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ON THE METRIZABILITY OF TVS-CONE METRIC SPACES

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Abstract. Metric spaces are cone metric spaces, and cone metric spaces are TVS-cone metric spaces. We prove that TVS-cone metric spaces are paracompact. A metrization theorem of TVS-cone metric spaces is obtained by a purely topological tools. We obtain that a homeomorphism f of a compact space is expansive if and only if f is TVS-cone expansive. In the end, for a TVS-cone metric topology, a concrete metric generating the topology is constructed.

1. Introduction

Cone metric spaces were introduced and discussed by Huang and Zhang in [**8**], in which every metric space is a cone metric space. In some results about metric spaces, can metric spaces be relaxed to cone metric spaces? This is an interesting question and many relevant results have been obtained (see [**1**,**8**,**12**,**20**], for example). Recently, Khani and Pourmahdian [**12**] proved that each cone metric space is metrizable, which shows that some improvements by relaxing metric spaces to cone metric spaces are trivial. This leads that more general cone metric spaces are discussed. In particular, it is interesting to consider certain topological groups in place of Banach spaces in the definition of cone metric spaces, which can serve as a topic for further studies [**12**]. In fact, Du [**6**] introduced and investigated TVS-cone metric spaces by replacing Banach spaces with topological vector spaces in the definition of cone metric spaces. In the past years, TVS-cone metric spaces have aroused many mathematical scholars' interests and some interesting results have been obtained (see [**5**, **6**, **10**, **17**], for example). However, the following question is still open.

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QUESTION 1.1. Are TVS-cone metric spaces metrizable?

As a partial answer for the above question, it is proved that each TVS-cone metric space is metrizable under assumption that the topological vector space is locally convex and Hausdorff (see [**5**, **6**, **10**], for example).

On the other hand, we notice that metrization problem for cone metric spaces and TVS-cone metric spaces were discussed by using "constructing method of concrete metric". For example, for a cone metric space (X, d), Khani and Pourmahdian [**12**] constructed a metric D on X by a symmetric function and the Frink Lemma's method such that D generates the same topology on X as the cone metric d; for a TVS-cone metric space (X, d), under assumption that the topological vector space is locally convex and Hausdorff, Du [**6**] used the nonlinear scalarization function ξ_e to construct a metric d_p on X such that the topology of (X, d_p) coincides with the topology of (X, d). Then, the following question arises naturally.

QUESTION 1.2. Can one solve metrization problem of TVS-cone metric spaces by a purely topological method?

Just as topological theories are enriched and deepen, some classical topological methods plays an important role in the development of topology. In 1948, Stone [**19**] proved one of the deepest and most important theorems we have about metric spaces: every metric space is paracompact. It would be difficult to overestimate the important role of this theorem in metrization theorem and the theory of generalized metrizable spaces [**7**]. One of the most interesting problems in general topology is the metrization problem. This is why the metrization problem of cone metric spaces and TVS-cone metric spaces cause attention once again. The general metrization problem was finally solved in the early 1950s independently by Nagata [**16**], Smirnov [**18**] and Bing [**3**]. It should be note that Stone's important result played a key role in the Nagata–Smirnov–Bing solution of the metrization problem.

Applying the technique of Stone's method, Michael [**13**–**15**] obtained a series of interesting characterizations for paracompact spaces in 1953, 1957 and 1959. It follows that closed mappings preserve T_2-paracompactness. As a further development of the technique of Stone's method, Burke [**4**] characterized subparacompact spaces and proved that closed mappings preserve subparacompactness in 1969. As an absolute gem of Stone's method, characterizations for submetacompact spaces were obtained by Junnila in 1978 [**9**], which makes that submetacompactness is preserved under closed mappings. After surveying developments for Stone's method, one can see that this method occupies an important place in topology and the breakthrough of general topology would not have been possible without Stone's paper.

This paper gives some properties of ordered topological vector spaces to investigate TVS-cone metric spaces. By Stone's sphere method, we give a purely topological proof for the metrization theorem of TVS-cone metric spaces without assumption that the topological vector space is locally convex and Hausdorff. More precisely, we use this method to prove that TVS-cone metric spaces are paracompact spaces, and then it is obtained that TVS-cone metric spaces are metrizable, which reveals the heart and soul of general topology. Also, as an application of metrization theorem of TVS-cone metric spaces, it is obtained that a homeomorphism f of a compact space is expansive if and only if f is TVS-cone expansive. In the end of this paper, for a TVS-cone metric topology, we also construct a concrete metric generating the topology.

