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ON THE GROWTH AND THE ZEROS OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. We investigate the growth of meromorphic solutions of homogeneous and nonhomogeneous higher order linear differential equations

$$f^{(k)}+\sum_{j=1}^{k-1}A_jf^{(j)}+A_0f=0\ \ (k\geqslant 2),$$

$$f^{(k)}+\sum_{j=1}^{k-1}A_jf^{(j)}+A_0f=A_k\ \ (k\geqslant 2),$$

where $A_j(z)$ $(j=0,1,\dots,k)$ are meromorphic functions with finite order. Under some conditions on the coefficients, we show that all meromorphic solutions $f\not\equiv 0$ of the above equations have an infinite order and infinite lower order. Furthermore, we give some estimates of their hyper-order, exponent and hyper-exponent of convergence of distinct zeros. We improve the results due to Kwon; Chen and Yang; Belaïdi; Chen; Shen and Xu.

1. Introduction and statement of results

In this paper, we use the standard notations of Nevanlinna's value distribution theory [**10, 12, 15**]. In addition, we use the notations $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zeros and the poles of a meromorphic function f, $\rho(f)$ and $\mu(f)$ to denote respectively the order and the lower order of f. The hyper-order $\rho_2(f)$, the hyper-exponent $\lambda_2(f)$ of convergence of zeros and the hyper-exponent $\overline{\lambda}_2(f)$ of convergence of distinct zeros of f are defined respectively by

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r},\quad \lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N(r,1/f)}{\log r},$$

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$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r,1/f)}{\log r}.$$

First, we recall the following definitions. The linear measure of a set $E \subset (0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset (1,+\infty)$ is defined by $\operatorname{lm}(F) = \int_1^{+\infty} \frac{1}{t}\chi_F(t)\,dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset (0,+\infty)$ is defined by

$$\overline{\operatorname{dens}}\, E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r}.$$

The upper logarithmic density of a set $F \subset (1,+\infty)$ is defined by

$$\overline{\log\operatorname{dens}}(F) = \limsup_{r\to+\infty} \frac{\operatorname{lm}(F\cap[1,r])}{\log r}.$$

Proposition 1.1. *For all $H \subset [1,+\infty)$ the following statements hold:*
i) If $\operatorname{lm}(H) = \infty$, then $m(H) = \infty$;
ii) If $\overline{\operatorname{dens}}\, H > 0$, then $m(H) = \infty$;
iii) If $\overline{\log\operatorname{dens}}\, H > 0$, then $\operatorname{lm}(H) = \infty$.

Proof. i) Since we have $\chi_H(t)/t \leqslant \chi_H(t)$ for all $t \in H \subset [1,+\infty)$, then

$$m(H) \geqslant \operatorname{lm}(H).$$

We can easily prove the results ii) and iii) by applying the definition of the limit and the properties $m(H\cap[0,r]) \leqslant m(H)$ and $\operatorname{lm}(H\cap[1,r]) \leqslant \operatorname{lm}(H)$. □

In [**11**], Kwon investigated the growth of second order equations and obtained the following result.

Theorem 1.1. [**11**] *Let H be a set of complex numbers satisfying $\overline{\operatorname{dens}}\{|z| : z \in H\} > 0$, and let $A(z)$ and $B(z)$ be entire functions such that for real constants $\alpha\ (>0)$, $\beta\ (>0)$,*

$$|A(z)| \leqslant \exp\{o(1)|z|^{\beta}\} \quad \text{and} \quad |B(z)| \geqslant \exp\{(1+o(1))\alpha|z|^{\beta}\}$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of the equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.1}$$

has infinite order and $\rho_2(f) \geqslant \beta$.

In [**6**], Chen and Yang have studied the growth of solutions of (1.1) and obtained the following result.

Theorem 1.2. [**6**] *Let H be a set of complex numbers satisfying $\overline{\operatorname{dens}}\{|z| : z \in H\} > 0$, and let $A(z)$ and $B(z)$ be entire functions with $\rho(A) \leqslant \rho(B) = \rho < +\infty$ such that for real constant $C\ (>0)$ and for any given $\varepsilon > 0$,*

$$|A(z)| \leqslant \exp\{o(1)|z|^{\rho-\varepsilon}\} \quad \text{and} \quad |B(z)| \geqslant \exp\{(1+o(1))C|z|^{\rho-\varepsilon}\}$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) has infinite order and $\rho_2(f) = \rho(B)$.

