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SEMI-BASIC 1-FORMS AND COURANT STRUCTURE FOR METRIZABILITY PROBLEMS

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Abstract. The metrizability of sprays, particularly symmetric linear connections, is studied in terms of semi-basic 1-forms using the tools developed by Bucataru and Dahl in [**2**]. We introduce a type of metrizability in relationship with the Finsler and projective metrizability. The Lagrangian corresponding to the Finsler metrizability, as well as the Bucataru–Dahl characterization of Finsler and projective metrizability are expressed by means of the Courant structure on the big tangent bundle of TM. A byproduct of our computations is that a flat Riemannian metric, or generally an R-flat Finslerian spray, yields two complementary, but not orthogonally, Dirac structures on $T^{\mathrm{big}}TM$. These Dirac structures are also Lagrangian subbundles with respect to the natural almost symplectic structure of $T^{\mathrm{big}}TM$.

1. Introduction

The question whether a given symmetric linear connection ∇ on the manifold M is the Levi-Civita connection of a Riemannian metric is very important from both mathematical (where it appears as a reciprocal of the fundamental theorem of Riemannian geometry) and physical reasons [**15**]. So, it has a rich history (which is possible to begin with [**9**]) pointed out for example in [**8**], where the conditions for local metrizability as well as the global problem are discussed in detail. Recently, there is a growing interest for this question in control theory [**12**] and more generally, under the name of *inverse problem*, in geometrization of general and special Lagrangian systems [**14, 16**].

Here we introduce a type of metrizability which involves the geometry of the tangent bundle. Namely, we search for a metric g not on the base manifold M but on the tangent manifold TM by imposing that g is covariant constant with respect

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to the dynamical covariant derivative of the spray associated to ∇. A condition for this kind of metrizability is then derived by using the Finsler metrizability of sprays characterized by Bucataru and Dahl in [**2**] in terms of semi-basic 1-forms. The initial metrizability problem is recovered in terms of global smoothness of this 1-form and also there are considered several relationships with the symplectic geometry of TM induced by a 2-homogeneous Lagrangian L of Finsler type. In fact, the possibility to use the symplectic structures was the motivation for our choice of [**2**] as the main point of beginning the present study instead of classical holonomy methods, [**17**] or [**19**].

We express the above Finsler (and also the projective) metrizability of a general spray S and the corresponding Lagrangian L by means of the Courant structure on the big tangent bundle of TM. Essential terms in the Courant bracket for several objects are based on the coefficients of the Jacobi operator of S, as well as on the coefficients of the curvature of the nonlinear connection induced by S. For example, even in the Riemannian case we derive new results: the flatness of the Riemannian metric is equivalent with the Courant isotropy of the pair (the horizontal distribution, vertical dual of the Poincaré–Cartan 1-form). The same flatness yields a half-Dirac structure on $T^{\mathrm{big}}TM$ while the above subbundle together with (the vertical distribution, the Poincaré–Cartan 1-form) are isotropic with respect to the natural almost symplectic structure of the big tangent bundle of TM.

Moreover, we obtain a Dirac structure $V_D(S)$=(the vertical distribution, its annihilator) for every Lagrangian spray S i.e., S gives the Euler–Lagrange equations of a regular Lagrangian L; the regularity means that the Hessian of $L(x, y)$ with respect to the vertical variables y is nondegenerate. With respect to the dual point of view, the pair $H_D(S)$=(the horizontal distribution, its annihilator) is a Dirac structure on the $T^{\mathrm{big}}TM$ if and only if the Finslerian spray S is R-flat which means the vanishing of the curvature of its nonlinear connection; this means in the particular case of Riemannian geometry that the metric is flat. In conclusion, for a flat Riemannnian metric the big tangent bundle of TM is the complementary (but not orthogonally) sum of two Dirac structures, both being also Lagrangian subbundles for the almost symplectic structure of $T^{\mathrm{big}}TM$ as well as Lie algebroids over TM. Let us remark that some classes of Dirac structures naturally associated to Lagrangian systems appear also in [**20**] and the above results are already obtained for the Riemannian case in [**8**].

