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ON THE GENERALIZED SUPERSTABILITY OF nth ORDER LINEAR DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

Jinghao Huang, Qusuay H. Alqifiary, and Yongjin Li

Abstract. We establish the generalized superstability of differential equations of nth-order with initial conditions and investigate the generalized superstability of differential equations of second order in the form of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and the superstability of linear differential equations with constant coefficients with initial conditions.

1. Introduction

In 1940, Ulam [**28**] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers [**7**] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: Let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X_1 \to X_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \quad (x, y \in X_1),$$

there exists a unique additive function $A : X_1 \to X_2$ with the property

$$\|f(x) - A(x)\| \leqslant \varepsilon \quad (x \in X_1).$$

A generalized solution to Ulam's problem for approximately linear mappings was proved by Rassias in 1978 [**22**]. He considered a mapping $f : E_1 \to E_2$ such that $t \mapsto f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geqslant 0$ and $0 \leqslant p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p)$$

for any $x, y \in E_1$. After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions [**4**, **8**, **12**]. A generalization of Ulam's problem was recently proposed by

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replacing functional equations with differential equations: The differential equation $\varphi\big(f, y, y', \dots, y^{(n)}\big) = 0$ has the Hyers–Ulam stability if for given $\varepsilon > 0$ and a function y such that $\big|\varphi(f, y, y', \dots, y^{(n)})\big| \leqslant \varepsilon$, there exists a solution y_0 of the differential equation such that $|y(t) - y_0(t)| \leqslant K(\varepsilon)$ and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

Obłoza seems to be the first author who has investigated the Hyers–Ulam stability of linear differential equations [**18**, **19**]. Thereafter, Alsina and Ger published their paper [**1**], which handles the Hyers–Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \leqslant \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leqslant 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers–Ulam stability of linear differential equations of the first and higher orders with constant coefficients in [**16**, **26**, **27**] and in [**17**], respectively. Furthermore, Jung has also proved the Hyers–Ulam stability of linear differential equations [**9**–**11**]. Rus investigated the Hyers–Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators [**24**, **25**]. Recently, the Hyers–Ulam stability problems of linear differential equations of the first and second orders with constant coefficients were studied by using the method of integral factors [**15**, **29**]. The results given in [**10**, **15**, **16**] have been generalized by Cimpean and Popa [**3**] and by Popa and Raşa [**20**, **21**] for the linear differential equations of nth order with constant coefficients. Furthermore, the Laplace transform method was recently applied to the proof of the Hyers–Ulam stability of linear differential equations [**23**].

In 1979, Baker, Lawrence and Zorzitto [**2**] proved a new type of stability of the exponential equation $f(x + y) = f(x)f(y)$. More precisely, they proved that if a complex-valued mapping f defined on a normed vector space satisfies the inequality $|f(x + y) - f(x)f(y)| \leqslant \delta$ for some given $\delta > 0$ and for all x, y, then either f is bounded or f is exponential. Such a phenomenon is called the superstability of the exponential equation, which is a special kind of Hyers–Ulam stability. It seems that the results of Găvruţa, Jung and Li [**5**] are the earliest one concerning the superstability of differential equations.

Here we investigate the generalized superstability of linear differential equation of the nth order in the form

$$y^{(n)}(x) + \beta(x)y(x) = 0, \tag{1.1}$$

with initial conditions

$$y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0, \tag{1.2}$$

where $n \in \mathbb{N}^+$, $y \in C^n[a, b]$, $\beta \in C^0[a, b]$, $-\infty < a < b < +\infty$.

In addition to that we investigate the generalized superstability of differential equations of the second order in the form of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and the superstability of linear differential equations with constant coefficients.

First of all, we give the definition of superstability and generalized superstability with initial and boundary conditions.

Definition 1.1. Assume that for any function $y \in C^n[a,b]$, if y satisfies the differential inequality

$$\left|\varphi(f, y, y', \dots, y^{(n)})\right| \leqslant \epsilon$$

for all $x \in [a,b]$ and for some $\epsilon \geqslant 0$ with initial (or boundary) conditions, then either y is a solution of the differential equation

$$\varphi(f, y, y', \dots, y^{(n)}) = 0 \tag{1.3}$$

or $|y(x)| \leqslant K\epsilon$ for any $x \in [a,b]$, where K is a constant not depending on y explicitly. Then, we say that (1.3) has superstability with initial (or boundary) conditions.

