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RESIDUAL CHARACTER OF QUASILINEAR VARIETIES OF GROUPOIDS

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This paper is dedicated to Professor Jaroslav "Jarda" Ježek, coauthor, mentor and friend, who passed away five years ago.

Abstract. We consider the quasilinear varieties of groupoids which were characterized in [4] and find the residual character for all of them. Those varieties which are residually small turn out to be residually finite. We compute the residual bounds and find all subdirectly irreducible algebras in them.

1. Introduction

In this paper we complete the investigation of quasilinear varieties of groupoids started in [**3**] and [**4**] by investigating the residual character of those varieties. We have already determined the residual character of linear varieties in [**3**], so here we deal with quasilinear varieties which are not linear.

Finding the residual bound of finitely generated varieties is related to the Restricted Quackenbush Conjecture. The Restricted Quackenbush Conjecture claims that no finitely generated variety in a finite language has residual bound exactly ω (i.e. it is impossible for such varieties to contain only finite subdirectly irreducible algebras, but of unbounded size). All quasilinear varieties were shown to be finitely generated in [**4**]. Here we will prove that all quasilinear varieties have either a finite residual bound or are residually large. Thus, we establish that there are no counterexamples to the Restricted Quackenbush Conjecture among quasilinear varieties of groupoids. For those quasilinear varieties which are residually finite we find all subdirectly irreducible algebras that contain them, thus semantically describing them up to Horn formulae (rather than just identities). For the residually

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large quasilinear varieties, we had to invent new constructions to prove residual largeness, which may be useful in future research.

We introduce some of the definitions, notation and basic facts about universal algebra which we will need in this paper. This is not comprehensive, the reader is expected to know standard universal algebra (e.g. free algebras, varieties etc.) in order to follow the arguments in our paper. Good textbooks are [**2, 7, 1**].

Let $\mathcal{F}$ be an algebraic language. We say that an $\mathcal{F}$-term is linear if it is a variable, or equal to $f(x_1,\dots,x_n)$, where f is an operation symbol of arity n and $x_1,\dots,x_n$ are variables (thus constant symbols are also linear terms, being nullary operation symbols).

In the previous papers [**4**] and [**3**] we introduced equational theories which are represented on linear terms, as defined below.

Definition 1.1. An equational theory E is *linear* if any term is E-equivalent to a unique linear term. E is *quasilinear* if it is idempotent and each term is E-equivalent to a linear term (not necessarily unique). The variety of all models of a linear (quasilinear) equational theory is called a linear (quasilinear) variety.

Note that we called these properties $*$-linear and $*$-quasilinear in [**3**] and [**4**]. This was because we needed to distinguish n-linear and n-quasilinear for the equational theories which have the above properties just for terms with up to n variables. Now we don't need these weaker properties and thus we simplify our terminology.

We recall some basic facts about simple and subdirectly irreducible algebras. An algebra is *simple* if it has more than one element and only the two trivial congruences, the equality and full relation. An algebra is *subdirectly irreducible* if it has more than one element and the least congruence which is not the equality relation. The least congruence different from the equality relation is called the *monolith* of a subdirectly irreducible algebra. A class of algebras is *residually small* if it has a cardinal bound on the size of subdirectly irreducible algebras in the class. The class of algebras is *residually large* otherwise. The property of residual smallness/largeness is called the residual character of the class of algebras in question. The *residual bound* of a class of algebras $\mathcal{K}$, denoted by $\operatorname{resb}(\mathcal{K})$ is the least cardinal κ such that all subdirectly irreducible algebras in $\mathcal{K}$ are of size strictly smaller than κ, if $\mathcal{K}$ is residually small, or $\operatorname{resb}(\mathcal{K})=\infty$ if $\mathcal{K}$ is residually large.