2. Semibasic 1-forms for the metrizability of linear connections

Fix M a smooth $n \geqslant 2$ dimensional manifold with the tangent bundle TM and cotangent bundle T^*M. Local coordinates on M will be denoted by $x = (x^i)$, $1 \leqslant i \leqslant n$ while the induced coordinates on TM will be denoted by $(x, y) = (x^i, y^i)$. Let $\{0\}$ be the zero-section of the tangent bundle and $\mathcal{X}(TM)$ the Lie algebra of vector fields on TM. A main structure of TM is the *tangent structure* $J = \frac{\partial}{\partial y^i} \otimes dx^i$.

Fix also a symmetric linear connection ∇ on M and let $S_\nabla \in \mathcal{X}(TM)$ its *semispray*; if ∇ has the local coefficients $\Gamma^i_{jk}(x)$, then S_∇ has the local expression

$S_\nabla = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ with

$$G^i = \frac{1}{2}\Gamma^i_{jk} y^j y^k. \tag{2.1}$$

We have

$$J(S_\nabla) = \mathbb{C} \tag{2.2}$$

with $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$ the *Liouville vector field* of TM. This vector field is useful in characterizations of *homogeneous* objects on $TM \setminus \{0\}$, [**2**, p. 163]; namely let k be an integer, then a vector field X is k-homogeneous if $\mathcal{L}_{\mathbb{C}} X = (k-1)X$, a p-form ω is k-homogeneous if $\mathcal{L}_{\mathbb{C}}\omega = k\omega$ and a $(1,1)$-tensor field L is k-homogeneous if $\mathcal{L}_{\mathbb{C}} L = (k-1)L$. Here $\mathcal{L}_Z$ is the Lie derivative with respect to the vector field Z; the vector fields $\left\{\frac{\partial}{\partial y^i}\right\}$ are 0-homogeneous.

In fact S_∇ is more than a semispray; it is a *spray* due to the 2-homogeneity of the coefficients G^i of (2.1). Then S_∇ has the reduced form $S_\nabla = y^i \frac{\delta}{\delta x^i}$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$$

with $N^i_j = \frac{\partial G^i}{\partial y^j} = \Gamma^i_{jk} y^k$. We derive a new local basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ in $\mathcal{X}(TM)$, called *Berwald*, which is adapted to our computations and has the dual basis $\left(dx^i, \delta y^i = dy^i + N^i_j dx^j\right)$. This decomposition of the iterated tangent bundle $T(TM)$ produces [**2**, p. 165]:

i1) *the horizontal projector* $h = \frac{\delta}{\delta x^i} \otimes dx^i$,

ii1) *the vertical projector* $v = \frac{\partial}{\partial y^i} \otimes \delta y^i$, and the following distributions and codistributions:

i2) *the horizontal distribution* $H(TM) = \operatorname{span}\left\{\frac{\delta}{\delta x^i}\right\} = h(\mathcal{X}(TM))$ and its annihilator $H^*(TM) = \operatorname{span}\{\delta y^i\}$,

ii2) *the vertical distribution* $V(TM) = \operatorname{span}\left\{\frac{\partial}{\partial y^i}\right\} = v(\mathcal{X}(TM))$ and its annihilator $V^*(TM) = \operatorname{span}\{dx^i\}$.

Therefore S_∇ is a *horizontal vector field* while $\mathbb{C}$ is a *vertical vector field*.

Let ∇_S be the *dynamical covariant derivative* induced on TM by S_∇, [**2**, p. 167]. Inspired by the defining property of the Levi-Civita connection, we introduce:

Definition 2.1. The symmetric linear connection ∇ is said to be *tangent metrizable* if there exists a Riemannian metric g on TM, of Sasaki type

$$g = g_{ij} dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$$

such that $\nabla_S g = 0$ which means *the tangent Christoffel formula*

$$S(g_{ij}) = N^k_i g_{kj} + N^k_j g_{ki}. \tag{2.3}$$

Our approach in finding characterizations of tangent metrizability follows closely the techniques of Bucataru and Dahl from [**2**], where, among others, the following type of metrizability is studied:

Definition 2.2. [**2**, p. 178] The spray S is *Finsler metrizable* if there exists a 2-homogeneous Lagrangian $L : TM \to \mathbb{R}$ such that $\mathcal{L}_S(d_J L) = dL$. The spray is called *projective metrizable* if the above Lagrangian is 1-homogeneous.