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^*$ denote the set of all natural numbers, the set of all integral numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively.

2. Preliminaries

Definition 2.1. [**6**, **10**] Let E be a topological vector space with its zero vector θ. A subset P of E is called a TVS-cone in E if the following are satisfied.

(1) P is nonempty and closed in E.
(2) $\alpha, \beta \in P$ and $a, b \in \mathbb{R}^+ \Rightarrow a\alpha + b\beta \in P$.
(3) $\alpha, -\alpha \in P \Rightarrow \alpha = \theta$.

Remark 2.1. (1) Let P be a TVS-cone of a topological vector space E. Then the zero vector $\theta \in P - P^\circ$, where P° denotes the interior of P in E. In fact, pick $\alpha, \beta \in P$; then $(\alpha+\beta)/n \in P$ for each $n \in \mathbb{N}$ from Definition 2.1(2). Note that $(\alpha+\beta)/n \to \theta$ when $n \to \infty$. So $\theta \in P$ because P is closed from Definition 2.1(1). On the other hand, pick $\gamma \in E \setminus \{\theta\}$, then $\gamma/n \to \theta$ and $-\gamma/n \to \theta$ when $n \to \infty$. If $\theta \in P^\circ$, then there is $n \in \mathbb{N}$ such that $\{\gamma/n, -\gamma/n\} \subseteq P^\circ \subseteq P$. By Definition 2.1(3), $\gamma/n = \theta$. This contradicts that $\gamma \neq \theta$. So $\theta \notin P^\circ$.

(2) If Definition 2.1(2) is replaced by "$\alpha, \beta \in P$ and $a, b \in \mathbb{R}^* \Rightarrow a\alpha + b\beta \in P$", then we obtain the original definition of the TVS-cone introduced in [**6**, **10**], where $\mathbb{R}^*$ denotes the set of all nonnegative real numbers. By above (1), Definition 2.1 and the original definition of the TVS-cone are equivalent, which shows that the former is formally an improvement of the latter.

Definition 2.2. [**6**, **10**] Let P be a TVS-cone in a topological vector space E. Some partial orderings $\leqslant$, $<$ and $\ll$ on E with respect to P are defined as follows. For each $\alpha, \beta \in E$,

(1) $\alpha \leqslant \beta$ if $\beta - \alpha \in P$. (2) $\alpha < \beta$ if $\alpha \leqslant \beta$ and $\alpha \neq \beta$. (3) $\alpha \ll \beta$ if $\beta - \alpha \in P^\circ$.

Then a pair (E, P) is called an ordered topological vector space.

For an ordered topological vector space (E, P), unless otherwise specified, we always suppose that E is a topological vector space with its zero vector θ and P is a TVS-cone in E with nonempty interior P°.

Remark 2.2. For the sake of conveniences, we also use notations $\geqslant$, $>$ and $\gg$ in an ordered topological vector space (E, P). The meanings of these notations are clear and the following hold.

(1) $\alpha \geqslant \beta \Leftrightarrow \alpha - \beta \geqslant \theta \Leftrightarrow \alpha - \beta \in P$.
(2) $\alpha > \beta \Leftrightarrow \alpha - \beta > \theta \Leftrightarrow \alpha - \beta \in P \setminus \{\theta\}$.
(3) $\alpha \gg \beta \Leftrightarrow \alpha - \beta \gg \theta \Leftrightarrow \alpha - \beta \in P^\circ$.
(4) $\alpha \gg \beta \Rightarrow \alpha > \beta \Rightarrow \alpha \geqslant \beta$.

LEMMA 2.1. *Let (E, P) be an ordered topological vector space. Then the following hold.*

(1) *If $\alpha \gg \theta$, then $r\alpha \gg \theta$ for each $r \in \mathbb{R}^+$.*
(2) *If $\alpha \gg \theta$, then $\alpha \gg \alpha/2 \gg \cdots \gg \alpha/n \gg \cdots \gg \theta$.*
(3) *If $\alpha_1 \gg \beta_1$ and $\alpha_2 \geqslant \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.*
(4) *If $\alpha \gg \beta \geqslant \gamma$ or $\alpha \geqslant \beta \gg \gamma$, then $\alpha \gg \gamma$.*
(5) *If $\alpha \gg \theta$ and $\beta \in E$, then there is $n \in \mathbb{N}$ such that $\beta/n \ll \alpha$*
(6) *If $\alpha \gg \theta$ and $\beta \gg \theta$, then there is $\gamma \gg \theta$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.*
(7) *If $\varepsilon \gg \theta$ and $\theta \leqslant \alpha \leqslant \varepsilon/n$ for each $n \in \mathbb{N}$, then $\alpha = \theta$.*

PROOF. For $r \in \mathbb{R}^+$, $\alpha \in E$ and $B \subseteq E$, rB and $\alpha + B$ denote $\{r\beta : \beta \in B\}$ and $\{\alpha + \beta : \beta \in B\}$, respectively.

