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GRADIENT RICCI SOLITONS ON ALMOST KENMOTSU MANIFOLDS

Yaning Wang, Uday Chand De, and Ximin Liu

Abstract. If the metric of an almost Kenmotsu manifold with conformal Reeb foliation is a gradient Ricci soliton, then it is an Einstein metric and the Ricci soliton is expanding. Moreover, let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution and $h \neq 0$. If the metric g of M^{2n+1} is a gradient Ricci soliton, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, also, the Ricci soliton is expanding with $\lambda = 4n$.

1. Introduction

In 1972, Kenmotsu in [**12**] introduced a special class of almost contact metric manifolds, which is known as Kenmotsu manifolds nowadays. Since then, many authors have investigated Kenmotsu manifolds by using various meaningful geometric conditions. Almost Kenmotsu manifolds were first introduced by Janssens and Vanhecke in [**11**], generalizing the class of Kenmotsu manifolds. Recently, almost Kenmotsu manifolds were investigated by some authors in [**5, 6, 13, 14**].

On the other hand, in 1982, Hamilton in [**9**] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial_t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

A Ricci soliton (see [**10**]) is a generalization of the Einstein metric (that is, the Ricci tensor is a constant multiple of the Riemannian metric g) and is defined on

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a Riemannian manifold (M, g) by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} + \lambda g = 0 \tag{1.2}$$

for some constant λ and a vector field V. Clearly, a Ricci soliton with V zero or a Killing vector field reduces to an Einstein equation. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Compact Ricci solitons are the fixed points of the Ricci flow projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.2) becomes

$$\nabla\nabla f = \mathrm{Ric} + \lambda g. \tag{1.3}$$

Following Perelman [**15**], we know that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

Ricci solitons on contact metric manifods, three-dimensional trans-Sasakian manifolds and $N(k)$-quasi-Einstein manifolds were studied by Ghosh [**8**], Turan, De, and Yildiz [**18**] and Crasmareanu [**3**], respectively. With regard to the studies of Ricci solitons on Kenmotsu manifolds, we refer the reader to De and Matsuyama [**4**] and Ghosh [**7**], respectively. Moreover, Ricci solitons on f-Kenmotsu manifolds were studied by Călin and Crasmareanu [**2**]. As far as we know, there are no studies on Ricci solitons on almost Kenmotsu manifolds. The object of this paper is to investigate gradient Ricci solitons on almost Kenmotsu manifolds under some geometric conditions. In fact, we mainly obtain the following results in Section 3.

Theorem 1.1. *If the metric of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation is a gradient Ricci soliton, then one of the following cases occurs:*

case 1*: $n = 1$, M^3 is a three dimensional Kenmotsu manifold of constant sectional curvature -1 and the gradient Ricci soliton is expanding with $\lambda = 2$;*

case 2*: $n > 1$, M^{2n+1} is an Einstein manifold with the Ricci operator $Q = -2n\,\mathrm{id}$ and the gradient Ricci soliton is expanding with $\lambda = 2n$.*

The above theorem is a generalization of Theorem 4.1 of [**4**] (see Corollary 3.3 in Section 3). Moreover, gradient Ricci solitons on almost Kenmotsu manifolds with ξ belonging to certain nullity distribution and $h \neq 0$ are classified as follows:

Theorem 1.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution and $h \neq 0$. If the metric g of M^{2n+1} is a gradient Ricci soliton, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold. Moreover, the Ricci soliton is expanding with $\lambda = 4n$ and the gradient of the potential function is an eigenvector field of h' with eigenvalue -1.*

The paper is organized as follows. In Section 2, we recall some well known basic formulas and properties of almost Kenmotsu manifolds. In Section 3, by applying some results proved by Pastore and Saltarelli, we completely classify the gradient Ricci solitons on almost Kenmotsu manifolds with conformal Reeb foliation.

Finally, by using some results obtained by Dileo and Pastore, we also obtain the classification of gradient Ricci solitons on almost Kenmtsu manifolds with $h \neq 0$ whose Reeb vector field ξ belongs to the $(k,\mu)'$-nullity distribution.

2. Almost Kenmotsu manifolds

Following [**5, 6, 12**], we first recall some basic notions and properties of almost Kenmotsu manifolds. An almost contact structure (see Blair [**1**]) on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ,ξ,η), where ϕ is a $(1,1)$-type tensor field, ξ a global vector field (which is called the characteristic vector field) and η a 1-form, such that

$$\phi^2 = -\operatorname{id} + \eta\otimes\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity mapping. This implies that $\phi(\xi)=0$, $\eta\circ\phi=0$ and $\operatorname{rank}(\phi)=2n$. A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ,ξ,η) if $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$for any vector fields $X,Y\in\Gamma(TM)$, where $\Gamma(TM)$ denotes the Lie algebra of all differentiable vector fields on M^{2n+1}.