Here d_J is the exterior derivative induced by J on forms of TM; then $d_J L(X) = dL(JX) = (JX)(L)$ for any $X \in \mathcal{X}(TM)$. The fundamentals of Finsler geometry are excellently exposed in [**1**].

The characterization of Finsler metrizability is obtained in [**2**] in terms of semi-basic 1-forms i.e., 1-forms on TM with local expression $\theta = \theta_i(x, y)dx^i$:

Theorem 2.1. [**2**, p. 177] *The spray S is Finsler metrizable if and only if there exists a 1-homogeneous semi-basic 1-form θ on the slit tangent bundle $TM \setminus \{0\}$ such that the following Helmoltz conditions hold*
(1) θ is d_h-closed: $d_h\theta = 0$,
(2) θ is d_J-closed: $d_J\theta = 0$,
(3) $d\theta$ is covariant constant with respect to ∇_S: $\nabla_S(d\theta) = 0$.
Then θ is called the Poincaré–Cartan 1-form of the Lagrangian L.

Let us point out that our condition (2.3) corresponds to the relation (3) above. Therefore we apply Theorem 2.1 and to this aim let $\omega = d\theta$. Recall also:

Definition 2.3. [**2**, p. 164] The semibasic 1-form θ is called *nondegenerate* if ω is a symplectic form on $TM \setminus \{0\}$.

Recall that a pair (ω, K) with a symplectic form and a compatible almost complex structure yields a Riemannian metric g through the formula [**4**, p. 86]:

$$g(\cdot, \cdot) = \omega(\cdot, K\cdot). \tag{2.4}$$

The required compatibility is [**4**, p. 90]:

$$\omega(KX, KY) = \omega(X, Y). \tag{2.5}$$

But ∇ (equivalently S_∇) yields an almost complex structure $\mathbb{F}$, [**2**, p. 166]:

$$\mathbb{F}\Big(\frac{\delta}{\delta x^i}\Big) = -\frac{\partial}{\partial y^i}, \quad \mathbb{F}\Big(\frac{\partial}{\partial y^i}\Big) = \frac{\delta}{\delta x^i} \tag{2.6}$$

or, in the Berwald basis

$$\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.$$

Now, we are able to state the first main result of this note.

Proposition 2.1. *Let ∇ be a symmetric linear connection on M such that the associated spray S_∇ is Finsler metrizable via the semibasic 1-form θ. If θ is nondegenerate i.e., the determinant* $\det\big(\frac{\partial \theta_i}{\partial y^j}\big)$ *is nonvanishing, then ∇ is tangent metrizable.*

Proof. We need to prove two facts.

(I) ω and $\mathbb{F}$ are compatible, but this is a consequence of relations (1) and (2) from Theorem 2.1. Indeed, after [**2**, p. 172], we have $\omega = a_{ij}dx^j \wedge dx^i + 2g_{ij}\delta y^j \wedge dx^i$ and $d_h\theta = a_{ij}dx^j \wedge dx^i$, $d_J\theta = (g_{ij} - g_{ji})dx^j \wedge dx^i$ with

$$a_{ij} = \frac{1}{2}\Big(\frac{\delta\theta_i}{\delta x^j} - \frac{\delta\theta_j}{\delta x^i}\Big), \quad g_{ij} = \frac{1}{2}\frac{\partial\theta_i}{\partial y^j}.$$

A direct computation gives

$$\text{(i)} \qquad \omega\Big(\mathbb{F}\frac{\delta}{\delta x^i}, \mathbb{F}\frac{\delta}{\delta x^j}\Big) = 0 = a_{ij} = \omega\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big),$$

$$\text{(ii)} \qquad \omega\Big(\mathbb{F}\frac{\partial}{\partial y^i}, \mathbb{F}\frac{\partial}{\partial y^j}\Big) = a_{ij} = 0 = \omega\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big),$$

$$\text{(iii)} \qquad \omega\Big(\mathbb{F}\frac{\delta}{\delta x^i}, \mathbb{F}\frac{\partial}{\partial y^j}\Big) = -g_{ji} = -g_{ij} = \omega\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\Big).$$

(II) $\nabla_S g = 0$. The identity (3) reads $\nabla_S\omega = 0$ and we have $\nabla_S\mathbb{F} = 0$ from [**2**, p. 169]; these two relations yield $\nabla_S g = 0$ because of (2.4).

