots, z_n$ be pairwise distinct complex numbers and $\mathcal{F}$ be the set of all complex-valued functions defined on a set $A \supseteq \{z_0, \dots, z_n\}$. In most books on numerical analysis, the divided differences of a function $f \in \mathcal{F}$ are defined recursively:

$$[z_0; f] = f(z_0), \quad \dots,$$
$$[z_0, \dots, z_n; f] = \frac{[z_1, \dots, z_n; f] - [z_0, \dots, z_{n-1}; f]}{z_n - z_0}, \qquad n = 1, 2, \dots.$$

For the sake of clarity, we will also use the alternative notation $[z_0, \dots, z_n; f(t)]_t$ to represent the divided difference $[z_0, \dots, z_n; f]$. The divided difference $[z_0, \dots, z_n; f]$ can equivalently be defined as the coefficient of z^n of the interpolating polynomial $L[z_0, \dots, z_n; f]$.

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The polynomial $L[z_0, \ldots, z_n; f]$ can be written in the Lagrange–Waring form

$$L[z_0, \ldots, z_n;\, f](z) = \sum_{k=0}^{n} \frac{\ell_n(z)}{z - z_k} \cdot \frac{f(z_k)}{\ell_n'(z_k)}, \tag{1.1}$$

where $\ell_n(z) = (z-z_0)\cdots(z-z_n)$. From (1.1) we obtain the following representation of the divided difference

$$[z_0, \ldots, z_n;\, f] = \sum_{k=0}^{n} \frac{f(z_k)}{\ell_n'(z_k)}. \tag{1.2}$$

Let $e_i \colon \mathbb{C} \to \mathbb{C}$ denote the monomial functions $e_i(z) = z^i$, $i = 0, 1, \ldots$. In our analysis we will focus on a recurrence formula for sequences of divided differences of the form $[z_0, \ldots, z_n;\, e_m f]$, $m = 0, 1, \ldots$.

Popoviciu [**19**] proved that

$$[z_0, \ldots, z_n;\, e_{n+r}] = \sum z_0^{k_0} \ldots z_n^{k_n}, \tag{1.3}$$

where the sum runs over all non-negative $k_0, \ldots, k_n$ with $k_0 + \cdots + k_n = r$, where r is a non-negative integer. Popoviciu's formula (1.3) was rediscovered in 1981 by Neuman [**18**]. Another representation for the divided differences of monomials in terms of the complete Bell polynomials was given in [**2**]. Maybe the technique used in [**17**] could be useful in finding an alternative explicit formula for $[z_0, \ldots, z_n;\, e_m f]$.

We note that, obtaining a recurrence formula *of order* n (the same as the number of knots) for the sequence $[z_0, \ldots, z_n;\, e_m f]$, $m = 0, 1, \ldots$, is quite simple and well known. However, the problem of finding a recurrence formula of a *given order* m (*independent of* n) is not so simple, and this is the main aim of the paper. This is similar to finding a recurrence formula of order m for the sequence of determinants of order $n+1$:

$$\begin{vmatrix} 1 & z_0 & \cdots & z_0^{n-1} & z_0^m f(z_0) \\ 1 & z_1 & \cdots & z_1^{n-1} & z_1^m f(z_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_n & \cdots & z_n^{n-1} & z_n^m f(z_n) \end{vmatrix}, \qquad m = 0, 1, \ldots$$

2. Main results

Let $D \subseteq \mathbb{C}$ and $P, Q \colon D \to \mathbb{C}^*$ be two $m+1$ times differentiable functions. To simplify the proof of the main result we consider the function $\mathbf{F} \colon D \to \mathbb{C}^*$, $\mathbf{F} = \frac{P}{Q}$. Taking into account the identity $\left(\frac{P}{Q}\right)' = \frac{P}{Q}\left(\frac{P'}{P} - \frac{Q'}{Q}\right)$, and using the Leibniz formula for the high order derivative of a product, we obtain the following recurrence formula of order m for the high order derivatives of $\mathbf{F}$, which may be interesting in itself,

$$\frac{\mathbf{F}^{(m+1)}}{(m+1)!} = \frac{1}{m+1} \sum_{k=0}^{m} \frac{\mathbf{F}^{(k)}}{k!} \, \frac{1}{(m-k)!} \left(\frac{P'}{P} - \frac{Q'}{Q}\right)^{(m-k)}. \tag{2.1}$$

In order to simplify the notations we will suppose that $[z_0, \dots, z_n; f] \neq 0$, but mention that the current method works for $[z_0, \dots, z_n; f] = 0$ also, with only slight modifications.

