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A TRANSCENDENCE CRITERION FOR CONTINUED FRACTION EXPANSIONS IN POSITIVE CHARACTERISTIC

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Abstract. We exhibit a family of transcendental continued fractions of formal power series over a finite field through some specific irregularities of its partial quotients.

1. Introduction

A well-known open question in diophantine approximation suggested by Khintchine in [**5**] asks whether an irrational algebraic number x of degree > 2 has a continued fraction expansion whose sequence of partial quotients is unbounded. The answer to this conjecture remains a hard matter. Several transcendence criteria for continued fractions that have been established recently gave a partial solution to this question. In [**2**] Baker proved that if $x = [a_0, a_1, a_2, \dots]$ such that $a_n = a_{n+1} = \cdots = a_{n+\lambda(n)-1}$, for infinitely many positive integers n where $\lambda(n)$ is a sequence of integers verifying certain increasing properties, then x is transcendental. The proof of this result is based on Liouville's and Roth's theorems.

Recently and based on the Schmidt Subspace Theorem, Adamczewski and Bugeaud [**1**] improved the result of Baker.

In 1967, Schmidt [**12**] demonstrated that any positive irrational number which is very well approximated by quadratic numbers is either quadratic or transcendental. This result has been used in several works.

However, for formal power series over a finite field, we have some examples of algebraic formal series of degree $\geqslant 3$ whose sequence of the degrees of the partial quotients is bounded, as well as examples whose partial quotients take an infinity of values.

In 1976, Baum and Sweet [**3**] gave the first example of algebraic formal series of degree 3 in $\mathbb{F}_2((X^{-1}))$ whose partial quotients have only a finite number of values. This work was pursued in [**7**] by Mills and Robbins who provided an example of algebraic formal series over $\mathbb{F}_2((X^{-1}))$ whose sequence of partial quotients is

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unbounded. Moreover, Robbins gave a family of cubic formal power series with bounded partial quotients [**11**].

In 2004, Mkaouar [**9**] gave a new transcendence criteria of formal power series over a finite field that is based on the degree of its partial quotients.

THEOREM 1.1. [**9**] *Let $f \in \mathbb{F}_q((X^{-1}))$ be an irrational formal series which is not quadratic such that*

$$f = [\overbrace{a_1, \ldots, a_1}^{n_1}, \overbrace{a_2, \ldots, a_2}^{n_2}, \overbrace{a_3, \ldots, a_3}^{n_3}, \ldots],$$

where a_i are blocks of consecutive partial quotients. Let r_i be the sum of degrees of partial quotients of block a_i. If

$$\lim_{\infty} \frac{n_i r_i}{n_{i-1} r_{i-1}} = \limsup n_i = +\infty,$$

then f is transcendental.

In [**4**], Hbaib, Mkaouar and Tounsi constructed a family of transcendental continued fractions over $\mathbb{F}_q((X^{-1}))$ from an algebraic formal power series of degree more than 2.

THEOREM 1.2. [**4**] *Let g be an algebraic formal power series such that $\deg(g) > 0$ and $f = [B_1, B_2, \ldots]$ where B_i are finite blocks of partial quotients whose the first n_i-terms are those of the continued fraction expansion of g. Let d_i denote the sum of degrees of B_i and δ_i the sum of degrees of the first n_i-terms of B_i. If*

$$\liminf_{s \to +\infty} \frac{1}{\delta_s} \sum_{j=1}^{s-1} d_j = 0,$$

then f is transcendental or quadratic.

Our main purpose here is twofold: to improve the last results and to give a new transcendence criteria depending only on the length of specific blocks appearing in the sequence of partial quotients. The present paper is organized as follows: in Section 2, we define the field of formal series and the continued fraction expansions over this field. In Section 3, we state the main transcendence criterion and we present some lemmas that we will use to prove our result. We close this section with the proof of our main theorem (see Theorem 3.1) and an example to illustrate the limit of our result.

