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SOME REMARKS ON ALMOST MENGER SPACES AND WEAKLY MENGER SPACES

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Abstract. A space X is *almost Menger (weakly Menger)* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \{\overline{V} : V \in \mathcal{V}_n\} = X$ (respectively, $\overline{\bigcup_{n\in\mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\}} = X$). We investigate the relationships among almost Menger spaces, weakly Menger spaces and Menger spaces, and also study topological properties of almost Menger spaces and weakly Menger spaces.

1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *Menger* [**5, 2**] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an open cover of X. As generalization of Menger spaces, Kočinac [**4**] defined a space X to be *almost Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \{\overline{V} : V \in \mathcal{V}_n\} = X$. Pansera [**6**] defined a space X to be *weakly Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\overline{\bigcup_{n\in\mathbb{N}} \bigcup \{V : V \in \mathcal{V}_n\}} = X$. Clearly, every Menger space is almost Menger and every almost Menger space is weakly Menger, but the converses do not hold (see Examples 2.1 and 2.2). On the study of weakly Menger spaces, almost Menger spaces and Menger spaces, the readers can see the references [**2, 3, 4, 5, 6**].

Here we investigate the relationships among almost Menger spaces, weakly Menger spaces and Menger spaces, and also study topological properties of almost Menger spaces and weakly Menger spaces.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal

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is the initial ordinal and an ordinal is the set of smaller ordinals. Every ordinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [**1**].

2. Some examples

In this section, we give some examples showing the relationships among weakly Menger spaces, almost Menger spaces and Menger spaces. Kocev [**3**] showed the following result.

PROPOSITION 2.1. [**3**] *If X is a regular almost Menger space, then X is Menger.*

In the following, we give an example showing that Proposition 2.1 is not true for Urysohn spaces.

EXAMPLE 2.1. There exists an Urysohn almost Menger space X which is not Menger.

PROOF. Let $A = \{a_\alpha : \alpha < \omega_1\}$, $B = \{b_i : i \in \omega\}$ and $Y = \{\langle a_\alpha, b_i\rangle : \alpha < \omega_1, i \in \omega\}$ and let $X = Y \cup A \cup \{a\}$ where $a \notin Y \cup A$. We topologize X as follows: every point of Y is isolated; a basic neighborhood of $a_\alpha \in A$ for each $\alpha < \omega_1$ takes the form $U_{a_\alpha}(i) = \{a_\alpha\} \cup \{\langle a_\alpha, b_j\rangle : j \geqslant i\}$ where $i \in \omega$ and a basic neighborhood of a takes the form $U_a(\alpha) = \{a\} \cup \bigcup \big\{\langle a_\beta, b_i\rangle : \beta > \alpha,\ i \in \omega\big\}$ where $\alpha < \omega_1$. Clearly, X is an Urysohn space. Moreover X is not regular, since the point a can not be separated from the closed set $\{a_\alpha : \alpha < \omega_1\}$. Since $\{a_\alpha : \alpha < \omega_1\}$ is an uncountable discrete closed set of X, X is not Lindelöf, thus X is not Menger, since every Menger space is Lindelöf.

We show that X is almost Menger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. There exists some $U_1 \in \mathcal{U}_1$ such that $a \in U_1$. By the definition of topology of X, there exists a $\beta < \omega_1$ such that $U_a(\beta) \subseteq U_1$, then

$$\{a_\alpha : \alpha > \beta\} \cup \{a\} \cup \{\langle a_\alpha, b_i\rangle : \alpha > \beta, i \in \omega\} \subseteq \overline{U_1}.$$

On the other hand, the subset $C = \bigcup_{\alpha \leqslant \beta} \big(a_\alpha \cup \{\langle a_\alpha, b_i\rangle : i \in \omega\}\big)$ is countable by the definition of X. Thus we may enumerate C as $\{c_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, we can find $U_{n+1} \in \mathcal{U}_{n+1}$ such that $c_n \in U_{n+1}$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{U_n\}$. Then the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is almost Menger. □

For a Tychonoff space X, let βX denote the Čech–Stone compactification of X. Recall that a space X is *almost Lindelöf* [**8**] if for every open cover $\mathcal{U}$ of X there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\big\{\overline{V} : V \in \mathcal{V}\big\} = X$. Clearly, every almost Menger space is almost Lindelöf.

EXAMPLE 2.2. There exists a Tychonoff weakly Menger space which is not almost Menger.

PROOF. Let D be a discrete space of cardinality ω_1, let

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product of βD and $\omega + 1$.

We show that X is weakly Menger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \omega$, $\beta D \times \{n\}$ is compact, there exists a finite subset $\mathcal{V}_{n+1}$ of $\mathcal{U}_{n+1}$ such that $\beta D \times \{n\} \subseteq \bigcup \mathcal{V}_{n+1}$. Thus we get a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\beta D \times \omega \subseteq \bigcup_{n\in\mathbb{N}}(\bigcup \mathcal{V}_n)$. Since $\beta D \times \omega$ is a dense subset of X, $X = \overline{\bigcup_{n\in\mathbb{N}}(\bigcup \mathcal{V}_n)}$, which shows that X is weakly Menger.