Let us remark that (i) and (ii) mean that the distributions $H(TM)$ and $V(TM)$ are Lagrangian for ω.

The matrix of ω in the dual Berwald basis $\big(dx^i, \delta y^j\big)$ is

$$\omega = \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and then ω is non-degenerate if and only if $\det(g) \neq 0$. The matrix of $\mathbb{F}$ is

$$\mathbb{F} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and then the corresponding metric g is of Sasaki type

$$g = \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix}. \qquad \square$$

In this framework, lifting to the tangent bundle the geodesic equations of ∇ i.e., the flow equations of S_∇ we arrive at exactly the Euler–Lagrange equations of the Lagrangian $L_{\nabla,\theta} = g(S,S) = g(\mathbb{C},\mathbb{C}) = \omega(\mathbb{C}, \mathbb{F}(\mathbb{C}))$ which can be thought of as *the energy of* g. A dual relation to (2.2) is

$$\mathbb{F}(\mathbb{C}) = S_\nabla \tag{2.7}$$

and then

$$L_{\nabla,\theta} = \omega(\mathbb{C}, S_\nabla) = d\theta(\mathbb{C}, S_\nabla) = \frac{1}{2}\big\{\mathbb{C}(\theta(S_\nabla)) - S_\nabla(\theta(\mathbb{C})) - \theta([\mathbb{C}, S_\nabla])\big\}.$$

Since θ is semibasic, we have $\theta(\mathbb{C}) = 0$ while the 2-homogeneity of S_∇ reads

$$[\mathbb{C}, S_\nabla] = S_\nabla$$

and then

$$L_{\nabla,\theta} = \mathbb{C}\Big(\frac{1}{2}\theta(S_\nabla)\Big) - \frac{1}{2}\theta(S_\nabla)$$

which recover the classical formula for the Hamiltonian H associated to a Lagrangian L

$$H = y^i \frac{\partial L}{\partial y^i} - L = \mathbb{C}(L) - L.$$

But again the 2-homogeneity of S_∇: $\mathbb{C}(\theta(S_\nabla)) = 2\theta(S_\nabla)$ gives the final expression $L_{\nabla,\theta} = \frac{1}{2}\theta(S_\nabla) = \frac{1}{2}\theta_i(x, y)y^i$ which is 2-homogeneous since θ is 1-homogeneous. A similar formula can be derived by using *the* $\mathbb{F}$*-adjoint of* θ

$$\theta^* = \mathbb{F}^*(\theta) = \frac{1}{2}\theta_i \delta y^i$$

($\mathbb{F}^*$ is the version of $\mathbb{F}$ on 1-forms), namely $L_{\nabla,\theta} = \theta^*(\mathbb{C})$. Let us point out that the Euler characterization of 1-homogeneity for θ

$$\theta_i = \frac{\partial \theta_i}{\partial y^j} y^j$$

means that $\theta_i = 2g_{ij}y^j$, and then L has the well-known expression $L = g_{ij}y^iy^j$.

The final answer to the metrizability problem. Now, we are ready for a return to the initial problem namely the metrizability of ∇. If we assume the smoothness of our tools on the whole TM, then the 1-homogeneity of θ means that the components θ_i are homogeneous polynomials of degree 1 in y and then g is in fact a Riemannian metric on the base manifold M i.e., $g = g(x)$. In conclusion, our proposed approach for the metrizability problem of a linear connection ∇ consists in two steps:

(1) we lift from M to the tangent bundle TM and study the Finsler metrizability of S_∇ obtaining the metric g,

(2) we return to the basis manifold M under the homogeneity hypothesis, obtaining the desired metric $g = g(x)$.