(1) Let $\alpha \gg \theta$, i.e., $\alpha \in P^\circ$. Then there is an open neighborhood B of α in E such that $B \subseteq P$. If $r \in \mathbb{R}^+$, then $rB \subseteq P$ from Definition 2.1(2). Note that $r\alpha \in rB$ and rB is an open subset of E. So $r\alpha \in P^\circ$, i.e., $r\alpha \gg \theta$.

(2) Let $\alpha \gg \theta$. For each $n \in \mathbb{N}$, $\alpha/n \gg \theta$ by (1). Furthermore, $\alpha/n - \alpha/(n+1) = \alpha/((n(n+1)) \gg \theta$, and so $\alpha/n \gg \alpha/(n+1)$.

(3) Let $\alpha_1 \gg \beta_1$ and $\alpha_2 \geqslant \beta_2$. Then $\alpha_1 - \beta_1 \gg \theta$ and $\alpha_2 - \beta_2 \geqslant \theta$, i.e., $\alpha_1 - \beta_1 \in P^\circ$ and $\alpha_2 - \beta_2 \in P$. So there is an open neighborhood B of $\alpha_1 - \beta_1$ in E such that $B \subseteq P$. Note that $(\alpha_2 - \beta_2) + B$ is an open subset of E, and $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \in (\alpha_2 - \beta_2) + B \subseteq P$ from Definition 2.1(2). So $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \in P^\circ$, i.e., $(\alpha_2 - \beta_2) + (\alpha_1 - \beta_1) \gg \theta$. It follows that $(\alpha_1 + \alpha_2) - (\beta_1 + \beta_2) \gg \theta$, i.e., $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.

(4) Let $\alpha \gg \beta \geqslant \gamma$ or $\alpha \geqslant \beta \gg \gamma$. Then $\alpha - \beta \gg \theta$ and $\beta - \gamma \geqslant \theta$, or $\alpha - \beta \geqslant \theta$ and $\beta - \gamma \gg \theta$. By (3), $\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma) \gg \theta + \theta = \theta$. So $\alpha \gg \gamma$.

(5) Let $\alpha \gg \theta$ and $\beta \in E$. It is clear that $\{\alpha - \beta/n\} \to \alpha \in P^\circ$ when $n \to \infty$. So there is $n \in \mathbb{N}$ such that $\alpha - \beta/n \in P^\circ$, i.e., $\alpha - \beta/n \gg \theta$. It follows that $\beta/n \ll \alpha$.

(6) Let $\alpha \gg \theta$ and $\beta \gg \theta$. By (5), there is $n \in \mathbb{N}$ such that $\beta/n \ll \alpha$. Put $\gamma = \beta/(n+1)$, then $\gamma \gg \theta$ by (1). By (2), $\gamma \ll \beta/n \ll \alpha$ and $\gamma \ll \beta/n \leqslant \beta$. It follows that $\gamma \ll \alpha$ and $\gamma \ll \beta$.

(7) Let $\varepsilon \gg \theta$. If $\theta \leqslant \alpha \leqslant \varepsilon/n$ for each $n \in \mathbb{N}$, then $\varepsilon/n - \alpha \geqslant \theta$, i.e., $\varepsilon/n - \alpha \in P$. Let $n \to \infty$, then $\{\varepsilon/n - \alpha\} \to -\alpha$. Hence $-\alpha \in P$ because P is closed by Definition 2.1(1). Note that $\alpha \in P$. So $\alpha = \theta$ by Definition 2.1(3). $\square$

DEFINITION 2.3. [**6**,**12**] Let (E, P) be an ordered topological vector space and let X be a nonempty set. A mapping $d : X \times X \to E$ is called a TVS-cone metric and (X, E, P, d) is called a TVS-cone metric space if the following is satisfied.