2. Field of formal series $\mathbb{F}_q((X^{-1}))$

Let $\mathbb{F}_q$ be a field with $q > 1$ elements of characteristic $p > 0$, $\mathbb{F}_q[X]$ the ring of polynomials with coefficient in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ the field of rational functions. Let $\mathbb{F}_q((X^{-1})) = \{f = \sum_{n \geqslant n_0} b_n X^{-n} \mid b_n \in \mathbb{F}_q,\ n_0 \in \mathbb{Z}\}$ be the field of formal power series. Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0, \\ 0 & \text{for } f = 0. \end{cases}$$

Thus, $|\cdot|$ is not an archimedean absolute value over $\mathbb{F}_q((X^{-1}))$, that is $|f+g| \leqslant \max(|f|,|g|)$ and $|f+g| = \max(|f|,|g|)$ if $|f| \neq |g|$. By analogy with the real case, we have a chain-fraction algorithm in $\mathbb{F}_q((X^{-1}))$. A formal power series $f = \sum_{n \geqslant n_0} b_n X^{-n}$ has a unique decomposition as $f = [f] + \{f\}$ with $[f] \in \mathbb{F}_q[X]$ and $|\{f\}| < 1$. The polynomial $[f]$ is called the polynomial part of f and $\{f\}$ is called the fractional part of f. We can write for any $f \in \mathbb{F}_q((X^{-1}))$

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots],$$

where $a_0 = [f]$ and $a_i = [f_i] \in \mathbb{F}_q[X]$ with $\deg(a_i) \geqslant 1$ for any $i \geqslant 1$ and $f_i = 1/\{f_{i-1}\}$. The sequence $(a_i)_{i \geqslant 0}$ is called the partial quotients of f and we denote by $f_n = [a_n, a_{n+1}, \ldots]$ the n-th complete quotient of f.

Remarks. 1) If $(\deg(a_i))_{i \geqslant 0}$ is bounded, then f is said to have a bounded continued fraction expansion.

2) The expansion is finite if and only if $f \in \mathbb{F}_q(X)$.

3) The sequence of partial quotients of f is ultimately periodic if and only if f is quadratic over $\mathbb{F}_q(X)$.

Now, we define two sequences of polynomials $(P_n)_{n \geqslant 0}$ and $(Q_n)_{n \geqslant 0}$ by

$$P_0 = a_0, \quad Q_0 = 1, \quad P_1 = a_0 a_1 + 1, \quad Q_1 = a_1$$
$$P_n = a_n P_{n-1} + P_{n-2}, \quad Q_n = a_n Q_{n-1} + Q_{n-2}, \quad \text{for } n \geqslant 2.$$

We easily check that

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}, \quad \text{for } n \geqslant 1,$$
$$\frac{P_n}{Q_n} = [a_0, a_1, a_2, \ldots, a_n], \quad \text{for } n \geqslant 0.$$

P_n/Q_n is called the n^{th}convergent of f and it satisfies

$$\lim_{n \to \infty} \frac{P_n}{Q_n} = f = [a_0, a_1, \ldots, a_n, \ldots].$$

With the nonarchimedean absolute value, we find the following important equality

$$\left| f - \frac{P_n}{Q_n} \right| = \left| \frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} \right| = |Q_n Q_{n+1}|^{-1} = |a_{n+1}|^{-1} |Q_n|^{-2}.$$

Let f be an algebraic formal power series of minimal polynomial $P(Y) = A_m Y^m + A_{m-1} Y^{m-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[X]$. Set $H(f) = \max_{0 \leqslant i \leqslant m} |A_i|$ and $\sigma(f) = A_m$.

Recall from [**8**] that a polynomial $P \in \mathbb{F}_q[X][Y]$ is said to be reduced if $\deg(A_{m-1}) > \deg(A_i)$ for any $i \neq m-1$, and an algebraic formal power series is reduced if its minimal polynomial is reduced and $[f] \neq 0$.

In [**4**], the authors gave the following lemma which identifies the reduced formal power series.

Lemma 2.1. *Let f be an algebraic formal power series of degree d and P its minimal polynomial. We denote by $f_1, \dots, f_{d-1}$ the conjugates of f in the algebraic closure of $\mathbb{F}_q((X^{-1}))$. Then f is reduced if and only if $|f| > 1$ and $|f_i| < 1$, for all $i \in \{1, \dots, d-1\}$.*

3. Results

Before giving the main result, we need to introduce some notation. If $K_n = u_{\alpha_0} u_{\alpha_2} \dots u_{\alpha_n}$ is a finite block formed by $n+1$ polynomials, we denote by $|K_n|$ its length and by $\varphi(K_n)$ the maximal degree which appears in the terms of K_n, which means that $\varphi(K_n) = \max_{0 \leqslant i \leqslant n}(\deg(u_{\alpha_i}))$. If U_n, V_n are two finite blocks of polynomials, we write $U_n V_n$ for the block resulting by concatenation of them.