Let $\mathbf{A}$ be an algebra. We use the standard notation $\operatorname{Con}\mathbf{A}$ for the set of all congruences and $\mathbf{Con}\,\mathbf{A}$ for the lattice of all congruences of an algebra $\mathbf{A}$. For $X\subseteq A\times A$, $\operatorname{Cg}^{\mathbf{A}}(X)$ is the congruence of $\mathbf{A}$ generated by X. When $X=\{\langle a,b\rangle\}$, we use $\operatorname{Cg}^{\mathbf{A}}(a,b)$ as short notation for the principal congruence $\operatorname{Cg}^{\mathbf{A}}(X)$.

Let $\mathbf{A}$ be an algebra. A *basic translation* is either the identity map id_A or the operation $p(x)=f^{\mathbf{A}}(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)$, where $f\in\mathcal{F}$ is an operation symbol of arity $n\geqslant 1$, $1\leqslant i\leqslant n$ and $a_j\in A$ for all j. A *translation* is a composition of basic translations. The set of all translations of $\mathbf{A}$ is denoted by $\operatorname{Tr}A$. In the case of a groupoid $\mathbf{G}=(G;\cdot)$, the basic translations are id_G, the maps $\lambda_a(x)=ax$ and $\rho_a(x)=xa$ for all $a\in G$.

In general, $\operatorname{Cg}^{\mathbf{A}}(X)$ is the transitive closure of the set

$$\big\{\langle u,v\rangle\in A^2 : (\exists\langle x,y\rangle\in X)(\exists p(x)\in\operatorname{Tr}A)\big(\{u,v\}=\{p(x),p(y)\}\big)\big\}.$$

Some authors call this a Mal'cev chain, each pair of the form $\{p(x), p(y)\}$ forming a link in the chain.

For shorter notation, we will write $x_1x_2x_3\dots x_n$ instead of $(((x_1x_2)x_3)\dots)x_n$ (the parentheses are grouped to the left), $x \cdot yz$ instead of $x(yz)$, etc. By a left-associated term we mean a term $x_1x_2\dots x_n$ where $n \geqslant 1$ and $x_1,\dots,x_n$ are variables. We finish the section by introducing some more notation.

Let G be a nonvoid set, and κ a cardinal. For any $a, b \in G$ and $j \in \kappa$, the element $a^\kappa_{j\mapsto b}$ of G^κ is defined by

$$a^\kappa_{j\mapsto b}(i) = \begin{cases} a, & \text{for all } i \in \kappa \smallsetminus \{j\}; \\ b, & \text{for } i = j. \end{cases}$$

Let $\alpha \in G^\kappa$ and $e, f \in G$. Then the element $\alpha_{e\mapsto f}$ in G^κ is defined by

$$\alpha_{e\mapsto f}(i) = \begin{cases} \alpha(i), & \text{if } \alpha(i) \neq e; \\ f, & \text{if } \alpha(i) = e. \end{cases}$$

2. Our result, statement and easy cases

We recall the poset of all quasilinear varieties of groupoids, which was discovered in [**4**].