(1) $d(x, y) \geqslant 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Let (X, E, P, d) be a TVS-cone metric space. The following notations are used in this paper, where $x \in X$, $D \subseteq X$ and $\varepsilon \gg 0$.

(1) $B(x, \varepsilon) = \{y \in X : d(x, y) \ll \varepsilon\}$.
(2) $S(D, \varepsilon) = \bigcup\{B(x, \varepsilon) : x \in D\}$.

PROPOSITION 2.1. *Let (X, E, P, d) be a TVS-cone metric space. Put $\mathscr{B} = \{B(x, \varepsilon) : x \in X$ and $\varepsilon \gg 0\}$; then $\mathscr{B}$ is a base for some topology on X.*

PROOF. It is clear that $X = \bigcup \mathscr{B}$. Let $B(x, \alpha), B(y, \beta) \in \mathscr{B}$ and $z \in B(x, \alpha) \cap B(y, \beta)$. Since $z \in B(x, \alpha)$, $d(x, z) \ll \alpha$. Put $\gamma_1 = \alpha - d(x, z)$; then $\gamma_1 \gg 0$. We claim that $B(z, \gamma_1) \subseteq B(x, \alpha)$. In fact, if $u \in B(z, \gamma_1)$, then $d(z, u) \ll \gamma_1$, hence $d(x, u) \leqslant d(x, z) + d(z, u) \ll d(x, z) + \gamma_1 = \alpha$, and so $u \in B(x, \alpha)$. Using the same way, we can obtain that there is $\gamma_2 \gg 0$ such that $B(z, \gamma_2) \subseteq B(y, \beta)$. By Lemma 2.1(6), there is $\gamma \gg 0$ such that $\gamma \ll \gamma_1$ and $\gamma \ll \gamma_2$. Let $v \in B(z, \gamma)$; then $d(z, v) \ll \gamma \ll \gamma_1$ and $d(z, v) \ll \gamma \ll \gamma_2$, so $v \in B(z, \gamma_1) \cap B(z, \gamma_2) \subseteq B(x, \alpha) \cap B(y, \beta)$. This proves that $B(z, \gamma) \subseteq B(x, \alpha) \cap B(y, \beta)$. Note that $z \in B(z, \gamma) \in \mathscr{B}$. Consequently, $\mathscr{B}$ is a base for a topology on X. In fact, put $\mathscr{T} = \{U \subseteq X :$ there is $\mathscr{B}' \subseteq \mathscr{B}$ such that $U = \bigcup \mathscr{B}'\}$; then $\mathscr{T}$ is a topology on X and $\mathscr{B}$ is a base for $\mathscr{T}$. □

Let (X, E, P, d) be a TVS-cone metric space. We always suppose that X is a topological space endowed with the topology $\mathscr{T}$ described above.

Now we give Michael's theorem for characterizations of paracompact spaces and the classical Nagata–Smirnov metrization theorem.

THEOREM 2.1. [**13**] *A regular space X is paracompact if and only if each open cover of X has a σ-discrete open refinement.*

THEOREM 2.2. [**16, 18**] *A regular space X is metrizable if and only if X has a σ-locally finite base.*

3. A metrization theorem

LEMMA 3.1. *Let (X, E, P, d) be a TVS-cone metric space. Then X is regular.*

PROOF. Let $x, y \in X$ such that $x \neq y$; then $d(x, y) > \theta$. By Lemma 2.1(7), there is $\varepsilon \gg \theta$ such that "$d(x, y) \leqslant \varepsilon$" does not hold, and so "$d(x, y) \ll \varepsilon$" does not hold by Remark 2.2(4). It follows that $x \notin B(y, \varepsilon)$ and $y \notin B(x, \varepsilon)$. This proves that X is a T_1-space.

Let F be a closed subset of X and $x \in X \setminus F$. Then there is $\varepsilon \gg \theta$ such that $B(x, 2\varepsilon) \cap F = \emptyset$. Put $U = \bigcup \{B(y, \varepsilon) : y \in F\}$. It is clear that U is an open subset of X containing F. We claim that $U \cap B(x, \varepsilon) = \emptyset$, hence X is regular. In fact, if not, then there are $y \in F$ and $z \in B(y, \varepsilon) \cap B(x, \varepsilon)$. Thus, $d(x, y) \leqslant d(x, z) + d(z, y) \ll \varepsilon + \varepsilon = 2\varepsilon$, hence $y \in B(x, 2\varepsilon)$. This contradicts $B(x, 2\varepsilon) \cap F = \emptyset$. □