Definition 1.2. Assume that for any function $y \in C^n[a,b]$, if y satisfies the differential inequality

$$\left|\varphi(f, y, y', \dots, y^{(n)})\right| \leqslant \varphi(x)$$

for all $x \in [a,b]$ and for some function $\varphi : [a,b] \to [0,\infty)$ with initial(or boundary) conditions, then either y is a solution of the differential equation (1.3) or $|y(x)| \leqslant \Phi(x)$ for any $x \in [a,b]$, where $\Phi : I \to [0,\infty)$ is a function not depending on y explicitly. Then, we say that (1.3) has generalized superstability with initial (or boundary) conditions.

2. Main results

In this section, given the closed interval $I = [a,b]$, we assume that $\varphi : I \to [0,\infty)$ and let $\mathbf{M}(p(x))$ denote $\max_{\tau\in[a,x]} |p(\tau)|$ for every $p \in C(I,\mathbb{R})$.

Theorem 2.1. *If* $|\beta(x)| < n!/(b-a)^n$ *for every* $x \in I$*, then* (1.1) *has generalized superstability with initial conditions* (1.2)*.*

Proof. Suppose that a function $y \in C^n(I,\mathbb{R})$ satisfies the inequality

$$\left|y^{(n)}(x) + \beta(x)y(x)\right| \leqslant \varphi(x),$$

for all $x \in I$,

By the Taylor formula, we have

$$y(x) = y(a) + y'(a)(x-a) + \cdots + \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Therefore,

$$|y(x)| = \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^n\right| \leqslant \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}$$

for every $x \in [a,b]$. Then,

$$\begin{aligned}\mathbf{M}(y(x)) &\leqslant \mathbf{M}\Big(\mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}\Big)\\ &\leqslant \mathbf{M}\big(\mathbf{M}(y^{(n)}(x))\big)\mathbf{M}\Big(\frac{(x-a)^n}{n!}\Big) = \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{M}(y(x)) &\leqslant \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}\\ &\leqslant \frac{(x-a)^n}{n!}\mathbf{M}\big(y^{(n)}(x)+\beta(x)y(x)\big)+\frac{(x-a)^n}{n!}\mathbf{M}|\beta(x)|\mathbf{M}(y(x))\\ &\leqslant \frac{(x-a)^n}{n!}\mathbf{M}(\varphi(x))+\frac{(b-a)^n}{n!}\max|\beta(x)|\mathbf{M}(y(x)).\end{aligned}$$

Let $C_1 = 1 - \frac{(b-a)^n}{n!}\max|\beta(x)|$. It is easy to see that

$$\mathbf{M}(y(x)) \leqslant \frac{(x-a)^n}{n!C_1}\mathbf{M}(\varphi(x)).$$

Moreover, $|y(x)| \leqslant \mathbf{M}(y(x))$, which completes the proof of our theorem. $\square$

In the following theorem, we investigate the generalized superstability of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{2.1}$$

with initial conditions

$$y(a) = 0 = y'(a), \tag{2.2}$$

where $y \in C^2[a,b]$, $p \in C[a,b]$, $q \in C^0[a,b]$, $-\infty < a < b < +\infty$.

THEOREM 2.2. *If* $\max\big\{q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\big\} < 2/(b-a)^2$, *then* (2.1) *has generalized superstability with initial conditions* (2.2).

PROOF. Suppose that $y \in C^2[a,b]$ satisfies the inequality

$$\big|y''(x) + p(x)y'(x) + q(x)y(x)\big| \leqslant \varphi(x). \tag{2.3}$$

Let

$$u(x) = y''(x) + p(x)y'(x) + q(x)y(x), \tag{2.4}$$

for all $x \in [a,b]$, and define $z(x)$ by

$$y(x) = z(x)\exp\bigg(-\frac{1}{2}\int_a^x p(\tau)d\tau\bigg). \tag{2.5}$$

By a substitution (2.5) in (2.4), we obtain

$$z''(x) + \bigg(q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\bigg)z(x) = u(x)\exp\bigg(\frac{1}{2}\int_a^x p(\tau)d\tau\bigg).$$