An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form Φ of an almost contact metric structure is defined by $\Phi(X,Y)=g(X,\phi Y)$ for any vector fields X and Y on M^{2n+1}. We may define on the product manifold $M^{2n+1}\times\mathbb{R}$ an almost complex structure J by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1}\times\mathbb{R}$. From Blair [**1**], the normality of an almost contact structure is expressed by the vanishing of the tensor $N_\phi=[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (see [**11**]). It is known [**12**] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I\times_f M^{2n}$ (where M^{2n} is a Kählerian manifold, I is an open interval with coordinate t and $f=ce^t$ for some positive constant c).

Now let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold. We consider two tensor fields $l=R(\cdot\,,\xi)\xi$ and $h=\frac{1}{2}\mathcal{L}_\xi\phi$ on M^{2n+1}, where R denotes the curvature tensor and $\mathcal{L}$ is the Lie differentiation. Following [**13**], the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi=0,\ \ l\xi=0,\ \ \operatorname{tr}h=0,\ \ \operatorname{tr}(h\phi)=0,\ \ h\phi+\phi h=0. \tag{2.2}$$

We also have the following formulas presented in [**5, 6, 13**]:

$$\nabla_X\xi = -\phi^2X-\phi hX\ (\Rightarrow \nabla_\xi\xi=0), \tag{2.3}$$

$$\phi l\phi - l = 2(h^2-\phi^2), \tag{2.4}$$

$$\operatorname{tr}(l) = S(\xi,\xi) = g(Q\xi,\xi) = -2n-\operatorname{tr}h^2, \tag{2.5}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.6}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where S, Q and ∇ denote the Ricci curvature tensor, the Ricci operator with respect to g and the Levi-Civita connection of g, respectively.

On an almost contact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha \operatorname{id} + \beta\eta \otimes \xi,$$

where both α and β are smooth functions, then M is called an η-Einstein manifold. Clearly, an η-Einstein manifold with $\beta = 0$ and α a constant is an Einstein manifold. An η-Einstein manifold is said to be proper η-Einstein if $\beta \neq 0$.

3. Proofs of main results

We assume that the metric g of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a gradient soliton; then equation (1.3) becomes

$$\nabla_Y Df = QY + \lambda Y \tag{3.1}$$

for any $Y \in \Gamma(TM)$, where D denotes the gradient operator of g. It follows from (3.1) that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$. Before giving the detailed proof of our Theorem 1.1, we first present the following result which is directly deduced from Theorem 5.1 and Remark 5.1 of [**14**]. Throughout this paper, we denote by $\mathcal{D}$ the distribution defined by $\mathcal{D} = \ker \eta$.

LEMMA 3.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an η-Einstein almost Kenmotsu manifold with $h = 0$, then one of the following cases occurs:*

case 1: $n = 1$, the Ricci operator of M^3 is $Q = -(\beta + 2)\operatorname{id} + \beta\eta \otimes \xi$ and $\xi(\beta) = -2\beta$;

case 2: $n > 1$, $\beta = 0$, the Ricci operator is $Q = -2n \operatorname{id}$ and the integral submanifolds of the distribution $\mathcal{D}$ are Einstein almost Kählerian Ricci-flat hypersurfaces;

case 3: $n > 1$, β is not a constant, $X(\beta) = 0$ for any $X \in \mathcal{D}$ and $\xi(\beta) = -2\beta$. Hence, the Ricci operator is $Q = -2n \operatorname{id} + \beta\phi^2$, where β is locally given by $\beta = ce^{-2t}$ for some constant $c \neq 0$.

From Lemma 3.1 we see that for an η-Einstein almost Kenmotsu manifold M^{2n+1} with $h = 0$, if either α or β is a constant then M^{2n+1} is an Einstein manifold with $Q = -2n \operatorname{id}$.