These results were improved by Belaïdi in [**1, 2**] by considering more general conditions to higher order linear differential equations with entire coefficients. Recently in [**8**], Chen extended the previous results by studying the zeros and the growth of meromorphic solutions of equation (1.1) and the non-homogeneous equation $f'' + A(z)f' + B(z)f = F$ when $A(z), B(z), F(z)$ are meromorphic functions.

Here we consider for $k \geqslant 2$ the homogeneous and the non-homogeneous linear differential equations

$$f^{(k)} + \sum_{j=1}^{k-1} A_j f^{(j)} + A_0 f = 0, \tag{1.2}$$

$$f^{(k)} + \sum_{j=1}^{k-1} A_j f^{(j)} + A_0 f = A_k, \tag{1.3}$$

where $A_j(z)$ $(j = 0, 1, \dots, k)$ $(A_0 \not\equiv 0$ and $A_k \not\equiv 0)$ are meromorphic functions with finite order. We investigate the zeros and growth of meromorphic solutions of equations (1.2) and (1.3). The present article may be understood as an extension and improvement of Theorems 2.3 and 2.4 in the paper of Shen and Xu [**14**]. We improve the results due to Chen; Shen and Xu greatly and we give two corollaries in the case when $\rho = \max\{\rho(A_j) : j = 1, 2, \dots, k\} < \rho(A_0) = \sigma < 1/2$.

THEOREM 1.3. *Let $H \subset [0, +\infty)$ be a set with infinite linear measure, and let $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be meromorphic functions with finite order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $\rho = \max\{\rho(A_j) : j = 1, 2, \dots, k-1\} < \sigma$ and $|A_0(z)| \geqslant e^{\alpha r^{\sigma}}$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution $f \not\equiv 0$ of equation (1.2) satisfies $\mu(f) = \rho(f) = \infty$ and $\rho_2(f) \geqslant \sigma$. Furthermore, if $\lambda(1/f) < \infty$, then $\sigma \leqslant \rho_2(f) \leqslant \rho(A_0)$.*

REMARK 1.1. By using Proposition 1.1, we can obtain the same results in Theorem 1.3, while putting $H \subset [0, +\infty)$ to be a set with positive upper density (or while putting $H \subset [1, +\infty)$ to be a set with positive upper logarithmic density) (or while putting $H \subset [1, +\infty)$ to be a set with infinite logarithmic measure) instead of putting H to be a set with infinite linear measure.

THEOREM 1.4. *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, \dots, k)$ $(A_k \not\equiv 0)$ be meromorphic functions with finite order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $\rho = \max\{\rho(A_j) : j = 1, 2, \dots, k\} < \sigma$ and $|A_0(z)| \geqslant e^{\alpha r^{\sigma}}$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution f with $\lambda(1/f) < \sigma$ of equation (1.3) is of infinite order and*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty, \quad \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f).$$

Furthermore, if $\lambda(1/f) < \min\{\mu(f), \sigma\}$, then $\rho_2(f) \leqslant \rho(A_0)$.

REMARK 1.2. It is clear that $\rho(A_0) = \beta \geqslant \sigma$ in Theorems 1.3 and 1.4. Indeed, suppose that $\rho(A_0) = \beta < \sigma$. Then, by using Lemma 2.2 of this paper, there exists a set $E_1 \subset (1, +\infty)$ that has a finite linear measure such that when $|z| = r \notin$

$[0,1]\cup E_1$, $r\to+\infty$, we have for any given ε $(0<\varepsilon<\sigma-\beta)$

$$|A_0(z)|\leqslant e^{r^{\beta+\varepsilon}}. \tag{1.4}$$

On the other hand, by the hypotheses of Theorems 1.3 and 1.4, there exist positive constants $\sigma>0$, $\alpha>0$ such that

$$|A_0(z)|\geqslant e^{\alpha r^{\sigma}} \tag{1.5}$$

as $|z|=r\in H$, $r\to+\infty$, where H is a set with $m(H)=\infty$. From (1.4) and (1.5), we obtain for $|z|=r\in H\setminus[0,1]\cup E_1$, $r\to+\infty$

$$e^{\alpha r^{\sigma}}\leqslant|A_0(z)|\leqslant e^{r^{\beta+\varepsilon}}$$

and by ε $(0<\varepsilon<\sigma-\beta)$ this is a contradiction as $r\to+\infty$. Hence $\rho(A_0)=\beta\geqslant\sigma$.