Theorem 3.1. *Let $f \in \mathbb{F}_q((X^{-1}))$ such that $f = [U_0 V_0 U_1 V_1 \dots U_n V_n \dots]$ where $(U_n)_{n \geqslant 0}$ and $(V_n)_{n \geqslant 0}$ are two sequences of finite blocks of polynomials such that*

1) *$U_i = P_i P_i^q P_i^{q^2} \dots P_i^{q^{\lambda_i - 1}}$, for any $i \geqslant 0$, with $P_i \in \mathbb{F}_q[X]$ of degree $\geqslant 1$.*
2) *The sequence $(|V_n|/|U_n|)_{n \geqslant 0}$ is bounded.*
3) *$(\lambda_i)_{i \geqslant 0}$ is an increasing sequence of positive integers.*
4) *$(\deg(P_i))_{i \geqslant 0}$ is bounded.*
5) *$\varphi(V_n) \leqslant \varphi(U_n)$, for all $n \geqslant 0$.*

If f satisfies

$$\limsup_{n \to \infty} \frac{q^{\lambda_n - \lambda_{n-1}}}{n \lambda_{n-1}} = +\infty,$$

then f is transcendental.

The proof of this theorem breaks into four lemmas.

Lemma 3.1. [**4**] *Let f be an algebraic formal power series of degree d such that $f = [a_1, a_2, \dots, a_t, h]$ where $a_1, \dots, a_t \in \mathbb{F}_q[X]$, $h \in \mathbb{F}_q((X^{-1}))$. If $|f| \geqslant 1$ and $|h| > 1$, then h is algebraic of degree d and*

$$H(h) \leqslant H(f) \left| \prod_{i=1}^{t} a_i \right|^{d-2}.$$

Lemma 3.2. *Let $P(Y) = A_n Y^n + A_{n-1} Y^{n-1} + \dots + A_0$ be a reduced polynomial with $A_i \in \mathbb{F}_q[X]$. If $A_0 \neq 0$, then P is irreducible.*

Proof. Let f be the unique root of P such that $|f| > 1$ and assume that $P(Y)$ is reducible; then $P(Y) = P_1(Y) P_2(Y)$ with $P_1, P_2 \in \mathbb{F}_q[X][Y]$. We suppose that $P_1(f) = 0$; then from Lemma 2.1, all the roots of P_2 have absolute values < 1, so the constant coefficient in P_2 is equal to 0, which is absurd because 0 is not a root of P. □

Lemma 3.3. [**10**] *Let $f = [a_0, a_1, \dots]$ and $g = [b_0, b_1, \dots]$ be two formal series having the same first $n+1$ partial quotients. Then*

$$|f - g| \leqslant \frac{1}{|Q_n|^2}.$$

LEMMA 3.4. [**4**] *Let f and g be two algebraic formal power series of degrees d and m respectively. If g is reduced and $f \neq g$, then*

$$|f-g| \geqslant \frac{1}{H(f)^m |g|^{d-2} |\sigma(g)|^{\max(m-1, m(d-m+2)-1)}}.$$

PROOF. Assume contrary that f is algebraic of degree $d > 2$. Let us use the notation: $\lambda_n = |U_n|$, $s_n = |V_n|$, for all $n \geqslant 0$ and $\alpha_n = \sum_{i=0}^{n-1}(\lambda_i + s_i)$, for all $n \geqslant 1$. Let g_n denote the continued fraction $[P_n, P_n^q, P_n^{q^2}, P_n^{q^3}, \ldots]$. An easy calculation ensures that g_n verifies the following equation

$$g_n^{q+1} - P_n g_n^q - 1 = 0.$$

Hence Lemma 3.2 guarantees that g_n is algebraic of degree $q+1$. Let $f_{\alpha_n} = [U_n V_n U_{n+1} V_{n+1} \ldots]$ denote the α_n^{th} complete quotient of f. Since $(\deg(P_i))_{i \geqslant 0}$ is bounded, then for sufficiently large n, $g_n \neq f_{\alpha_n}$. On the other hand, it follows from Lemma 3.1 that f_{α_n} is algebraic of degree $d > 2$. Therefore, according to Lemma 3.4, we infer that

$$|f_{\alpha_n} - g_n| \geqslant H(f_{\alpha_n})^{-q-1} |g_n|^{2-d}.$$

Moreover, by using again Lemma 3.1, we can check, for sufficiently large n that

$$|f_{\alpha_n} - g_n| \geqslant H(f)^{-q-1} |P_n|^{2-d} \left| \prod_{i=0}^{\alpha_n - 1} a_i \right|^{(d-2)(-q-1)}, \tag{3.1}$$

where $(a_i)_{i \geqslant 0}$ is the sequence of partial quotients of f.