PROOF OF THEOREM 1.1. From Pastore and Saltarelli [**14**], we see that the Reeb foliation of an almost Kenmotsu manifold is conformal if and only if $h = 0$. Using $h = 0$ in (2.3) and (2.6) we have

$$\nabla_X \xi = X - \eta(X)\xi \tag{3.3}$$

for any $Y \in \Gamma(TM)$ and

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{3.4}$$

for any $X, Y \in \Gamma(TM)$, respectively. Replacing X by ξ in (3.4) gives an equation; then taking the inner product of the resulting equation with Df and making use of the curvature properties, we obtain

$$g(R(\xi, Y)Df, \xi) = -Y(f) + \eta(Y)\xi(f) \tag{3.5}$$

for any $Y \in \Gamma(TM)$. On the other hand, by a straightforward calculation, we may obtain from (3.2) that

$$g(R(\xi, Y)Df, \xi) = 0 \tag{3.6}$$

for any $Y \in \Gamma(TM)$, where $Q\xi = -2n\xi$ (which is deduced from equation (3.4)) and (3.3) are used. Clearly, from (3.5) and (3.6) we have

$$Df = \xi(f)\xi.$$

Substituting the above equation into (3.1) gives

$$QY = (\xi(f) - \lambda)Y + (Y(\xi(f)) - \xi(f)\eta(Y))\xi \tag{3.7}$$

for any $Y \in \Gamma(TM)$. Then taking the inner product of relation (3.7) with ξ and making use of $Q\xi = -2n\xi$ we obtain

$$Y(\xi(f)) = (\lambda - 2n)\eta(Y) \tag{3.8}$$

for any $Y \in \Gamma(TM)$. Finally, putting (3.8) into (3.7) we get

$$Q = (\xi(f) - \lambda)\,\mathrm{id} + (\lambda - \xi(f) - 2n)\eta \otimes \xi \tag{3.9}$$

This means that M^{2n+1} is an η-Einstein manifold. In this context, it follows from Lemma 3.1 that $\xi(\beta) = -2\beta$ for $n \geqslant 1$, applying this equation on (3.9) we have $\xi(\xi(f)) = 2(\lambda - 2n - \xi(f))$. On the other hand, replacing Y by ξ in equation (3.8) implies that $\xi(\xi(f)) = \lambda - 2n$. Consequently, it follows that $\xi(f) = \frac{\lambda - 2n}{2}$ being a constant. This means that case 3 of Lemma 3.1 can not occur. Taking into account equation (3.8) we obtain $\lambda = 2n$ and hence $\xi(f) = 0$, then we see from (3.9) that $Q = -2n\,\mathrm{id}$. It is well known that the curvature tensor of a three dimensional Riemannian manifold (M^3, g) is given by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X \\ & - g(QX, Z)Y - \frac{r}{2}\big(g(Y, Z)X - g(X, Z)Y\big) \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, where r denotes the scalar curvature of M^3. It is also well known that an almost Kenmotsu manifold of dimension 3 with $h = 0$ is a Kenmotsu manifold (see [**5**]). Then putting $Q = -2\,\mathrm{id}$ in the above equation proves case 1 of Theorem 1.1. The proof of case 2 of Theorem 1.1 follows from the above arguments. □

Remark 3.1. We observe from [**6, 14**] that for an almost Kenmotsu manifold, the following four conditions are equivalent: (1) the Reeb foliation is conformal; (2) ξ belongs to the k-nullity distribution; (3) ξ belongs to the (k, μ)-nullity distribution; (4) the tensor field h vanishes. Therefore, for an almost Kenmotsu manifold whose Riemanian metric is a gradient Ricci soliton, under one of the above four conditions, the conclusion of Theorem 1.1 still holds.

From Proposition 3.2 of [**5**], we know that an almost Kenmotsu manifold is a Kenmotsu manifold if and only if the integral manifolds of the distribution $\mathcal{D}$ are Kählerian and $h=0$. Thus, the following result follows from Theorem 1.1.

Corollary 3.1. *If the metric g of a Kenmotsu manifold M^{2n+1} is a gradient Ricci soliton, then M^{2n+1} is an Einstein manifold with $Q=-2n\,\mathrm{id}$ and the gradient Ricci soliton is expanding with $\lambda=2n$.*

We remark that Corollary 3.1 is just Theorem 4.1 of [**4**], which means that our Theorem 1.1 extends the corresponding results shown in [**4**].

In what follows, we shall consider almost Kenmotsu manifolds for which ξ belongs to the $(k,\mu)'$-nullity condition (see [**6**]), that is,

$$R(X,Y)\xi = k(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)h'X-\eta(X)h'Y) \tag{3.10}$$

for any $X,Y\in\Gamma(TM)$ where $k,\mu\in\mathbb{R}$ and $h'=h\circ\phi$. It follows from Proposition 4.1 of [**6**] that, for an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution, $\mu=-2$. Replacing Y by ξ in (3.10) gives $lX=k(X-\eta(X)\xi)+\mu h'X$, then making use of (2.1) and (2.2) we get $\phi l\phi X=-k(X-\eta(X)\xi)+\mu h'X$. Substituting this equation into (2.4) gives