Corollary 1.1. *Let $A_j(z)$ $(j=0,1,\dots,k)$ $(A_k\not\equiv0)$ be meromorphic functions having only finitely many poles such that $\rho=\max\{\rho(A_j):j=1,2,\dots,k\}<\rho(A_0)=\sigma<1/2$. Then every meromorphic solution $f\not\equiv0$ of equation* (1.2) *satisfies $\mu(f)=\rho(f)=\infty$ and $\rho_2(f)=\sigma$. Furthermore, every meromorphic solution f of equation* (1.3) *satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=\infty,\quad \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)$$

and if $\lambda(1/f)<\mu(f)$, then $\rho_2(f)\leqslant\rho(A_0)$.

Corollary 1.2. *Let $A_j(z)$ $(j=0,1,\dots,k)$ $(A_k\not\equiv0)$ be meromorphic functions such that $\lambda(1/A_0)<\mu(A_0)\leqslant\rho(A_0)=\sigma<1/2$ and $\rho=\max\{\rho(A_j):j=1,2,\dots,k\}<\sigma$. Then every meromorphic solution $f\not\equiv0$ of equation* (1.2) *whose all poles are of uniformly bounded multiplicity satisfies $\mu(f)=\rho(f)=\infty$ and $\rho_2(f)=\sigma$. Furthermore, every meromorphic solution f of equation* (1.3) *whose all poles are of uniformly bounded multiplicity satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=\infty,\quad \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)$$

and if $\lambda(1/f)<\mu(f)$, then $\rho_2(f)\leqslant\rho(A_0)$.

2. Auxiliary lemmas

Lemma 2.1. [**9**] *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha>1$, $\varepsilon>0$ be given constants. Then there exists a set $E_0\subset[0,\infty)$ that has finite linear measure and there exists a constant $c>0$, such that for all z satisfying $|z|=r\notin E_0$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leqslant c[T(\alpha r,f)r^{\varepsilon}\log T(\alpha r,f)]^j (j\in\mathbb{N}).$$

Lemma 2.2. [**5**] *Let $f(z)$ be a meromorphic function of order $\rho(f)=\rho<+\infty$. Then for any given $\varepsilon>0$, there exists a set $E_1\subset(1,+\infty)$ that has finite linear measure and finite logarithmic measure such that when $|z|=r\notin[0,1]\cup E_1$, $r\to+\infty$, we have $|f(z)|\leqslant\exp\{r^{\rho+\varepsilon}\}$.*

Let $g(z)=\sum_{n=0}^{\infty}a_n z^n$ be an entire function. We define by $\mu(r)=\max\{|a_n|r^n;\ n=0,1,\cdots\}$ the maximum term of g, and define by $\nu_g(r)=\max\{m;\ \mu(r)=|a_m|r^m\}$ the central index of g.

Lemma 2.3. [**7**] *Let $f(z) = g(z)/d(z)$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions such that*

$$\mu(g) = \mu(f) = \mu \leqslant \rho(g) = \rho(f) \leqslant +\infty,$$
$$\lambda(d) = \rho(d) = \lambda(1/f) = \beta < \mu.$$

Let z be a point with $|z| = r$ at which $|g(z)| = M(r, g)$ and $\nu_g(r)$ denote the central index of $g(z)$. Then there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure $\operatorname{lm}(E_2) < \infty$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, we have

$$\frac{f^{(n)}(z)}{f(z)} = \Big(\frac{\nu_g(r)}{z}\Big)^n (1 + o(1)) \ \ (n \geqslant 1 \text{ is an integer}).$$

Lemma 2.4. [**6**] *Let $g(z)$ be an entire function of infinite order, with the hyper-order $\rho_2(g) = \sigma$. Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_g(r)}{\log r} = \sigma.$$

Lemma 2.5. [**13**] *Let $g(z)$ be an entire function of infinite order. Denote $M(r, g) = \max\{|g(z)| : |z| = r\}$, then for any sufficiently large number $\lambda > 0$, and any $r \in H_0 \subset (1, +\infty)$*

$$M(r, g) > c_1 \exp\{c_2 r^{\lambda}\},$$

where $\operatorname{lm}(H_0) = \infty$ and c_1, c_2 are positive constants.

Lemma 2.6. *Suppose that $k \geqslant 2$ and $h_0, h_1, \dots, h_k$ ($h_0 \not\equiv 0$) are meromorphic functions. Let $\rho = \max\{\rho(h_j) : j = 0, 1, \dots, k\} < \infty$ and let f be a meromorphic solution of infinite order of the equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_0 f = h_k \tag{2.1}$$

with $\lambda(1/f) = \mu < \mu(f)$. Then $\rho_2(f) \leqslant \rho$.