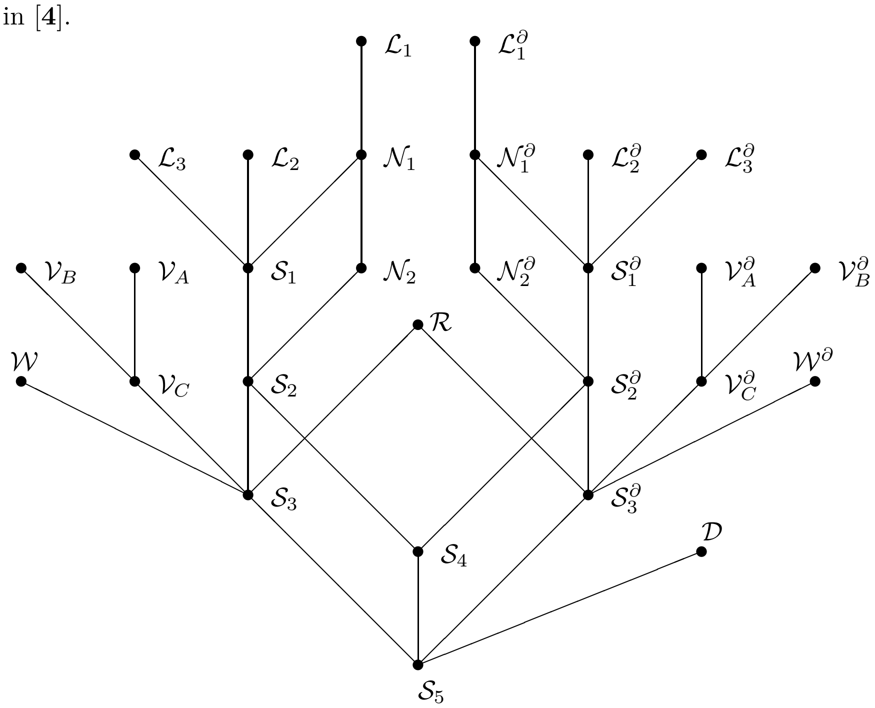

FIGURE 1. The poset of all $*$-quasilinear varieties.

In this paper we prove that these varieties have the residual characters as depicted in Table 1. First we get the easy cases out of the way.

TABLE 1. Residual bounds of idempotent $*$-quasilinear varieties.

Variety	Residual bound
$\mathcal{S}_5$	2
$\mathcal{S}_3,\ \mathcal{S}_3^\partial,\ \mathcal{S}_4,\ \mathcal{R},$	3
$\mathcal{S}_2,\ \mathcal{S}_2^\partial,\ \mathcal{V}_C,\ \mathcal{V}_C^\partial,\ \mathcal{D},$	4
$\mathcal{W},\ \mathcal{W}^\partial$	5
$\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3,\ \mathcal{N}_1,\ \mathcal{N}_2,\ \mathcal{S}_1,\ \mathcal{V}_A,\ \mathcal{V}_B,$ $\mathcal{L}_1^\partial,\ \mathcal{L}_2^\partial,\ \mathcal{L}_3^\partial,\ \mathcal{N}_1^\partial,\ \mathcal{N}_2^\partial,\ \mathcal{S}_1^\partial,\ \mathcal{V}_A^\partial,\ \mathcal{V}_B^\partial,$	residually large

If $\mathbf{A} = \langle A; \cdot\rangle$ is a groupoid, then $\mathbf{A}^\partial = \langle A; *\rangle$ such that $x * y = y \cdot x$ for all $x, y \in A$, while $\mathcal{V}^\partial$ stands for the variety $\{\mathbf{A}^\partial : \mathbf{A} \in \mathcal{V}\}$. Clearly, $\mathcal{V}$ and $\mathcal{V}^\partial$ have the same residual bounds. Recall that all subdirectly irreducible groupoids in the variety $\mathcal{S}_2$ (the variety of idempotent semigroups which satisfy $xyz \approx xzy$) were described in [**6**]. Thus we know that $\mathrm{resb}(\mathcal{S}_2) = 4$, and that its subvarieties $\mathcal{S}_3$ (the left-zero semigroups, i.e. all models of $xy \approx x$) and $\mathcal{S}_4$ (semilattices) have only the two-element subdirectly irreducible algebras. Also, it is well known (and easy to prove) that the variety of rectangular bands $\mathcal{R}$ has only the two-element left-zero semigroup and the two-element right-zero semigroup as nontrivial subdirectly irreducibles. Therefore, $\mathrm{resb}(\mathcal{S}_4) = \mathrm{resb}(\mathcal{S}_3) = \mathrm{resb}(\mathcal{R}) = 3$. The trivial variety $\mathcal{S}_5$ of one-element groupoids is conventionally given the residual bound 2.