To show that X is not almost Menger it is enough to show that X is not almost Lindelöf, since every almost Menger space is almost Lindelöf. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $U_\alpha = \{d_\alpha\} \times (\omega + 1)$. For each $n \in \omega$, let $V_n = \beta D \times \{n\}$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_n : n \in \omega\}$$

of X. It is not difficult to see that $\cup\mathcal{V} = \cup\{\overline{V} : V \in \mathcal{V}\}$ for each a countable subset $\mathcal{V}$ of $\mathcal{U}$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$ and let $\alpha_0 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \omega_1$, since $\mathcal{V}$ is countable. If we pick $\alpha' > \alpha_0$, then $\langle d_{\alpha'}, \omega\rangle \notin \bigcup\{\overline{V} : V \in \mathcal{V}\}$, since $U_{\alpha'}$ is the only element of $\mathcal{U}$ containing $\langle d_{\alpha'}, \omega\rangle$ and $\cup\mathcal{V} = \cup\{\overline{V} : V \in \mathcal{V}\}$. □

Remark 2.1. Pansera [**6**] also constructed an example showing that there exists a Tychonoff weakly Menger space that is not almost Menger [**6**, Example 6]. However we include Example 2.2 here, since it is simpler than his construction and we use it later in the text.

3. Behavior with respect to subspaces, images and products

A subset B of a space X is *regular open (regular closed)* if $B = \overline{B}^o$ (resp., $B = \overline{B^o}$). Kocev [**3**] proved the following result, we include the proof for the sake of completeness.

Proposition 3.1. *A space X is almost Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open subsets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that foe each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup\{\overline{V} : V \in \mathcal{V}_n\} = X$.*

Proof. $\Rightarrow$: This is obvious.

$\Leftarrow$: Let $\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open cover of X. For each $c \in \mathbb{N}$, let $\mathcal{U}'_n = \{\overline{U}^o : n \in \mathbb{N}\}$. Then $(\mathcal{U}'_n$ is a cover of X by regular open subsets. There exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup\{\overline{\overline{V}^o} : V \in \mathcal{V}_n\} = X$. Since $\overline{\overline{V}^o} = \overline{V}$ for open V, thus the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is almost Menger. □

Similar to the proof of Proposition 3.1, we can prove the following result for weakly Menger spaces.

Proposition 3.2. *A space X is weakly Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open subsets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\overline{\bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\}} = X$.*

From Example 2.1, it is not difficult to see that the closed subset of a Urysohn almost Menger space need not be almost Menger. The following example shows that a regular closed subspace of a Urysohn almost Menger spaces need not be almost Menger.

Example 3.1. There exist an Urysohn almost Menger space X having a regular closed subset which is not almost Menger.

Proof. Let S_1 be the same space X of Example 2.1. Then S_1 is almost Menger. Let S_2 be the same space X of Example 2.2. Then S_2 is not almost Menger.

We assume that $S_1 \cap S_2 = \emptyset$. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. Let $\varphi : D \times \{\omega\} \to A$ be a bijection defined by $\varphi(\langle d_\alpha, \omega\rangle) = a_\alpha$ for each $\alpha < \omega_1$. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega\rangle$ with $\varphi(\langle d_\alpha, \omega\rangle)$ for each $\alpha < \omega_1$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map and $Y = \pi(S_2)$. Then Y is a regular closed subset of X, since $\overline{Y^o} = Y$ by the construction of X. Since Y is homeomorphic to S_2, thus Y is not almost Menger.

Now we show X is almost Menger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since $\pi(S_1)$ is almost Menger, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and

$$\pi(S_1) \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \{\overline{V} : V \in \mathcal{V}'_n\}.$$

On the other hand, for each $n \in \omega$, since $\pi(\beta D \times \{n\})$ is a compact subset of X, there exists a finite subfamily $\mathcal{V}''_{n+1}$ of $\mathcal{U}_{n+1}$ such that $\pi(\beta D \times \{n\}) \subseteq \bigcup \mathcal{V}''_n$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \mathcal{V}'_n \cup \mathcal{V}''_n$. Then the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is almost Menger. $\square$

In the following, we give a positive result, which can be easily proved.

Proposition 3.3. *If X is an almost Menger space, then every open and closed subset of X is almost Menger.*

From Example 2.2, it is not difficult to see that a closed subset of a Tychonoff weakly Menger space need not be weakly Menger. However we have the following positive result.

Proposition 3.4. *Every regular closed subset of a weakly Menger space is weakly Menger.*

Proof. Let X be a weakly Menger space and F be a regular closed subset of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of F. For each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$, there exists an open subset $V_{(n,U)}$ of X such that $V_{(n,U)} \cap F = U$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \{V_{(n,U)} : U \in \mathcal{U}_n\} \cup \{X \setminus F\}$, $\mathcal{U}'_n$ is an open cover of X. Then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of open covers of X. There exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and $\overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}'_n} = X$, since X is weakly Menger. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \mathcal{V}'_n \setminus \{X \setminus F\}$. Then $F^o \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n}$.