$$h'^2=(k+1)\phi^2\ (\Leftrightarrow h^2=(k+1)\phi^2). \tag{3.11}$$

Let $X\in\mathcal{D}$ be an eigenvector field of h' orthogonal to ξ with the corresponding eigenvalue γ, thus from (3.10) we have that $\gamma^2=-(k+1)$. It follows that $k\leqslant -1$ and $\gamma=\pm\sqrt{-k-1}$. It is known [**6**] that M^{2n+1} is locally isometric to the warped product of an $(n+1)$-dimensional hyperbolic space of constant sectional curvature $k-2\gamma$ and a flat n-dimensional space. By Proposition 4.2 of [**6**], if $h'\neq 0$ ($\Leftrightarrow h\neq 0$), the present first and third authors proved the following result.

Lemma 3.2 (Lemma 3.2 of [**20**]). *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h'\neq 0$. Then the Ricci operator Q of M^{2n+1} is given by*

$$Q=-2n\,\mathrm{id}+2n(k+1)\eta\otimes\xi-2nh', \tag{3.12}$$

where $k<-1$, moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Proof of Theorem 1.2. By applying Lemma 3.2 we have

$$\begin{aligned}(\nabla_Y Q)X &= 2n(k+1)\eta(X)(Y+h'Y)-2n(\nabla_Y h')X\\ &\quad + 2n(k+1)(g(X,Y)-2\eta(X)\eta(Y)+g(h'X,Y))\xi\end{aligned}$$

for any $X,Y\in\Gamma(TM)$. It follows from the above equation that

$$\begin{aligned}(\nabla_X Q)Y-(\nabla_Y Q)X = &-2n((\nabla_X h')Y-(\nabla_Y h')X)\\ &-2n(k+1)(\eta(X)(Y+h'Y)-\eta(Y)(X+h'X))\end{aligned} \tag{3.13}$$

for any $X,Y\in\Gamma(TM)$. Replacing X by ξ in (3.10) gives an equation, then taking the inner product of the resulting equation with Df and making use of the curvature properties we obtain

$$g(R(\xi,Y)Df,\xi)=kg(Df-\xi(f)\xi,Y)-2g(Df,h'Y) \tag{3.14}$$

for any $Y \in \Gamma(TM)$, where $\mu = -2$ is used. On the other hand, replacing X by ξ in (3.2) and making use of (3.13) we see

$$R(\xi, Y)Df = -2n((\nabla_\xi h')Y - (\nabla_Y h')\xi) - 2n(k+1)(Y + h'Y - \eta(Y)\xi)$$

for any $Y \in \Gamma(TM)$. Hence, it follows from the above equation and (2.2)-(2.3) that

$$g(R(\xi, Y)Df, \xi) = 0 \tag{3.15}$$

for any $Y \in \Gamma(TM)$. Comparing (3.14) with (3.15) we get

$$k(Df - \xi(f)\xi) = 2h'(Df). \tag{3.16}$$

By $h'\xi = 0$, the action of h' on the above equation and making use of (3.16) imply

$$h'^2(Df) = \frac{k^2}{4}(Df - \xi(f)\xi).$$

Putting the above equation into (3.11) and taking into account $k < -1$ we know

$$(k+2)^2(Df - \xi(f)\xi) = 0. \tag{3.17}$$

Clearly, it follows from (3.17) that either $k = -2$ or $Df = \xi(f)\xi$ Now we prove that the latter case can not occur. In fact, if we assume that the latter case is true, that is $Df = \xi(f)\xi$, putting this into (3.1) and making use of (2.3) we obtain

$$\nabla_Y Df = Y(\xi(f))\xi + \xi(f)(Y - \eta(Y)\xi + h'Y)$$

for any $Y \in \Gamma(TM)$. Using the above equation in (3.1) we have

$$QY = (\xi(f) - \lambda)Y + (Y(\xi(f)) - \xi(f)\eta(Y))\xi + \xi(f)h'Y$$

for any $Y \in \Gamma(TM)$. Comparing the above equation with (3.12) we see that

$$(2n+\xi(f))h'Y = (\lambda-\xi(f)-2n)Y + (2n(k+1)\eta(Y)+\xi(f)\eta(Y)-Y(\xi(f)))\xi \tag{3.18}$$

for any $Y \in \Gamma(TM)$. Next, we assume that $2n + \xi(f) = 0$; then it follows from (3.18) that $\lambda Y + 2nk\eta(Y)\xi = 0$ for any $Y \in \Gamma(TM)$. Let $Y \in \mathcal{D}$ in this equation we get $\lambda = 0$, consequently, it follows that $k = 0$; this contradicts the hypothesis $k < -1$ ($\Leftrightarrow h \neq 0$). Hence, by $2n + \xi(f) \neq 0$, (3.18) implies that for any vector field Y orthogonal to ξ we have $Y(\xi(f)) = 0$ and