We also know that the variety $\mathcal{S}_1$ of idempotent semigroups, which satisfy $xyx \approx xy$, contains subdirectly irreducible algebras of size at least κ where κ is any cardinal (cf. Section 3 of [**6**], and replace the set of natural numbers with κ). Thus all varieties which contain $\mathcal{S}_1$ are residually large, and those are $\mathcal{N}_1$, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

The variety $\mathcal{D}$ is the class of all models of the identities

(D1) $xx \approx x$ (D2) $xy \approx yx$

(D3) $x \cdot xy \approx y$ (D4) $xy \cdot zt \approx xz \cdot yt$

It is proved in [**4**], Theorem 9.7, that the variety $\mathcal{D}$ is $*$-quasilinear and generated by the three-element groupoid $\mathbf{G}_7$.

$\mathbf{G}_7$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Since the equational base of the variety $\mathcal{D}$ contains the identities $(D1)$ $xx \approx x$, $(D2)$ $xy \approx yx$, $(D3)$ $x \cdot xy \approx y$, it follows that $\mathcal{D}$ is a variety of Steiner quasigroups. Actually it is the least such nontrivial variety. The subdirectly irreducible Steiner quasigroups were investigated in the influential paper [**9**], but since the variety $\mathcal{D}$ is not generated by the so-called planar ones, we would have to work through the proofs given in [**9**] to realize that $\mathrm{resb}(\mathcal{D}) = 4$.

The easier way is to realize that $(D1)-(D4)$ are alternative axioms for the subvariety denoted by $\mathcal{V}_3$ of the variety of SIE groupoids in the terminology of B. Roszkowska-Lech. This $\mathcal{V}_3$ is a variety which is axiomatized by $(D1)$, $(D4)$ and

$$\text{(D2}') \ x \approx y \cdot xy \qquad\qquad \text{(D3}') \ xyy \approx x$$

in place of $(D2)$ and $(D3)$. We only need to prove $(D2')$ and $(D3')$ in $\mathcal{D}$, though it can be proved that $(D2)$ and $(D3)$ hold in $\mathcal{V}_3$, too. $(D3')$ follows trivially from commutativity and $(D3)$, while $x \approx_{(D3)} y \cdot yx \approx_{(D2)} y \cdot xy$. Then we invoke [**10**], where it is proved that all algebras in $\mathcal{V}_k$ for odd k are polynomially equivalent to Abelian groups which additionally satisfy $kx = 0$. Thus the only subdirectly irreducible algebra in $\mathcal{D} = \mathcal{V}_3$ is the one polynomially equivalent to the three-element Abelian group, and $\mathrm{resb}(\mathcal{D}) = 4$. The reader is referred to papers [**11**] and [**12**] for more details, and to [**5**] for yet another approach to the same variety.

It remains to find the residual bounds of $\mathcal{N}_2$, $\mathcal{V}_A$, $\mathcal{V}_B$, $\mathcal{V}_C$ and $\mathcal{W}$, which we will do in the following sections.

3. The varieties $\mathcal{N}_2$, $\mathcal{V}_A$ and $\mathcal{V}_B$ are residually large

We first consider the variety $\mathcal{N}_2$, given by its equational base

(N1) $xx \approx x$ (N2) $x \cdot yx \approx xy$

(N3) $x \cdot xyz \approx x \cdot yz$ (N4) $x \cdot yz \approx x \cdot zy$

LEMMA 3.1. *The variety $\mathcal{N}_2$ is generated by the subdirectly irreducible groupoid $\mathbf{G}_{\mathcal{N}_2}$*