Proof. We assume that f is a meromorphic solution of infinite order $\rho(f) = \infty$ of equation (2.1). We can rewrite (2.1) as

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |h_0| + \sum_{s=1}^{k-1} |h_s| \left|\frac{f^{(s)}}{f}\right| + \left|\frac{h_k}{f}\right|. \tag{2.2}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions such that

$$\mu(f) = \mu(g) \leqslant \rho(f) = \rho(g) = \infty, \lambda(d) = \rho(d) = \lambda(1/f) = \mu < \mu(f)$$
$$\rho_2(f) = \rho_2(g).$$

By Lemma 2.3, there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$ at which $|g(z)| = M(r, g)$, we have

$$\frac{f^{(n)}(z)}{f(z)} = \Big(\frac{\nu_g(r)}{z}\Big)^n (1 + o(1)) \quad (n \geqslant 1). \tag{2.3}$$

Since $\rho(h_k(z)d(z)) \leqslant \rho_1 = \max\{\mu, \rho\}$, then by Lemma 2.2, for any $\varepsilon > 0$ there exists a set $E_1 \subset (1, +\infty)$ with a finite logarithmic measure such that

$$(2.4) \qquad |h_k(z)d(z)| \leqslant e^{r^{\rho_1+\varepsilon}}$$

holds for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, $r \to +\infty$. On the other hand, by Lemma 2.5 for a sufficiently large number $\lambda > \rho_1$, there exist a set $H_0 \subset (1, +\infty)$ with $\mathrm{lm}(H_0) = \infty$ and positive constants c_1, c_2 such that for any $r \in H_0$

$$(2.5) \qquad M(r, g) > c_1 \exp\{c_2 r^{\lambda}\}.$$

By (2.4) and (2.5), for any given ε with $0 < \varepsilon < \lambda - \rho_1$ and for all z satisfying $|z| = r \in H_0 \setminus ([0, 1] \cup E_1)$, $r \to +\infty$ at which $|g(z)| = M(r, g)$, we have

$$(2.6) \qquad \Big|\frac{h_k(z)}{f(z)}\Big| = \Big|\frac{d(z)h_k(z)}{g(z)}\Big| \leqslant \frac{e^{r^{\rho_1+\varepsilon}}}{c_1 e^{c_2 r^{\lambda}}} \to 0, \quad r \to +\infty.$$

Since $\rho = \max\{\rho\ (h_j) : j = 0, 1, \dots, k\} < \infty$, then by Lemma 2.2, we have

$$(2.7) \qquad |h_j(z)| \leqslant e^{r^{\rho+\varepsilon}} \quad (j = 0, 1, \dots, k)$$

holds for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, $r \to +\infty$. Substituting (2.3), (2.6) and (2.7) into (2.2), we obtain for all z satisfying $|z| = r \in H_0 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \to +\infty$ at which $|g(z)| = M(r, g)$,

$$\Big|\frac{\nu_g(r)}{z}\Big|^k |1 + o(1)| \leqslant e^{r^{\rho+\varepsilon}} + \sum_{s=1}^{k-1} e^{r^{\rho+\varepsilon}} \Big|\frac{\nu_g(r)}{z}\Big|^s |1 + o(1)| + o(1).$$

So, we get

$$(2.8) \qquad |\nu_g(r)|^k |1 + o(1)| \leqslant (k + 1)r^k e^{r^{\rho+\varepsilon}} |\nu_g(r)|^{k-1} |1 + o(1)|.$$

Then, by Lemma 2.4, we obtain from (2.8) that $\rho_2(g) = \rho_2(f) \leqslant \rho + \varepsilon$. Since ε $(0 < \varepsilon < \lambda - \rho_1)$ being arbitrary, then we get $\rho_2(f) \leqslant \rho$. $\square$

Lemma 2.7. [**3**] *Let $f(z)$ be an entire function of order ρ, where $0 < \rho(f) = \rho < 1/2$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H_1 \subset [0, +\infty)$ with $\overline{\mathrm{dens}}\, H_1 \geqslant 1-2\rho$ such that $|f(z)| \geqslant \exp\{r^{\rho-\varepsilon}\}$ for all z satisfying $|z| = r \in H_1$.*

Lemma 2.8. [**7**] *Suppose that $h(z)$ is a meromorphic function with $\lambda(1/h) < \mu(h) \leqslant \rho(h) = \sigma < 1/2$. Then for any $\varepsilon > 0$, there exists a set $H_2 \subset (1, +\infty)$ that has a positive upper logarithmic density such that for all z satisfying $|z| = r \in H_2$, we have $|h(z)| \geqslant \exp\{(1 + o(1))r^{\sigma-\varepsilon}\}$.*