LEMMA 3.2. *Let (X, E, P, d) be a TVS-cone metric space, $x \in X$ and $\varepsilon \gg \theta$. Then $\{B(x, \varepsilon/n) : n \in \mathbb{N}\}$ is a neighborhood base at x in X.*

PROOF. Let $x \in U$ with U open in X. Then there is $\alpha \gg \theta$ such that $x \in B(x, \alpha) \subseteq U$. By Lemma 2.1(5), there is $k \in \mathbb{N}$ such that $\varepsilon/k \ll \alpha$. It follows that $x \in B(x, \varepsilon/k) \subseteq B(x, \alpha) \subseteq U$. This proves that $\{B(x, \varepsilon/n) : n \in \mathbb{N}\}$ is a neighborhood base at x in X. □

Theorem 3.1. *Let (X, E, P, d) be a TVS-cone metric space. Then X is a paracompact space.*

Proof. Let $\mathscr{U}$ be an open cover of X. By the well ordering principle, we can assume that $\mathscr{U} = \{U_\alpha : \alpha < \kappa\}$, where κ is an ordinal. Pick $\varepsilon \gg \theta$. For each $n \in \mathbb{N}$, $\alpha < \kappa$, put $K_{n,\alpha} = U_\alpha \smallsetminus (S(X \smallsetminus U_\alpha, \varepsilon/n) \cup \bigcup\{U_\beta : \beta < \alpha\})$ and $\mathscr{V} = \bigcup\{\mathscr{U}_n : n \in \mathbb{N}\}$, where $\mathscr{U}_n = \{S(K_{n,\alpha}, \varepsilon/(3n)) : \alpha < \kappa\}$.

Claim 1: $\mathscr{V}$ is an open refinement of $\mathscr{U}$.

It is clear that each element of $\mathscr{V}$ is an open subset of X. Let $x \in X$. Put $\alpha_0 = \min\{\alpha < \kappa : x \in U_\alpha\}$, then $\alpha_0 < \kappa$ and $x \in U_{\alpha_0} \smallsetminus \bigcup\{U_\beta : \beta < \alpha_0\}$. By Lemma 3.2, there is $n_0 \in \mathbb{N}$ such that $B(x, \varepsilon/n_0) \subseteq U_{\alpha_0}$. If $y \in X \smallsetminus U_{\alpha_0}$, then $y \notin B(x, \varepsilon/n_0)$, and so $x \notin B(y, \varepsilon/n_0)$. Consequently, $x \notin S(X \smallsetminus U_{\alpha_0}, \varepsilon/n_0)$. It follows that $x \in K_{n_0,\alpha_0} \subset S(K_{n_0,\alpha_0}, \varepsilon/(3n_0)) \in \mathscr{U}_{n_0}$. This shows that $\mathscr{V}$ is a cover of X. Let $n \in \mathbb{N}$ and $\alpha < \kappa$. If $x \in S(K_{n,\alpha}, \varepsilon/(3n))$, then there is $y \in K_{n,\alpha}$ such that $x \in B(y, \varepsilon/(3n))$. Thus, $y \notin S(X \smallsetminus U_\alpha, \varepsilon/n)$ and $y \in B(x, \varepsilon/(3n))$. It follows that $x \notin X \smallsetminus U_\alpha$, i.e., $x \in U_\alpha$. This shows that $S(K_{n,\alpha}, \varepsilon/(3n)) \subseteq U_\alpha$ for each $n \in \mathbb{N}$ and $\alpha < \kappa$. By the above, $\mathscr{V}$ is an open refinement of $\mathscr{U}$.

Claim 2: $\mathscr{U}_n$ is a discrete family in X for each $n \in \mathbb{N}$.