Examples 2.1. (1) Let on $M = \mathbb{R}^2$ the symmetric linear connection with the only nonzero components, [**19**, p. 513]

$$\Gamma^1_{11} = \frac{x^1}{(x^1)^2 + 1}, \quad \Gamma^2_{22} = \frac{x^2}{(x^2)^2 + 1}.$$

In the cited paper, it is proved that ∇ is the Levi-Civita connection for a metric g on the base manifold and this g is computed. In the present framework we derive that ∇ is tangent metrizable with the semibasic 1-form

$$\theta^1 = 2b_2[(x^1)^2 + 1]y^1 + b_1\sqrt{[(x^1)^2 + 1][(x^2)^2 + 1]}\, y^2,$$
$$\theta^2 = b_1\sqrt{[(x^1)^2 + 1][(x^2)^2 + 1]}\, y^1 + 2b_3[(x^2)^2 + 1]y^2$$

with b_i real parameters.

(2) Examples of non-Finsler metrizable sprays are in [**3**].

It is natural to ask about $\omega^* = d\theta^*$. A straightforward computation yields

$$\omega^* = \frac{1}{2}R^k_{ij}\theta_k dx^i \wedge dx^j + \frac{1}{2}\theta_{i|j} dx^i \wedge \delta y^j$$

where $\left(R^a_{ij}\right)$ are the components of the curvature of *nonlinear connection* $H(TM)$, [**2**, p. 166]

$$R^a_{ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i} \quad \text{and} \quad \theta_{j|i} = \frac{\delta \theta_j}{\delta x^i} - \frac{\partial N^k_i}{\partial y^j}\theta_k.$$

In Finsler (particularly Riemannian) geometry, by using the Cartan (particularly Levi-Civita) linear connection, the coefficient $\theta_{i|j}$ vanish and then $d\theta$ is the unique associated symplectic form. A Finslerian spray S with vanishing curvature R is called R-*flat* [**7**]. So, for an R-flat Finsler metric (particularly flat Riemannian metric) we have that θ^* is closed.

Let us end this section by making several relationships with some previous works.

(a) Usually, one seeks the solution of the metrizability problem bearing in mind the holonomy groups (or algebra) according to the first fundamental paper on this subject namely [**17**]. So, in [**19**, p. 514] are studied symmetric bilinear forms G with

$$G(AX, Y) + G(X, AY) = 0 \tag{2.8}$$

for all A in the holonomy group of a given point of M. Since $\mathbb{F}$ of (2.6) is an almost complex structure, the compatibility relation (2.5) reads:

$$g(\mathbb{F}X, Y) + g(X, \mathbb{F}Y) = 0 \tag{2.9}$$

and we remark that the formulae (2.8) and (2.9) are formally the same. The essential difference between these equations is that formula (2.9) is in a close relation to the curvature (and there is no such condition if the curvature is zero) while (2.8) is just expressing some symmetry of the metric (and survives even in the case when $R = 0$). In fact, formula (2.8) corresponds rather to one of the Helmholtz condition and its generalization involving the Jacobi endomorphism and its derivatives.

(b) In [**7**, p. 100] it is pointed out that $\mathbb{F}$ of (2.6) is not homogeneous and a new almost complex structure is considered

$$\mathbb{F}_S\Big(\frac{\delta}{\delta x^i}\Big) = -F\frac{\partial}{\partial y^i}, \quad \mathbb{F}_S\Big(\frac{\partial}{\partial y^i}\Big) = \frac{1}{F}\frac{\delta}{\delta x^i}$$

where F is the Finslerian fundamental function corresponding to the Lagrangian $L_{\nabla,\theta}$ constructed above via $L_{\nabla,\theta} = F^2$. This new almost complex structure is 0-homogeneous [**7**, p. 100], and a computation similar to that given in the proof of Proposition 2.1 yields that ω and $\mathbb{F}_S$ are still compatible. The formula corresponding to (2.7) is $\mathbb{F}_S(\mathbb{C}) = \frac{1}{F}S_\nabla$ and then the energy of the new metric $g_S = \omega(\cdot, \mathbb{F}_S\cdot)$ is

$$L_{\nabla,\theta,S} = \frac{1}{F}L_{\nabla,\theta} = F = \sqrt{\theta_i(x,y)y^i} = \sqrt{g_{ij}y^iy^j}.$$

This 1-homogeneous Lagrangian corresponds to the projective metrizability of S_∇.