Lemma 2.9. [**11**] *Let $g(z)$ be a nonconstant entire function of finite order. Then for any given $\varepsilon > 0$, there exists a set $H_3 \subset [0, +\infty)$ with $\overline{\mathrm{dens}}\, H_3 = 1$ such that $M(r, g) \geqslant \exp\{r^{\rho(g)-\varepsilon}\}$ for all z satisfying $|z| = r \in H_3$.*

Lemma 2.10. [**4**] *Let A_j $(j = 0, 1, \dots, k-1)$, $F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

LEMMA 2.11. *Let $H \subset [0,+\infty)$ be a set with a positive upper density (or of infinite linear measure), and let $h_j(z)$ $(j = 0,1,\dots,k)$ $(h_0 \not\equiv 0)$ be meromorphic functions with finite order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $|h_0(z)| \geqslant e^{\alpha r^\sigma}$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\{\rho(h_j) : j = 1,2,\dots,k\} < \sigma$, then every meromorphic solution $f \not\equiv 0$ of equation*

$$f^{(k)} + \sum_{j=1}^{k-1} h_j f^{(j)} + h_0 f = h_k \quad (k \geqslant 2), \tag{2.9}$$

is transcendental and satisfies $\rho(f) \geqslant \sigma$.

PROOF. Assume that $f \not\equiv 0$ is a meromorphic solution of (2.9) with $\rho(f) < \sigma$. It follows from (2.9) that

$$\frac{h_k}{f} - \frac{f^{(k)}}{f} - \sum_{j=1}^{k-1} h_j \frac{f^{(j)}}{f} = h_0. \tag{2.10}$$

Since $\rho(h_j) < \sigma$ $(j = 1,2,\dots,k)$ and $\rho(f) < \sigma$, then from (2.10) we obtain that the order of growth of h_0 is $\rho_1 = \rho(h_0) \leqslant \max\{\rho, \rho(f)\} < \sigma$. By Lemma 2.2, for any ε $(0 < \varepsilon < \sigma - \rho_1)$ there exists a set $E_1 \subset (1,+\infty)$ with a finite linear measure such that

$$|h_0(z)| \leqslant e^{r^{\rho_1+\varepsilon}} \tag{2.11}$$

holds for all z satisfying $|z| = r \notin [0,1] \cup E_1$, $r \to +\infty$. From the hypotheses of Lemma 2.11, there exists a set H with $\overline{\operatorname{dens}}\, H > 0$ (or $m(H) = \infty$), and there exist positive constants $\sigma > 0$, $\alpha > 0$ such that

$$|h_0(z)| \geqslant e^{\alpha r^\sigma} \tag{2.12}$$

holds for all z satisfying $|z| = r \in H$, $r \to +\infty$. By (2.11) and (2.12), we conclude that for all z satisfying $|z| = r \in H \setminus ([0,1] \cup E_1)$, $r \to +\infty$, we have $e^{\alpha r^\sigma} \leqslant e^{r^{\rho_1+\varepsilon}}$ and by ε $(0 < \varepsilon < \sigma - \rho_1)$ this is a contradiction as $r \to +\infty$. Consequently, any meromorphic solution $f \not\equiv 0$ of equation (2.9) is transcendental and satisfies $\rho(f) \geqslant \sigma$. □

3. Proofs of the results

PROOF OF THEOREM 1.3. Let $f \not\equiv 0$ be a meromorphic solution of (1.2). It follows from (1.2) that

$$|A_0| \leqslant \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1} |A_j| \left|\frac{f^{(j)}}{f}\right|. \tag{3.1}$$

By Lemma 2.11, we know that f is transcendental. By using Lemma 2.1, there is a set $E_0 \subset (0,+\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r[T(2r,f)]^{2k} \ (j = 1,2,\dots,k). \tag{3.2}$$

By Lemma 2.2, for any given ε $(0<\varepsilon<\sigma-\rho)$ there exists a set $E_1 \subset (1,+\infty)$ with finite linear measure such that

$$|A_j(z)| \leqslant e^{r^{\rho+\varepsilon}},\ j=1,2,\dots,k-1 \tag{3.3}$$

holds for all z satisfying $|z|=r\notin[0,1]\cup E_1$, $r\to+\infty$. Also, by the hypotheses of Theorem 1.3, there exists a set H with $m(H)=\infty$, such that for all z satisfying $|z|=r\in H$, $r\to+\infty$, we have

$$|A_0(z)| \geqslant e^{\alpha r^{\sigma}}. \tag{3.4}$$

Hence it follows from (3.1), (3.2), (3.3) and (3.4) that for all z satisfying $|z|=r\in H\setminus([0,1]\cup E_0\cup E_1)$, $r\to+\infty$, we have

$$e^{\alpha r^{\sigma}} \leqslant r[T(2r,f)]^{2k} + \sum_{j=1}^{k-1} e^{r^{\rho+\varepsilon}} r[T(2r,f)]^{2k} \leqslant kre^{r^{\rho+\varepsilon}}[T(2r,f)]^{2k}. \tag{3.5}$$