$$h'Y = \Big(\frac{\lambda}{2n + \xi(f)} - 1\Big)Y. \tag{3.19}$$

Combining (3.19) and $h' \circ \phi + \phi \circ h' = 0$, we get

$$\frac{\lambda}{2n + \xi(f)} - 1 = 0 \tag{3.20}$$

and thus $h'Y = 0$ for any $Y \in \mathcal{D}$. Therefore $h' = 0$, contradicting the hypothesis $h' \neq 0$. Consequently, we obtain from relation (3.17) that $k = -2$, making use of this equation in (3.11) we see that $h'^2 = -\phi^2$. Finally, it follows from Proposition 4.1 and Corollary 4.2 of [**6**] that M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

Now we denote by $[1]'$ and $[-1]'$ the distributions of the eigenvectors of h' orthogonal to ξ with eigenvalues 1 and -1 respectively. Notice that if $X \in [1]'$

then $\phi X \in [-1]'$, and thus both eigenvalues 1 and -1 have the same multiplicity n. Hence we may consider a local orthonormal ϕ-frame $\{\xi, E_i, \phi E_i\}$ for $1 \leqslant i \leqslant n$ with $E_i \in [1]'$. Suppose that $Df = \sum_{i=1}^n \alpha_i E_i + \sum_{i=1}^n \beta_i \phi E_i + \theta\xi$ (where α_i, β_i and θ are smooth functions on M^{2n+1}). Then we have $h'(Df) = \sum_{i=1}^n \alpha_i E_i - \sum_{i=1}^n \beta_i \phi E_i$. On the other hand, making use of $k = -2$ in equation (3.16), we obtain $h'(Df) = -\sum_{i=1}^n \alpha_i E_i - \sum_{i=1}^n \beta_i \phi E_i + (\xi(f) - \theta)\xi$. Obviously, we get $\alpha_i = 0$ for any $1 \leqslant i \leqslant n$ and $\theta = \xi(f)$. Thus, making use of $Df = \sum_{i=1}^n \beta_i \phi E_i + \xi(f)\xi$ in relation (3.1) and by a direct calculation we obtain an equation; applying (3.12) in (3.1) and making use of $k = -2$, we get another equation, comparing these two equations we have

$$\begin{aligned}(3.21)\quad &(\lambda - 2n)Y - 2n\eta(Y)\xi - 2nh'Y \\ &= \sum_{i=1}^n (Y(\beta_i)\phi E_i + \beta_i \nabla_Y \phi E_i) + Y(\xi(f))\xi + \xi(f)(Y - \eta(Y)\xi + h'Y)\end{aligned}$$

for any $Y \in \Gamma(TM)$. From the proof of Proposition 4.1 of [**6**], we have $\nabla_\xi \phi E_i \in [-1]'$; then letting $Y = \xi$ in (3.21) we obtain

$$\lambda - 4n = \xi(\xi(f)). \tag{3.22}$$

Also, from the proof of Proposition 4.1 of [**6**] we have $\nabla_{E_j} \phi E_i \in [-1]'$ for any $E_j \in [1]'$, then replacing Y by E_j in (3.21) we have

$$\lambda - 4n = 2\xi(f). \tag{3.23}$$

It follows from (3.22) and (3.23) that $\lambda = 4n$ and hence $\xi(f) = 0$. In this case, we may write $Df = \sum_{i=1}^n \beta_i \phi E_i$ and (3.16) becomes $h'(Df) = -Df$. Moreover, applying (3.12) we obtain from equation (3.1) that $\nabla Df = -2n(h' + \phi^2)$. This completes the proof. □

Remark 3.2. We see from Petersen and Wylie [**16, 17**] that under the hypothesis of our Theorem 1.2, M^{2n+1} turns out to be locally isometric to a rigid gradient Ricci soliton, which is the Riemannian product of the Einstein manifold $\mathbb{H}^{n+1}(-4)$ and the Gaussian soliton $\mathbb{R}^n$.

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Henan Engineering Laboratory
for Big Data Statistical Analysis and Optimal Control
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007
Henan
P. R. China
wyn051@163.com

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Department of Pure Mathematics
University of Calcutta
35, B.C. Road, Kolkata 700019
West Bengal
India
uc_de@yahoo.com

School of Mathematical Sciences
Dalian University of Technology
Dalian 116024
Liaoning
P. R. China
ximinliu@dlut.edu.cn