$\mathbf{G}_{\mathcal{N}_2}$	a	b	c	d
a	a	d	c	d
b	b	b	b	b
c	c	b	c	b
d	d	d	d	d

PROOF. One can directly verify that the groupoid $\mathbf{G}_{\mathcal{N}_2}$ satisfies the identities (1)–(4). The groupoid $\mathbf{G}_{\mathcal{N}_2}$ is subdirectly irreducible with the monolith equal to the congruence $\mathrm{Cg}^{\mathbf{G}_{\mathcal{N}_2}}(b, d)$. The groupoid $\mathbf{G}_{\mathcal{N}_2}$ is not a semigroup, since $acb \neq a \cdot cb$. Therefore, $\mathcal{V}(\mathbf{G}_{\mathcal{N}_2}) \not\subseteq \mathcal{S}_2$, but $\mathcal{V}(\mathbf{G}_{\mathcal{N}_2}) \subseteq \mathcal{N}_2$. From the fact that the varieties depicted in Figure 1 are a down-set (order ideal) in the lattice of all groupoid varieties, we deduce that $\mathcal{V}(\mathbf{G}_{\mathcal{N}_2})$ can be only $\mathcal{N}_2$. □

THEOREM 3.1. *The variety $\mathcal{N}_2$ is residually large.*

PROOF. The groupoid $\mathbf{G}_{\mathcal{N}_2}$ generates $\mathcal{N}_2$, according to Lemma 3.1. Let κ be a cardinal. Consider the groupoid $\mathbf{G} := (\mathbf{G}_{\mathcal{N}_2})^\kappa$ and denote by b^κ and d^κ the elements of the groupoid $\mathbf{G}$ such that $b^\kappa(i) = b$ and $d^\kappa(i) = d$ for all $i \in \kappa$. Define the relation ρ on the groupoid $\mathbf{G}^\kappa_{\mathcal{N}_2}$ by: $x\,\rho\,y$ iff one of the following three conditions hold:

(1) $x = y$,
(2) $\{x, y\} = \{b^\kappa, d^\kappa\}$,

(3) $\{x,y\}\cap\{b^\kappa,d^\kappa\}=\emptyset$, $d\in x(\kappa)$, $d\in y(\kappa)$, $x(\kappa)\not\subseteq\{a,d\}$, $y(\kappa)\not\subseteq\{a,d\}$ and for each $i\in\kappa$, either $\{x(i),y(i)\}=\{b,d\}$, or $x(i)=y(i)$.

The relation ρ defined above is an equivalence relation and has at least 2^κ equivalence classes. Next we are showing that ρ is a congruence of $\mathbf{G}$. It suffices to show that the following holds for all $x,y,z\in G$: If $x\rho y$, then $xz\rho yz$ and $zx\rho zy$.

Let $x\rho y$ and let $z\in G$ be arbitrary. By the definition of the relation ρ, it follows that $\{i\in\kappa : x(i)\neq y(i)\}=\{i\in\kappa : \{x(i),y(i)\}=\{b,d\}\}$. The fact that $ub=ud$ in $\mathbf{G}_{\mathcal{N}_2}$ for all $u\in\{a,b,c,d\}$ implies that $zx=zy$.

Now we show $\langle xz,yz\rangle\in\rho$. If $x=b^\kappa$ and $y=d^\kappa$, then $xz=x$ and $yz=y$, as b and d are left zeros of the groupoid $\mathbf{G}_{\mathcal{N}_2}$. Let x and y be distinct elements of G and $\{x,y\}\cap\{b^\kappa,d^\kappa\}=\emptyset$. Then $d\in x(\kappa)\cap y(\kappa)$ and since d is a left zero in $\mathbf{G}_{\mathcal{N}_2}$, thus $d\in xz(\kappa)\cap yz(\kappa)$. Since $x\rho y$, it follows that $\{b,c\}\cap x(\kappa)\neq\emptyset$, but since $\{b,c\}\cdot G=\{b,c\}$, it follows that $\{b,c\}\cap xz(\kappa)\neq\emptyset$, i.e. $xz(\kappa)\not\subseteq\{a,d\}$. Analogously we show that $yz(\kappa)\not\subseteq\{a,d\}$. If $x(i)\neq y(i)$, then $\{x(i),y(i)\}=\{b,d\}$, from which it follows that $\{xz(i),yz(i)\}=\{b,d\}$. Therefore, $\{xz(i),yz(i)\}\in\{\{a\},\{b\},\{c\},\{d\},\{b,d\}\}$, for all $i\in\kappa$. This proves that ρ is a congruence of $\mathbf{G}$.