Hence $F = \overline{F^o} \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n}$, since F is a regular closed subset of X. Thus

$$F = F\cap\overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n} = \mathrm{cl}_F\left(F\cap\left(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n\right)\right) = \mathrm{cl}_F\left(\bigcup_{n\in\mathbb{N}} \bigcup \{F\cap W : W \in \mathcal{W}_n\}\right).$$

For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{W \cap F : W \in \mathcal{W}_n\}$. Then $\mathcal{V}_n$ is a finite subset $\mathcal{U}_n$ and $F = \mathrm{cl}_F(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$, which shows that F is weakly Menger. $\square$

Kocev [**3**] proved the following result.

Proposition 3.5. *A continuous image of an almost Menger space is almost Menger.*

Similar to the Proposition 3.5, we can prove the following result.

Proposition 3.6. *A continuous image of a weakly Menger space is weakly Menger.*

Next we turn to consider preimages. To show that the preimage of an almost Menger (weakly Menger) space under a closed 2-to-1 continuous map need not be almost Menger (respectively, weakly Menger), we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Example 3.2. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is an Urysohn almost Menger space, but $A(X)$ is not almost Menger.

Proof. Let X be the space X of Example 2.1. Then X is almost Menger and has an infinite discrete closed subset $A = \{a_\alpha : \alpha < \omega_1\}$. Hence the Alexandroff duplicate $A(X)$ of X is not almost Menger, since $A \times \{1\}$ is an uncountable infinite discrete, open and closed set in $A(X)$ and every open and closed subset of an almost Menger space is almost Menger. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map. $\square$

If we use Example 2.2 instead of Example 2.1 in Example 3.2, we get the following result.

Example 3.3. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a Tychonoff weakly Menger space, but $A(X)$ is not weakly Menger.

Recall [**7**] that a mapping f from a space X to a space Y is called *almost open* if $f^{-1}(\overline{U}) \subset \overline{f^{-1}(U)}$ for each open subset U of Y.

Proposition 3.7. *If $f : X \to Y$ is an almost open and perfect continuous mapping and Y is an almost Menger space, then X is almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence open covers of X. Then for each $y \in Y$ and each $n \in \mathbb{N}$, there is a finite subfamily $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subset \bigcup \mathcal{U}_{n_y}$. Let $U_{n_y} = \bigcup \mathcal{U}_{n_y}$. Then $V_{n_y} = Y \setminus f(X \setminus U_{n_y})$ is an open neighborhood of y, since f is closed. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$, $\mathcal{V}_n$ is an open cover of Y. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of Y. There exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \{\overline{V} : V \in \mathcal{V}'_n\} = Y$, since Y is almost Menger. Without loss of generality, we may assume that $\mathcal{V}'_n = \{V_{n_{y_i}} : i \leqslant n'\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \bigcup_{i\leqslant n'} \mathcal{U}_{n_{y_i}}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Since f is almost open, then

$$\begin{aligned} X = f^{-1}\Big(\bigcup_{n\in\mathbb{N}} \bigcup \{\overline{V_{n_{y_i}}} : i \leqslant n'\}\Big) &= \bigcup_{n\in\mathbb{N}} \bigcup \{f^{-1}(\overline{V_{n_{y_i}}}) : i \leqslant n'\} \\ \subset \bigcup_{n\in\mathbb{N}} \bigcup \{\overline{f^{-1}(V_{n_{y_i}})} : i \leqslant n'\} &\subset \bigcup_{n\in\mathbb{N}} \bigcup \{\overline{U_{n_{y_i}}} : i \leqslant n'\} \\ = \bigcup_{n\in\mathbb{N}} \bigcup \Big\{\overline{\bigcup \mathcal{U}_{n_{y_i}}} : i \leqslant n'\Big\} &= \bigcup_{n\in\mathbb{N}} \bigcup \{\overline{U} : U \in \mathcal{U}'_n\}. \end{aligned}$$

Hence X is almost Menger. □

Similar to the proof of Proposition 3.7, we can prove the following result.

PROPOSITION 3.8. *If $f : X \to Y$ is an almost open and perfect continuous mapping and Y is a weakly Menger space, then X is weakly Menger.*

It is well known that the product of a Menger space and a compact space is Menger. For almost Menger spaces and weakly Menger spaces, since every open mapping is almost open, thus we have the following results by Propositions 3.7 and 3.8.

PROPOSITION 3.9. *If X is an almost Menger (weakly Menger) space and Y is a compact space, then $X \times Y$ is almost Menger (weakly Menger, respectively).*

It is clear that almost Menger (weakly Menger) property is countably additive. Thus we have the following result by Proposition 3.9.

PROPOSITION 3.10. *If X is an almost Menger (weakly Menger) space and Y is a σ-compact space, then $X \times Y$ is almost Menger (weakly Menger, respectively).*

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