By $0<\varepsilon<\sigma-\rho$, it follows from (3.5) that

$$\mu(f)=\rho(f)=\infty \text{ and } \rho_2(f)\geqslant\sigma. \tag{3.6}$$

Furthermore, if $\lambda(1/f)<\infty$, then f is a meromorphic solution of (1.2) with $\rho(f)=\mu(f)=\infty$, $\lambda(1/f)<\mu(f)$ and by Remark 1.2, we have $\max\{\rho(A_j): j=0,1,\dots,k-1\}=\rho(A_0)=\beta<\infty$. Thus, by Lemma 2.6, we get

$$\rho_2(f)\leqslant\rho(A_0). \tag{3.7}$$

By (3.6) and (3.7), we conclude that $\sigma\leqslant\rho_2(f)\leqslant\rho(A_0)$. $\square$

Proof of Theorem 1.4. Let f be a meromorphic solution of (1.3). Assume that $\rho(f)<\infty$. It follows from (1.3) that

$$|A_0| \leqslant \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}|A_j|\left|\frac{f^{(j)}}{f}\right| + \left|\frac{A_k}{f}\right|. \tag{3.8}$$

By Lemma 2.11, we know that f is transcendental with $\rho(f)\geqslant\sigma$. By the hypothesis $\lambda(1/f)<\sigma$ and Hadamard factorization theorem, we can write f as $f(z)=\frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with $\lambda(d)=\rho(d)=\lambda(1/f)<\sigma$, $\rho(f)=\rho(g)\geqslant\sigma$. By Lemma 2.9, for any given ε $(0<\varepsilon<\rho(f))$, there exists a set $H_3\subset[0,+\infty)$ with $\overline{\operatorname{dens}}\,H_3=1$ such that

$$M(r,g)\geqslant e^{r^{\rho(g)-\varepsilon}} \tag{3.9}$$

holds for all z satisfying $|z|=r\in H_3$. By (3.9), for all z satisfying $|z|\in H_3$ at which $|g(z)|=M(r,g)$, we get

$$|g(z)|\geqslant 1. \tag{3.10}$$

Then, by (3.10), we obtain

$$\left|\frac{A_k(z)}{f(z)}\right| = \left|\frac{d(z)A_k(z)}{g(z)}\right| \leqslant |d(z)A_k(z)|.$$

We have $\rho(d(z)A_k(z)) < \sigma$ and set

$$\rho_1 = \max\{\rho(A_j)(j = 1, 2, \ldots, k),\ \rho(d)\} < \sigma,$$

hence by using similar arguments as in the proof of Theorem 1.3, for any given ε $(0 < \varepsilon < \sigma - \rho_1)$ there exists a set $H_4 = H \cap H_3 \subset [0, +\infty)$ with positive upper density such that for all z satisfying $|z| = r \in H_4 \setminus ([0, 1] \cup E_0 \cup E_1)$, $r \to +\infty$, at which $|g(z)| = M(r, g)$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leqslant r[T(2r, f)]^{2k} \quad (j = 1, 2, \ldots, k), \tag{3.11}$$

$$\Big|\frac{A_k(z)}{f(z)}\Big| \leqslant e^{r^{\rho_1+\varepsilon}}, \tag{3.12}$$

$$|A_j(z)| \leqslant e^{r^{\rho_1+\varepsilon}}\ (j = 1, 2, \ldots, k - 1), \tag{3.13}$$

$$|A_0(z)| \geqslant e^{\alpha r^{\sigma}}. \tag{3.14}$$

Substituting (3.11)–(3.14) into (3.8), we obtain for all z satisfying $|z| = r \in H_4 \setminus ([0, 1] \cup E_0 \cup E_1)$, $r \to +\infty$, at which $|g(z)| = M(r, g)$, that

$$\begin{aligned} e^{\alpha r^{\sigma}} &\leqslant r[T(2r, f)]^{2k} + \sum_{j=1}^{k-1} e^{r^{\rho_1+\varepsilon}} r[T(2r, f)]^{2k} + e^{r^{\rho_1+\varepsilon}} \\ &\leqslant (k + 1)r[T(2r, f)]^{2k} e^{r^{\rho_1+\varepsilon}}. \end{aligned} \tag{3.15}$$

Hence by (3.15), we have $\rho(f) = \infty$. This is a contradiction which means that the assumption of $\rho(f) < \infty$ is not true. Hence, we conclude that $\rho(f) = \infty$.