Then it follows from inequality (2.3) that

$$\begin{aligned}\bigg|z''(x) + \Big(q(x) - \frac{1}{2}p'(x) - \frac{p^2(x)}{4}\Big)z(x)\bigg| &= \big|u(x)\big|\exp\bigg(\frac{1}{2}\int_a^x p(\tau)d\tau\bigg)\\ &\leqslant \varphi(x)\exp\bigg(\frac{1}{2}\int_a^x p(\tau)d\tau\bigg).\end{aligned}$$

From (2.2) and (2.5) we have $z(a) = 0 = z(b)$. It follows from Theorem 2.1 that there exists a constant $C_1 > 0$ such that

$$|z(x)| \leqslant \frac{(x-a)^n}{n!\,C_1}\mathbf{M}\bigg(\varphi(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\bigg).$$

From (2.5) we have

$$|y(x)| \leqslant \frac{(x-a)^n}{n!\,C_1}\mathbf{M}\bigg(\varphi(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\bigg)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right).$$

Thus (2.1) has generalized superstability with initial conditions (2.2). □

In the following theorems, we investigate the superstability of the differential equation

$$y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0 \tag{2.6}$$

with initial conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0, \tag{2.7}$$

where $y \in C^n(I,\mathbb{C})$, $a_i \in \mathbb{R}$ $(i = 0,1,\dots,n-1)$, $I = [a,b]$, $-\infty < a < b < +\infty$.

Lemma 2.1. *Assume that* $y \in C^1(I,\mathbb{C})$ *and* $C \in \big\{z \in \mathbb{C} \mid |z| < \frac{1}{b-a}\big\}$. *If*

$$|y'(x) - Cy(x)| \leqslant \varepsilon$$

with $y(a) = 0$*, then there exists a constant* $K > 0$ *such that* $|y(x)| \leqslant K\varepsilon$*.*

Proof. Let $y(x) = A(x)+iB(x)$, where i is the imaginary unit and $A(x), B(x) \in C^1(I,\mathbb{R})$. Since $y(a) = 0$, we have $A(a) = 0$ and $B(a) = 0$; so, similar to Theorem 2.1, we obtain

$$\begin{aligned}
\max|A(x)| &\leqslant (b-a)\max|A'(x) - CA(x)| + |C|\cdot(b-a)\max|A(x)|\\
&\leqslant (b-a)\max|y'(x) - Cy(x)| + |C|\cdot(b-a)\max|A(x)|\\
&\leqslant (b-a)\varepsilon + |C|\cdot(b-a)\max|A(x)|\\
\max|B(x)| &\leqslant (b-a)\varepsilon + |C|\cdot(b-a)\max|B(x)|.
\end{aligned}$$

Since $C \in \big\{z \in \mathbb{C} \mid |z| < \frac{1}{b-a}\big\}$, there exists a constant K such that

$$\max|y(x)| \leqslant \sqrt{\max|A(x)|^2 + \max|B(x)|^2} \leqslant K\varepsilon.$$ □

Theorem 2.3. *If all the roots of the characteristic equation are in the disc* $\big\{z \in \mathbb{C} \mid |z| < \frac{1}{b-a}\big\}$*, then* (2.6) *has superstability with initial conditions* (2.7).

Proof. Assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

Define $g_1(x) = y'(x) - \lambda_1 y(x)$ and $g_i(x) = g'_{i-1}(x) - \lambda_i g_{i-1}(x)$ $(i = 2,3,\cdots,n-1)$, thus

$$\big|g'_{n-1}(x) - \lambda_n g_{n-1}(x)\big| = \big|y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x)\big| \leqslant \varepsilon,$$

and $g_i(a) = 0$ for every $i = 1,2,\dots,n-1$.