Let $\zeta_1, \zeta_2, \dots, \zeta_n \in \mathbb{C}$ denote the roots of the polynomial $L[z_0, \dots, z_n; f]$, and consider the power sums:

$$s_k := s_k(z_0, \dots, z_n) := \sum_{j=0}^{n} z_j^k, \qquad k = 0, 1, \dots,$$

$$\sigma_k := \sigma_k(\zeta_1, \dots, \zeta_n) := \sum_{i=1}^{n} \zeta_i^k, \qquad k = 0, 1, \dots.$$

The following theorem is the main result of this paper.

Theorem 2.1. *The following recurrence formula of order m is satisfied*

$$[z_0, \dots, z_n;\, e_{m+1} f] = \frac{1}{m+1} \sum_{k=0}^{m} [z_0, \dots, z_n;\, e_k f](s_{m+1-k} - \sigma_{m+1-k}), \quad m = 0, 1, \dots \tag{2.2}$$

Proof. We will make use of the following formula

$$\left[z_0, \dots, z_n;\, \frac{f(t)}{z-t}\right]_t = \frac{L(z_0, \dots, z_n;\, f)(z)}{(z-z_0)\cdots(z-z_n)}, \quad z \in \mathbb{C} \setminus \{z_0, \dots, z_n\}, \tag{2.3}$$

which is a direct consequence of (1.1) and (1.2). We replace z with $\frac{1}{z}$ in (2.3) and define

$$\mathbf{F}(z) := \left[z_0, \dots, z_n;\, \frac{f(t)}{1-zt}\right]_t = [z_0, \dots, z_n; f]\, \frac{(1-z\,\zeta_1)\cdots(1-z\,\zeta_n)}{(1-zz_0)\cdots(1-zz_n)},$$

in a suitable neighborhood D of $z=0$. Let $P(z) = [z_0, \dots, z_n; f]\,(1-z\,\zeta_1)\cdots(1-z\,\zeta_n)$ and $Q(z) = (1-zz_0)\cdots(1-zz_n)$. We have

$$\frac{\mathbf{F}^{(k)}(0)}{k!} = [z_0, \dots, z_n;\, e_k f], \quad k = 0, 1, \dots \tag{2.4}$$

and

$$\left(\frac{P'}{P}\right)^{(m-k)}(0) = -(m-k)!\sigma_{m+1-k}, \quad \left(\frac{Q'}{Q}\right)^{(m-k)}(0) = -(m-k)!s_{m+1-k}.$$

The use of (2.4) and of (2.1) for $z=0$ completes the proof. □

Remark 2.1. In particular, for $f = e_n$, (2.2) becomes

$$[z_0, \dots, z_n;\, e_{n+m+1}] = \frac{1}{m+1} \sum_{k=0}^{m} [z_0, \dots, z_n;\, e_{n+k}] s_{m+1-k}, \quad m = 0, 1, \dots$$

Indeed, for $f = e_n$, we obtain $L[z_0, \dots, z_n;\, f] = e_n$ with roots $\zeta_1 = 0, \dots, \zeta_n = 0$.

We rewrite (2.2) in the form

$$[z_0, \dots, z_n;\, e_q f] = \frac{1}{q} \sum_{k=0}^{q-1} [z_0, \dots, z_n;\, e_k f]\,(s_{q-k} - \sigma_{q-k}), \quad q = 1, 2, \dots. \tag{2.5}$$

Let us suppose now that f possesses a derivative $f^{(n)}(0) \neq 0$.

REMARK 2.2. Coalescing all points $z_0, \dots, z_n$ to zero, equation (2.5) simplifies to a recurrence formula for the derivatives of f in terms of the roots of its Maclaurin polynomial,

$$(e_q f)^{(n)}(0) = -\frac{1}{q} \sum_{k=0}^{q-1} (e_k f)^{(n)}(0)\, \sigma_{q-k}, \quad q = 1, 2, \dots. \tag{2.6}$$

REMARK 2.3. By taking in (2.6)

$$f(t) = (t - \zeta_1) \cdots (t - \zeta_n) = \sum_{i=0}^{n} c_i t^{n-i},$$

we rediscover the classical *Newton identities* (also known as the Newton–Girard formulae):

$$c_0 = 1,$$
$$c_q = -\frac{1}{q} \sum_{k=0}^{q-1} c_k\ \sigma_{q-k}, \quad q = 1, 2, \dots, n.$$

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