Since $A_k \not\equiv 0$, then by Lemma 2.10, we obtain $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$. By (1.3), it is easy to see that if f has a zero at z_0 of order m, $m > k$, and A_j $(j = 0, 1, \ldots, k - 1)$ are analytic at z_0, then A_k must have a zero at z_0 of order $m - k$. Therefore, we get by $A_k \not\equiv 0$ that

$$N\Big(r, \frac{1}{f}\Big) \leqslant k\overline{N}\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{A_k}\Big) + \sum_{j=0}^{k-1} N(r, A_j). \tag{3.16}$$

On the other hand, (1.3) may be rewritten as

$$\frac{1}{f} = \frac{1}{A_k}\Big[\frac{f^{(k)}}{f} + \sum_{j=1}^{k-1} A_j \frac{f^{(j)}}{f} + A_0\Big].$$

So, we get

$$m\Big(r, \frac{1}{f}\Big) \leqslant m\Big(r, \frac{1}{A_k}\Big) + \sum_{j=0}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\Big(r, \frac{f^{(j)}}{f}\Big) + O(1). \tag{3.17}$$

By (3.16) and (3.17), we have

$$T\Big(r,\frac{1}{f}\Big) \leqslant k\overline{N}\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{A_k}\Big) + \sum_{j=0}^{k-1} N(r,A_j) + m\Big(r,\frac{1}{A_k}\Big) + \sum_{j=0}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\Big(r,\frac{f^{(j)}}{f}\Big) + O(1). \tag{3.18}$$

By (3.18) and the lemma of logarithmic derivative [**10**], there exists a set E of r of a finite linear measure such that for all $r \notin E$, we have

$$T(r,f) = T\Big(r,\frac{1}{f}\Big) + O(1) \leqslant k\overline{N}\Big(r,\frac{1}{f}\Big) + \sum_{j=0}^{k} T(r,A_j) + C\log(rT(r,f)), \tag{3.19}$$

where C is a positive constant. For sufficiently large r, we have

$$C\log(rT(r,f)) \leqslant \frac{1}{2}T(r,f). \tag{3.20}$$

$$T(r,A_j) \leqslant r^{\alpha+\varepsilon} \quad (j = 0,1,\dots,k), \tag{3.21}$$

where $\alpha = \max\{\rho(A_j)\ (j = 0,1,\dots,k)\}$. Then, for $r \notin E$ sufficiently large, by using (3.20) and (3.21), we conclude from (3.19) that for $r \notin E$ and r sufficiently large

$$T(r,f) \leqslant k\overline{N}\Big(r,\frac{1}{f}\Big) + (k+1)r^{\alpha+\varepsilon} + \frac{1}{2}T(r,f).$$

So, we get

$$T(r,f) \leqslant 2k\overline{N}\Big(r,\frac{1}{f}\Big) + 2(k+1)r^{\alpha+\varepsilon}, \quad r \notin E. \tag{3.22}$$

Hence, by (3.22), we get $\rho_2(f) \leqslant \overline{\lambda}_2(f)$ and then $\lambda_2(f) \geqslant \overline{\lambda}_2(f) \geqslant \rho_2(f)$. Since by definition, we have $\overline{\lambda}_2(f) \leqslant \lambda_2(f) \leqslant \rho_2(f)$, therefore

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f). \tag{3.23}$$

Furthermore, if $\lambda(1/f) < \min\{\mu(f),\sigma\}$, then f is a meromorphic solution of (1.3) with $\rho(f) = \infty$, $\lambda(1/f) < \mu(f)$ and by Remark 1.2, we have $\max\{\rho(A_j) : j = 0,1,\dots,k\} = \rho(A_0) = \beta < \infty$. Thus, by Lemma 2.6, we get

$$\rho_2(f) \leqslant \beta. \tag{3.24}$$

By (3.23) and (3.24), we conclude that $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \rho(A_0)$. □