Since the absolute value of $\lambda_n < \frac{1}{b-a}$ and $g_{n-1}(a) = 0$, it follows from Lemma 2.1 that there exists a $K_1 > 0$ such that $|g_{n-1}(x)| \leqslant K_1\varepsilon$. Recall $g_{n-1}(x) = g'_{n-2}(x) - \lambda_{n-1}g_{n-2}(x)$, we have $\left|g'_{n-2}(x) - \lambda_{n-1}g_{n-2}(x)\right| \leqslant K_1\varepsilon$. By an argument similar to the above and by induction, we can show that there exists a constant $K > 0$ such that $|y(x)| \leqslant K\varepsilon$. □

Remark 2.1. In the present paper, we have discussed the case that the solution f of a differential inequality is bounded. In fact, the case that f is the exact solution of the corresponding differential equation ($f(x) = 0 \leqslant 0\varepsilon$) is also included in it.

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References

1. C. Alsina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.
2. J. Baker, J. Lawrence, F. Zorzitto, *The stability of the equation $f(x+y) = f(x)f(y)$*, Proc. Am. Math. Soc. **74** (1979), 242–246.
3. D. S. Cimpean, D. Popa, *On the stability of the linear differential equation of higher order with constant coefficients*, Appl. Math. Comput. **217** (2010), 4141–4146.
4. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.
5. P. Găvruţa, S. Jung, Y. Li, *Hyers–Ulam stability for second-order linear differential equations with boundary conditions*, Electronic J. Diff. Equ. **2011** (2011), 1–5.
6. R. Ger, P. Šemrl, *The stability of the exponential equation*, Proc. Am. Math. Soc. **124** (1996), 779–787.
7. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Soc. USA **27** (1941), 222–224.
8. D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.
9. S.-M. Jung, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17** (2004), 1135–1140.
10. ______, *Hyers–Ulam stability of linear differential equations of first order, III*, J. Math. Anal. Appl. **311** (2005), 139–146.
11. ______, *Hyers–Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett. **19** (2006), 854–858.
12. ______, *Hyers–Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.
13. ______, *On the superstability of the Functional Equation $f(x^y) = yf(x)$*, Abh. Math. Sem. Univ. Hamburg **67** (1997), 315–322.
14. ______, *On the superstability of the functional equation $f(x_1 + \cdots + x_m) = f(x_1)\cdots f(x_m)$*, Comm. Korean Math. Soc. **14**(1) (1999), 75–80.
15. Y. Li, Y. Shen, *Hyers–Ulam stability of linear differential equations of second order*, Appl. Math. Lett. **23** (2010), 306–309.
16. T. Miura, S. Miyajima, S. E. Takahasi, *A characterization of Hyers–Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2003), 136–146.
17. ______, *Hyers–Ulam stability of linear differential operator with constant coefficients*, Math. Nachr. **258** (2003), 90–96.
18. M. Obłoza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. **13** (1993), 259–270.

19. ______, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat. **14** (1997), 141–146.
20. D. Popa, I. Raşa, *On the Hyers–Ulam stability of the linear differential equation*, J. Math. Anal. Appl. **381** (2011), 530–537.
21. ______, *Hyers–Ulam stability of the linear differential operator with non-constant coefficients*, Appl. Math. Comput. **219** (2012), 1562–1568.
22. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
23. H. Rezaei, S.-M. Jung, Th. M. Rassias, *Laplace transform and Hyers–Ulam stability of linear differential equations*, J. Math. Anal. Appl. **403** (2013), 244–251.
24. I. A. Rus, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory **10** (2009), 305–320.
25. ______, *Ulam stability of ordinary differential equations*, Stud. Univ. Babes-Bolyai Math. **54** (2009), 125–134.
26. S. E. Takahasi, T. Miura, S. Miyajima, *On the Hyers–Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. **39** (2002), 309–315.
27. S. E. Takahasi, H. Takagi, T. Miura, S. Miyajima, *The Hyers–Ulam stability constants of first order linear differential operators*, J. Math. Anal. Appl. **296** (2004), 403–409.
28. S. M. Ulam, *Problems in Modern Mathematics, Chapter VI*, Scince Editors, Wiley, New York, 1960.
29. G. Wang, M. Zhou, L. Sun, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21** (2008), 1024–1028.

Department of Mathematics
Sun Yat-sen University
Guangzhou, P.R. China
hjinghao@mail2.sysu.edu.cn
stslyj@mail.sysu.edu.cn

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College of Mathematics
University of Belgrade, Belgrade, Serbia

Computer Science and Mathematics College
University of Al-Qadisiyah, Al-Diwaniya, Iraq
qhaq2010@gmail.com