(c) If ω and $\mathbb{F}$ are not compatible, then, similar to relation (11) of [**13**, p. 8395], we replace ω with $\omega_s(\cdot,\cdot) = \frac{1}{2}\big\{\omega(\mathbb{F}\cdot, \mathbb{F}\cdot) + \omega(\cdot,\cdot)\big\}$.

(d) Also, from relation (29) from [**13**, p. 8396], we get that S_∇ is, at the same time:

– a Killing vector field for g,
– an infinitesimal symplectomorphism for ω,
– an infinitesimal automorphism for $\mathbb{F}$.

According to [**4**, p. 22] for an exact symplectic form $\omega = d\theta$, there exists a unique vector field ξ such that $i_\xi \omega = \theta$. For our case we have $\xi = \mathbb{C}$ so that we can say that all semisprays have an universal symplectic potential vector field namely$\mathbb{C}$.

(e) Again from [**7**, p. 100], the almost complex structure $\mathbb{F}$ is integrable if and only if the spray S_∇ is R-flat which means that ∇ is with vanishing curvature. In this case, the pair $(g, \mathbb{F})$ is a Kähler structure on $TM \setminus \{0\}$ for which $\omega = d\theta$ is the *Kähler* (or *fundamental*) *form*.

3. The relationship with the Courant structure of big *TM*. Associated Dirac structures

The interplay between vector fields and 1-forms of the previous section makes necessary the consideration of the manifold $T^{\mathrm{big}}M := TM \oplus T^*M$. This manifold is the total space of a natural vector bundle $\pi : T^{\mathrm{big}}M \to M$; so $T^{\mathrm{big}}M$ is called *the big tangent bundle* of M [**18**], and is endowed with *the Courant structure* $(\langle\, ,\rangle, [\, ,])$ [**6**]:

1. the (neutral) inner product $\big\langle (X, \alpha), (Y, \beta) \big\rangle = \frac{1}{2}\big(\beta(X) + \alpha(Y)\big)$
2. the (skew-symmetric) Courant bracket

$$\big[(X, \alpha), (Y, \beta)\big] = \Big([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2}\, d\big(\beta(X) - \alpha(Y)\big)\Big).$$

The same manifold $TM \oplus T^*M$ is called sometimes *the Pontryagin bundle* of M [**11**].

Going now with the similar structure by considering TM instead of M, we derive some ways to express the Lagrangian associated to a metrizable spray:

Proposition 3.1. *Let S be a spray which is Finsler metrizable through the semibasic* 1*-form θ. Then the corresponding* 2*-homogeneous Lagrangian is*

$$L = \big\langle (\mathbb{C}, \theta), (S, \theta) \big\rangle = \big\langle (\mathbb{C}, \theta^*), (S, \theta^*) \big\rangle = \frac{1}{2}\big\langle (\mathbb{C}, \theta), (S, \theta^*) \big\rangle.$$

Proof. The above formulae are direct consequences of the following expressions of L

$$\Big\langle \Big(\frac{\partial}{\partial y^i}, \theta\Big), (S, \theta) \Big\rangle = \Big\langle \Big(\frac{\delta}{\delta x^i}, \theta\Big), (S, \theta^*) \Big\rangle$$
$$= \Big\langle \Big(\frac{\partial}{\partial y^i}, \theta^*\Big), (\mathbb{C}, \theta) \Big\rangle = \Big\langle \Big(\frac{\delta}{\delta x^i}, \theta^*\Big), (\mathbb{C}, \theta^*) \Big\rangle. \qquad \square$$

Another interesting relation is

$$\Big\langle \Big(\frac{\partial}{\partial y^i}, \theta^*\Big), \Big(\frac{\delta}{\delta x^j}, \theta\Big) \Big\rangle = 0,$$

which means that the distributions $(V(TM), \theta^*)$, $(H(TM), \theta)$ are $\langle\, ,\rangle$-orthogonal on $T^{\mathrm{big}}TM$.