Let $x \in X$. It is enough to prove that $B(x, \varepsilon/(6n))$ intersects at most one member of $\mathscr{U}_n$. If not, then for some $\beta < \alpha < \kappa$, we would have

$$B(x, \varepsilon/(6n)) \cap S(K_{n,\beta}, \varepsilon/(3n)) \neq \emptyset,$$
$$B(x, \varepsilon/(6n)) \cap S(K_{n,\alpha}, \varepsilon/(3n)) \neq \emptyset.$$

Pick $y_1 \in B(x, \varepsilon/(6n)) \cap S(K_{n,\beta}, \varepsilon/(3n))$ and $y_2 \in B(x, \varepsilon/(6n)) \cap S(K_{n,\alpha}, \varepsilon/(3n))$. Then there are $z_1 \in K_{n,\beta}$ and $z_2 \in K_{n,\alpha}$ such that $y_1 \in B(z_1, \varepsilon/(3n))$ and $y_2 \in B(z_2, \varepsilon/(3n))$, hence $d(z_1, y_1) \ll \varepsilon/(3n)$ and $d(z_2, y_2) \ll \varepsilon/(3n)$. Note that $d(y_1, y_2) \leqslant d(y_1, x) + d(y_2, x) \ll \varepsilon/(3n)$. It follows $d(z_1, z_2) \leqslant d(z_1, y_1) + d(y_1, y_2) + d(y_2, z_2) \ll \varepsilon/n$. On the other hand, $K_{n,\alpha} \subseteq X \smallsetminus \bigcup\{U_\gamma : \gamma < \alpha\} \subseteq X \smallsetminus U_\beta$, so $S(K_{n,\alpha}, \varepsilon/n) \subseteq S(X \smallsetminus U_\beta, \varepsilon/n)$. Since $z_1 \in K_{n,\beta} \subseteq U_\beta \smallsetminus S(X \smallsetminus U_\beta, \varepsilon/n) \subseteq U_\beta \smallsetminus S(K_{n,\alpha}, \varepsilon/n)$, we have $z_1 \notin S(K_{n,\alpha}, \varepsilon/n)$. Note that $z_2 \in K_{n,\alpha}$. So $z_1 \notin B(z_2, \varepsilon/n)$. It follows that "$d(z_1, z_2) \ll \varepsilon/n$" does not hold, a contradiction. So $B(x, \varepsilon/(6n))$ intersects at most one member of $\mathscr{U}_n$.

By the above claims, $\mathscr{V}$ is a σ-discrete open refinement of $\mathscr{U}$. Consequently, X is a paracompact space by Lemma 3.1 and Theorem 2.1. □

Theorem 3.2. *Let (X, E, P, d) be a TVS-cone metric space. Then X is metrizable.*

Proof. Pick an $\varepsilon \gg \theta$. For each $x \in X$, put $\mathscr{B}(x) = \{B(x, \varepsilon/n) : n \in \mathbb{N}\}$; then $\mathscr{B}(x)$ is a neighborhood base at x in X by Lemma 3.2. For each $n \in \mathbb{N}$, put $\mathscr{U}_n = \{B(x, \varepsilon/n) : x \in X\}$; then $\mathscr{U}_n$ is an open cover of X. By Theorem 3.1, X is paracompact, and so $\mathscr{U}_n$ has a locally finite open refinement $\mathscr{B}_n$. Put $\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in \mathbb{N}\}$; then $\mathscr{B}$ is a σ-locally finite family consisting of open subsets of X. Note that X is regular. By Theorem 2.2, it suffices to prove that $\mathscr{B}$ is a base for X. Let $u \in U$ with U open in X. Since $\mathscr{B}(u)$ is a neighborhood base at u in X, there is $n_0 \in \mathbb{N}$ such that $B(u, \varepsilon/n_0) \subseteq U$. On the other hand, there is

$V \in \mathscr{B}_{2n_0} \subseteq \mathscr{B}$ such that $u \in V$ since $\mathscr{B}_{2n_0}$ is a cover of X. Note that $\mathscr{B}_{2n_0}$ is a refinement of $\mathscr{U}_{2n_0}$. So, there is $y \in X$ such that $u \in V \subseteq B(y, \varepsilon/(2n_0))$. It follows that $y \in B(u, \varepsilon/(2n_0))$, hence $B(y, \varepsilon/(2n_0)) \subseteq B(u, \varepsilon/n_0)$. Consequently, $V \in \mathscr{B}$ and $u \in V \subseteq B(y, \varepsilon/(2n_0)) \subseteq B(u, \varepsilon/n_0) \subseteq U$. So $\mathscr{B}$ is a base for X. □

4. An application

The following TVS-cone expansive homeomorphism is a natural generalization of expansive homeomorphism. As an application of Theorem 3.2, in this section, we prove that TVS-cone expansive homeomorphism and expansive homeomorphism are equivalent.