Proof of Corollary 1.1. Since A_0 is a meromorphic function with finitely many poles and $\rho(A_0) = \sigma$, then by the Hadamard factorization theorem, we can write A_0 as $A_0(z) = \frac{B(z)}{P(z)}$, where $B(z)$ is an entire function with $\rho(A_0) = \rho(B) = \sigma$ and $P(z)$ is a polynomial. Hence, by Lemma 2.7, for any given ε $(0 < \varepsilon < \sigma)$, there exists a set $H_1 \subset [0,+\infty)$ with $\overline{\operatorname{dens}} H_1 \geqslant 1 - 2\sigma > 0$ (by Proposition 1.1 we have

$m(H_1) = \infty$) such that $|B(z)| \geqslant e^{r^{\sigma-\varepsilon}}$ holds for all z, $|z| = r \in H_1$ and $r \to +\infty$. Hence

$$|A_0(z)| = |\frac{B(z)}{P(z)}| \geqslant \frac{e^{r^{\sigma-\varepsilon}}}{cr^m} \geqslant e^{r^{\sigma-2\varepsilon}}, \tag{3.25}$$

where $c > 0$ is a constant and $m = \deg P(z) \geqslant 1$. Since A_j $(j = 0, 1, \ldots, k)$ are meromorphic functions having finitely many poles and since the poles of f can only occur at the poles of A_j $(j = 0, 1, \ldots, k)$, then f must have finitely many poles. Therefore, $\lambda(1/f) = 0$. Then, for any given ε with $0 < 2\varepsilon < \sigma - \rho$, we have (3.25) and

$$\rho = \max\{\rho(A_j) : j = 1, 2, \ldots, k\} < \sigma - 2\varepsilon. \tag{3.26}$$

Furthermore, all meromorphic solutions $f \not\equiv 0$ of (1.2) or (1.3) satisfy

$$\lambda(1/f) < \sigma - 2\varepsilon. \tag{3.27}$$

Then, from (3.25), (3.26) and (3.27) by application of Theorem 1.3 for equation (1.2), we find that every meromorphic solution $f \not\equiv 0$ of (1.2) satisfies $\mu(f) = \rho(f) = \infty$ and $\sigma - 2\varepsilon \leqslant \rho_2(f) \leqslant \sigma$. By ε $(0 < 2\varepsilon < \sigma - \rho)$ being arbitrary, we obtain $\rho_2(f) = \sigma$. Furthermore, we conclude from Theorem 1.4 that every meromorphic solution f of equation (1.3) satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f)$ and if $\lambda(1/f) < \mu(f)$, then $\rho_2(f) \leqslant \rho(A_0)$. □

Proof of Corollary 1.2. Since A_0 is a meromorphic function with $\lambda(\frac{1}{A_0}) < \mu(A_0) \leqslant \rho(A_0) = \sigma < \frac{1}{2}$, then by Lemma 2.8, for any ε $(0 < \varepsilon < \sigma)$, there exists a set $H_2 \subset (1, +\infty)$ that has $\overline{\log dens} H_2 > 0$ (by Proposition 1.1 we have $m(H_2) = \infty$) such that for all z, $|z| = r \in H_2$, we have

$$|A_0(z)| \geqslant e^{(1+o(1))r^{\sigma-\varepsilon}}. \tag{3.28}$$

Obviously, the poles of f must occur at the poles of A_j $(j = 0, 1, \ldots, k)$, note that all poles of f are of uniformly bounded multiplicity, then by $\lambda(1/A_0) < \sigma$ and $\rho = \max\{\rho(A_j) : j = 1, 2, \ldots, k\} < \sigma$, we have $\lambda(\frac{1}{f}) < \sigma$. Then, for a given ε with $0 < \varepsilon < \min\{\sigma - \rho,\ \sigma - \lambda(\frac{1}{f})\}$, we have (3.28) and

$$\rho = \max\{\rho(A_j) : j = 1, 2, \ldots, k\} < \sigma - \varepsilon \quad \text{and} \quad \lambda(1/f) < \sigma - \varepsilon. \tag{3.29}$$

Then by (3.28) and (3.29) with application of Theorem 1.3 for equation (1.2), we find that every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicity of equation (1.2) satisfies $\mu(f) = \rho(f) = \infty$ and $\sigma - \varepsilon \leqslant \rho_2(f) \leqslant \sigma$. Since ε $(0 < \varepsilon < \min\{\sigma - \rho,\ \sigma - \lambda(1/f)\})$ being arbitrary, we obtain $\rho_2(f) = \sigma$. Furthermore, with application of Theorem 1.4 for equation (1.3), we find that every meromorphic solution f whose poles are of uniformly bounded multiplicity of equation (1.3) satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty, \ \ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f)$$

and if $\lambda(1/f) < \mu(f)$, then $\rho_2(f) \leqslant \rho(A_0)$. □

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