Furthermore, f_{α_n} and g_n have the same first λ_n partial quotients, hence Lemma 3.3 implies that

$$|f_{\alpha_n} - g_n| \leqslant |P_n P_n^q P_n^{q^2} \ldots P_n{}^{q^{\lambda_n - 1}}|^{-2}. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$|P_n P_n^q P_n^{q^2} \ldots P_n^{q^{\lambda_n - 1}}|^2 \leqslant H(f)^{q+1} |P_n|^{d-2} \left| \prod_{i=0}^{\alpha_n - 1} a_i \right|^{(d-2)(q+1)}$$

whence

$$2\deg(P_n)\Big(\frac{q^{\lambda_n} - 1}{q-1}\Big) \leqslant (q+1)\log_q H(f) + (q+1)(d-2) \sum_{i=0}^{\alpha_n - 1} \deg(a_i) + (d-2)\deg(P_n).$$

The fact that $(\deg(P_i))_{i \geqslant 0}$ is bounded yields the inequality

$$\limsup_{n \to \infty} \frac{q^{\lambda_n}}{\sum_{i=0}^{\alpha_n - 1} \deg(a_i)} \leqslant C, \quad \text{with } C = (q^2-1)(d-2).$$

Set $h = \sup_{i \geqslant 0}(\deg(P_i))$. As $\varphi(V_i) \leqslant \varphi(U_i)$ for all $i \geqslant 0$, we get $\deg(a_i) \leqslant q^{\lambda_{n-1} - 1} h$, for all $0 \leqslant i \leqslant \alpha_n$. Therefore

$$\limsup_{n \to \infty} \frac{q^{\lambda_n}}{q^{\lambda_{n-1} - 1} h \alpha_n} \leqslant C.$$

Since the sequence $(|V_i|/|U_i|)_{i\geqslant 0}$ is bounded, there exists $c > 0$ such that $s_i < c\lambda_i$ for all $i \geqslant 0$. Thus, $\alpha_n < (c+1)n\lambda_{n-1}$. Hence, we conclude that

$$\limsup_{n\to\infty} \frac{q^{\lambda_n-\lambda_{n-1}}}{n\lambda_{n-1}} < \infty, \quad \text{the desired contradiction.}$$

□

We close the paper with the following example.

EXAMPLE. Let $f \in \mathbb{F}_2((X^{-1}))$ such that $f = [U_0V_0U_1V_1 \dots U_nV_n \dots]$ where $U_i = [P_i, P_i^2, P_i^4, \dots, P_i^{2^{\lambda_i-1}}]$, with $P_i = X + i$ and $V_i = [X, X^2, X, X^2, \dots, X, X^2]$ of length $\lambda_i = (i+1)^2$, for all $i \geqslant 0$. Then f is transcendental because

$$\limsup_{n\to\infty} \frac{2^{2n+1}}{n^3} = +\infty.$$

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References

1. B. Adamczewski, Y. Bugeaud, *On the Maillet–Baker continued fractions*, J. Reine Angew. Math. **606** (2007), 105–121.
2. A. Baker, *Continued fractions of transcendental numbers*, Mathematika **9** (1962), 1–8.
3. L. E. Baum, H. M. Sweet, *Continued fractions of algebraic power series in characteristic 2*, Ann. Math. **103** (1976), 593–610.
4. M. Hbaib, M. Mkaouar, K. Tounsi, *Un critère de transcendance dans le corps des séries formelles* $\mathbb{F}_q((X^{-1}))$, J. Number Theory. **116** (2006), 140–149.
5. A. Khintchine, *Continued Fractions*, Gosudarstv. Izdat. Tech.-Teor. Lit. Moscow–Leningrad, 2nd edition, 1949, (In Russian).
6. J. Liouville, *Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductibles à des irrationnelles algébriques*, J. Math. Pures Appl. **16** (1851), 133–142.
7. W. H. Mills, D. P. Robbins, *Continued fractions for certain algebraic power series*, J. Number Theory. **23** (1986), 388–404.
8. M. Mkaouar, *Fractions continues et séries formelles algébriques réduites*, Port. Math. **58**(4) (2001), 439–448.
9. ———, *Transcendance de certaines fractions continues dans le corps des séries formelles*, J. Algebra. **281** (2004), 502–507.
10. O. Perron, *Die Lehre von den Kettenbruchen*, Teubner, Leipzig, 1929.
11. D. P. Robbins, *Cubic Laurent series in characteristic 2 with bounded partial quotients*, arXiv:math/9903092v1 [math.NT], 1999.
12. W. Schmidt, *On simultanous approximations of two algebraic numbers by rationals*, Acta Math. **119** (1967), 27–50.

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