Definition 4.1. Let $f : X \to X$ be a homeomorphism of a space X. f is called TVS-cone expansive if there are a TVS-cone metric d on X and $\varepsilon \gg \theta$ such that $x, y \in X$ with $x \neq y$ implies $d(f^n(x), f^n(y)) \gg \varepsilon$ for some $n \in \mathbb{Z}$. Here, ε is called an expansive cone-constant for f.

Remark 4.1. (1) If "TVS-cone metric", $\gg$ and θ in Definition 4.1 are replaced by "metric", $>$ and 0' respectively, then the definition of expansive homeomorphism of a space X is obtained [**2**].

(2) It is clear that every expansive homeomorphism is TVS-cone expansive.

Definition 4.2. [**11**] Let $f : X \to X$ be a homeomorphism of a space X. A finite open cover $\mathscr{U}$ of X is called a generator for f if for every bisequence $\{A_n : n \in \mathbb{Z}\}$ consisting of members of $\mathscr{U}$, $\bigcap\{f^{-n}(\bar{A}_n) : n \in \mathbb{Z}\}$ is at most one point.

Lemma 4.1. *Let f be a homeomorphism of a compact space X. If f is TVS-cone expansive, then f has a generator and X is TVS-cone metrizable.*

Proof. Let f be TVS-cone expansive. Then there are a TVS-cone metric d on X and an expansive cone-constant $\varepsilon \gg \theta$ for f. So X is TVS-cone metrizable. Put $\mathscr{U} = \{B(x, \varepsilon/2) : x \in X\}$. Then $\mathscr{U}$ is an open cover of X and has a finite subcover $\mathscr{V}$ of $\mathscr{U}$. We claim that $\mathscr{V}$ is a generator for f. In fact, if not, then there are a bisequence $\{A_n : n \in \mathbb{Z}\}$ consisting of members of $\mathscr{V}$ and $x, y \in \bigcap\{f^{-n}(\bar{A}_n) : n \in \mathbb{Z}\}$ with $x \neq y$. Note that ε is an expansive cone-constant for f. So there is $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) \gg \varepsilon$. Put $\eta = d(f^k(x), f^k(y)) - \varepsilon$, then $\eta \gg \theta$. Since $x, y \in f^{-k}(\bar{A}_k)$, then $f^k(x), f^k(y) \in \bar{A}_n = \overline{B(z, \varepsilon/2)}$ for some $z \in X$. Note that $B(f^k(x), \eta/2) \cap B(z, \varepsilon/2) \neq \emptyset$, we pick $u \in B(f^k(x), \eta/2) \cap B(z, \varepsilon/2)$; then $d(f^k(x), z) \leqslant d(f^k(x), u) + d(u, z) \ll \eta/2 + \varepsilon/2$. Similarly, $d(f^k(y), z) \ll \eta/2 + \varepsilon/2$. Thus, $d(f^k(x), f^k(y)) \leqslant d(f^k(x), z) + d(f^k(y), z) \ll \eta + \varepsilon = d(f^k(x), f^k(y))$. It follows that $d(f^k(x), f^k(y)) - d(f^k(x), f^k(y)) \gg \theta$, i.e., $\theta = d(f^k(x), f^k(y)) - d(f^k(x), f^k(y)) \in P^\circ$. This contradicts Remark 2.2. □

Lemma 4.2. [**11**] *Let f be a homeomorphism of a compact space X. Then f is expansive if and only if f has a generator and X is metrizable.*

Theorem 4.1. *Let f be a homeomorphism of a compact space X. Then the following are equivalent.*

(1) *f is expansive.* (2) *f is TVS-cone expansive.*

PROOF. (1) $\Rightarrow$ (2): It holds by Remark 4.1(2).

(2) $\Rightarrow$ (1): Let f be TVS-cone expansive. Then f has a generator and X is TVS-cone metrizable by Lemma 4.1. By Theorem 3.2, X is metrizable. It follows that f is expansive by Lemma 4.2. $\square$

5. A concrete metric on TVS-cone metric spaces

In this section, for a TVS-cone metric space, we construct a concrete metric topology coinciding with the TVS-cone metric topology, which came from the report of the referees.