The Courant bracket on $T^{\rm big}TM$ gives the following version of Theorem 5.4 from [**2**, p. 177]:

Proposition 3.2. *Let S be a spray on M. Then S is a Lagrangian vector field, induced by a k-homogeneous Lagrangian, if and only if there exists a $(k-1)$-homogeneous semibasic* 1*-form $\theta \in \Omega^1(TM \setminus \{0\})$ such that*

$$\big[(\mathbb{C},\theta),(S,\theta)\big] = \Big(S,(k-1)\theta + \Big(\frac{k}{2}-1\Big)\mathcal{L}_S\theta\Big).$$

It results in particular that S is:

– Finsler metrizable if and only if there exists a 1-homogeneous semibasic 1-form θ such that $\big[(\mathbb{C},\theta),(S,\theta)\big] = (S,\theta)$,

– projective metrizable if and only if there exists a 0-homogeneous semibasic 1-form θ such that $\big[(\mathbb{C},\theta),(S,\theta)\big] = \big(S,-\frac{1}{2}\mathcal{L}_S\theta\big)$.

Recall [**18**], that $[\,,]$-integrable, $\langle\,,\rangle$-isotropic subbundles of $T^{\rm big}M$ of maximal rank n are called *Dirac structures*; so, our next aim is to derive Dirac structures in the present framework.

With a computation similar to that of the first section we get

$$\mathcal{L}_{\frac{\delta}{\delta x^i}}\theta^* = -R^a_{ij}\theta_a dx^j + \theta_{i|j}\delta y^j$$

and then

$$\Big[\Big(\frac{\delta}{\delta x^i},\theta^*\Big),\Big(\frac{\delta}{\delta x^j},\theta^*\Big)\Big] = \Big(R^k_{ij}\frac{\partial}{\partial y^k},(R^b_{ja}-R^b_{ia})\theta_b dx^a + \big(\theta_{i|u}-\theta_{j|u}\big)\delta y^u\Big).$$

For a Finslerian (particularly Riemannian) spray S the above relation becomes

$$\Big[\Big(\frac{\delta}{\delta x^i},\theta^*\Big),\Big(\frac{\delta}{\delta x^j},\theta^*\Big)\Big] = \Big(R^k_{ij}\frac{\partial}{\partial y^k},(R^b_{ja}-R^b_{ia})\theta_b dx^a\Big).$$

So, for a R-flat Finslerian metric (particularly flat Riemannian metric) the subbundle $(H(TM),\theta^*)$ can be considered as a $\frac{1}{2}$-Dirac structure on $T^{\rm big}TM$.

From

$$\mathcal{L}_{\frac{\delta}{\delta x^i}}dx^j = \mathcal{L}_{\frac{\partial}{\partial y^i}}dx^j = 0,\ \ \mathcal{L}_{\frac{\delta}{\delta x^i}}\delta y^j = -R^j_{ia}dx^a - \frac{\partial N^j_i}{\partial y^a}\delta y^a,\ \ \mathcal{L}_{\frac{\partial}{\partial y^i}}\delta y^j = \frac{\partial N^j_i}{\partial y^a}\delta y^a$$

we compute

$$\begin{aligned}\Big[\Big(\frac{\delta}{\delta x^i},dx^j\Big),\Big(\frac{\delta}{\delta x^k},dx^l\Big)\Big] = \Big(R^a_{ik}\frac{\partial}{\partial y^a},0\Big),\ &\Big[\Big(\frac{\partial}{\partial y^i},\delta y^j\Big),\Big(\frac{\partial}{\partial y^k},\delta y^l\Big)\Big]\\ &= \Big(0,\Big(\frac{\partial N^l_i}{\partial y^a}-\frac{\partial N^j_k}{\partial y^a}\Big)\delta y^a\Big)\end{aligned}$$

so $(H(TM),V^*(TM))$ is Courant closed if and only if the Finslerian spray S (particularly Riemannian metric g) is R-flat (flat) and $\big(V(TM),H^*(TM)\big)$ is always Courant closed.