THEOREM 5.1. *Let (X, E, P, d) be a TVS-cone metric space. Let $\varepsilon_0 \gg \theta$. Define $D : X \times X \to \mathbb{R}^*$ by $D(x, y) = \inf\{r \in \mathbb{R}^* : d(x, y) \ll r\varepsilon_0\}$ for all $x, y \in X$. Then D is a metric on X.*

PROOF. (1) Let $x \in X$. For any $r \in \mathbb{R}^+$, $d(x, x) = \theta \ll r\varepsilon_0$, so $D(x, x) = \inf\{r \in \mathbb{R}^* : d(x, x) \ll r\varepsilon_0\} = \inf \mathbb{R}^+ = 0$. Let $x, y \in X$ and $D(x, y) = 0$. If $d(x, y) \neq \theta$, then there is $r_0 \in \mathbb{R}^+$ such that $d(x, y) \leqslant r_0\varepsilon_0$ does not hold by Lemma 2.1(7). It follows that $d(x, y) \ll r\varepsilon_0$ does not hold for all $r < r_0$. Thus, $D(x, y) = \inf\{r \in \mathbb{R}^* : d(x, x) \ll r\varepsilon_0\} \geqslant r_0$. This contradicts $D(x, y) = 0$, so $d(x, y) = 0$. Consequently, $x = y$.

(2) It is clear that $D(x, y) = D(y, x)$ for all $x, y \in X$.

(3) Let $x, y, z \in X$. Then $d(x, y) \leqslant d(x, z) + d(z, y)$. So

$$
\begin{aligned}
D(x, y) &= \inf\{r \in \mathbb{R}^* : d(x, y) \ll r\varepsilon_0\} \leqslant \inf\{r \in \mathbb{R}^* : d(x, z) + d(z, y) \ll r\varepsilon_0\} \\
&\leqslant \inf\{r \in \mathbb{R}^* : d(x, z) \ll r\varepsilon_0\} + \inf\{r \in \mathbb{R}^* : d(z, y) \ll r\varepsilon_0\} \\
&= D(x, z) + D(z, y).
\end{aligned}
$$

By (1), (2), (3), D is a metric on X. $\square$

THEOREM 5.2. *Let (X, E, P, d) be a TVS-cone metric space, $\mathscr{T}$ be the topology on X described in Proposition 2.1 and D be the metric described above. For $x \in X$ and $r \in \mathbb{R}^+$, put $B_D(x, r) = \{y \in X : D(x, y) < r\}$, and $\mathscr{F} = \{B_D(x, r) : x \in X \text{ and } r \in \mathbb{R}^+\}$; then $\mathscr{F}$ is a base for the topology $\mathscr{T}$.*

PROOF. Let $B(x, \varepsilon) \in \mathscr{B}$, where $\mathscr{B}$ is the base for the topology $\mathscr{T}$ described in Proposition 2.1. By Lemma 2.1(5), there is $n \in \mathbb{N}$ such that $\varepsilon_0/n \ll \varepsilon$. We prove that $B_D(x, 1/n) \subseteq B(x, \varepsilon)$. Let $y \in B_D(x, 1/n)$. Then $D(x, y) < 1/n$, i.e., $\inf\{t \in \mathbb{R}^* : d(x, y) \ll t\varepsilon_0\} < 1/n$. If $d(x, y) \ll \varepsilon_0/n$ does not hold, then $d(x, y) \ll t\varepsilon_0$ does not hold for all $t < 1/n$. This results that $D(x, y) = \inf\{t \in \mathbb{R}^* : d(x, y) \ll t\varepsilon_0\} \geqslant 1/n$, which contradicts $D(x, y) < 1/n$. This shows that $d(x, y) \ll \varepsilon_0/n < \varepsilon$. So $y \in B(x, \varepsilon)$. Thus, we proved that $B_D(x, 1/n) \subseteq B(x, \varepsilon)$.

On the other hand, let $B_D(x, r_0) \in \mathscr{F}$. Then $B(x, r_0\varepsilon_0) \subseteq B_D(x, r_0)$. In fact, if $y \in B(x, r_0\varepsilon_0)$, then $d(x, y) \ll r_0\varepsilon_0$. It follows that $D(x, y) = \inf\{r \in \mathbb{R}^* : d(x, y) \ll r\varepsilon_0\} \leqslant r_0$, so $y \in B_D(x, r_0)$. This proves that $B(x, r_0\varepsilon_0) \subseteq B_D(x, r_0)$. Consequently, $\mathscr{F}$ is a base for the topology $\mathscr{T}$. $\square$

Remark 5.1. Let (X, E, P, d) be a TVS-cone metric space and D be the metric described in Theorem 5.1. By Theorem 5.2, the topology $\mathscr{T}_D$ generated by D coincides with the topology $\mathscr{T}$ generated by d.

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