Another main result of this section is:

Theorem 3.1. *$V_D(S) = (V(TM),V^*(TM))$ is a Dirac structure on $T^{\rm big}TM$. $H_D(S) = (H(TM),H^*(TM))$ is a Dirac structure on $T^{\rm big}TM$ if and only if the*

Finslerian spray S, in particular the Riemannian metric g, is R-flat, in particular is flat.

PROOF. We have

$$\Big\langle\Big(\frac{\partial}{\partial y^i}, dx^j\Big), \Big(\frac{\partial}{\partial y^k}, dx^l\Big)\Big\rangle = 0, \quad \Big\langle\Big(\frac{\delta}{\delta x^i}, \delta y^j\Big), \Big(\frac{\delta}{\delta x^k}, \delta y^l\Big)\Big\rangle = 0$$

respectively

$$\Big[\Big(\frac{\partial}{\partial y^i}, dx^j\Big), \Big(\frac{\partial}{\partial y^k}, dx^l\Big)\Big] = (0, 0),$$

$$\Big[\Big(\frac{\delta}{\delta x^i}, \delta y^j\Big), \Big(\frac{\delta}{\delta x^k}, \delta y^l\Big)\Big] = \Big(R^a_{ik}\frac{\partial}{\partial y^a}, (R^j_{ka} - R^l_{ia})dx^a + \Big(\frac{\partial N^j_k}{\partial y^a} - \frac{\partial N^l_i}{\partial y^a}\Big)\delta y^a\Big)$$

which yields the conclusions. □

It results that an R-flat Finslerian spray or a flat Riemannian metric yields a decomposition of $T^{\mathrm{big}}TM$ in complementary Dirac structures:

$$T^{\mathrm{big}}TM = V_D(S) \oplus H_D(S) \tag{3.1}$$

in analogy with the decomposition of the iterated tangent bundle

$$T(TM) = V(TM) \oplus H(TM).$$

From

$$\Big\langle\Big(\frac{\partial}{\partial y^k}, dx^l\Big), \Big(\frac{\delta}{\delta x^i}, \delta y^j\Big)\Big\rangle = \frac{1}{2}\big(\delta^l_i + \delta^j_k\big)$$

the decomposition (3.1) is not $\langle,\rangle$-orthogonal. Recall, after [**10**] that a pair of complementary Dirac subspaces is called *a reflector*. So, the decomposition (3.1) provides $T^{\mathrm{big}}TM$ with a reflector.

Recall also that if L is a Dirac structure on M, then the triple $(L, [\,,]|_L, pr : L \to TM)$ is a Lie algebroid on M [**6**, p. 645]. Therefore, we get two Lie algebroids over TM provided by a flat Riemannian metric, or more generally R-flat Finsler spray.

4. The relationship with the almost symplectic structure of big TM

In the first section we have a symplectic structure on TM associated with our framework; let us remark that the big tangent bundle has a nondegenerate skew-symmetric 2-form [**18**] $\Omega\big((X,\alpha),(Y,\beta)\big) = \frac{1}{2}\big(\beta(X) - \alpha(Y)\big)$. Our aim in this section is to use this almost symplectic structure of big tangent bundle for our computations.

From

$$\Omega\Big(\Big(\frac{\partial}{\partial y^i}, \theta\Big), \Big(\frac{\partial}{\partial y^j}, \theta\Big)\Big) = 0, \quad \Omega\Big(\Big(\frac{\delta}{\delta x^i}, \theta^*\Big), \Big(\frac{\delta}{\delta x^j}, \theta^*\Big)\Big) = 0$$

it results that $(V(TM), \theta)$ and $(H(TM), \theta^*)$ are Ω-isotropic subbundles of $T^{\mathrm{big}}TM$, while from

$$\Omega\Big(\Big(\frac{\partial}{\partial y^i}, dx^j\Big), \Big(\frac{\partial}{\partial y^k}, dx^l\Big)\Big) = 0, \quad \Omega\Big(\Big(\frac{\delta}{\delta x^i}, \delta y^j\Big), \Big(\frac{\delta}{\delta x^k}, \delta y^l\Big)\Big) = 0$$

we get that the complementary Dirac structures $V_D(S)$ and $H_D(S)$ are Ω-Lagrangian subbundles of $T^{\mathrm{big}}TM$.

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