Note: the set $\{b,d\}^\kappa$ is the union of two ρ-classes. One is $\{b^\kappa,d^\kappa\}$, and the other one is all other elements of $\{b,d\}^\kappa$. In other words, for all $x,y\in G$, if $x(\kappa)=y(\kappa)=\{b,d\}$, then $x\rho y$.

Next we show that $\mathbf{G}/\rho$ is subdirectly irreducible. Let the congruence $\theta\in\operatorname{Con}\mathbf{G}$ be such that $\rho\subsetneq\theta$.

We want to show that there exists $p\in G$ which satisfies $p(\kappa)=\{b,d\}$ such antecedent that $\langle b^\kappa,p\rangle\in\theta$. Considering the discussion on ρ restricted to $\{b,d\}^\kappa$, this would imply that every such θ has $\{b,d\}^\kappa$ in a single θ-class, and the congruence $(\rho\vee\operatorname{Cg}^{\mathbf{G}}(b^\kappa,p))/\rho$ (which is the same no matter which p is chosen) would be the monolith of $\mathbf{G}/\rho$.

Let $x,y\in G$ be such that $\langle x,y\rangle\in\theta\setminus\rho$. We consider all cases:

Case 1: There exists $j\in\kappa$ such that $\{x(j),y(j)\}\in P_2(\{a,b,c,d\})\setminus\{b,d\}$, where $P_2(X)$ is the set of all subsets of X with exactly two elements. Then $b^\kappa_{j\mapsto a}x(i)=b^\kappa_{j\mapsto a}y(i)=b$ for all $i\in\kappa\setminus\{j\}$ and $\{b^\kappa_{j\mapsto a}x(j),b^\kappa_{j\mapsto a}y(j)\}\in P_2(\{a,c,d\})$. If $c\in\{b^\kappa_{j\mapsto a}x(j),b^\kappa_{j\mapsto a}y(j)\}$, then

$$\{b^\kappa_{j\mapsto a}xd^\kappa,b^\kappa_{j\mapsto a}yd^\kappa\}=\{b^\kappa,b^\kappa_{j\mapsto d}\},$$

therefore, $\langle b^\kappa,b^\kappa_{j\mapsto d}\rangle\in\theta$. If $\{b^\kappa_{j\mapsto a}x(j),b^\kappa_{j\mapsto a}y(j)\}=\{a,d\}$, then

$$\{b^\kappa_{j\mapsto a}xc^\kappa d^\kappa,b^\kappa_{j\mapsto a}yc^\kappa d^\kappa\}=\{b^\kappa,b^\kappa_{j\mapsto d}\},$$

and hence $\langle b^\kappa,b^\kappa_{j\mapsto d}\rangle\in\theta$. Thus, $b^\kappa_{j\mapsto d}$ can be chosen for p.

Case 2: For each $i\in\kappa$,

$$\{x(i),y(i)\}\in\{\{a\},\{b\},\{c\},\{d\},\{b,d\}\}.$$

Our assumptions about x and y imply that $x\neq y$, $\{x,y\}\neq\{b^\kappa,d^\kappa\}$.

If $x=b^\kappa$, then $y(\kappa)=\{b,d\}$, so $b^\kappa=x\,\theta\,y$ and y can be chosen for the element p which we need. If $x=d^\kappa$, then $b^\kappa\rho\,d^\kappa=x\,\theta\,y$ and y can be chosen for p, as $\rho\subseteq\theta$.

Let $\{x,y\}\cap\{b^\kappa,d^\kappa\}=\emptyset$. Suppose that $d\notin x(\kappa)$ and $d\notin y(\kappa)$. Assumptions of Case 2 imply that $x=y$, a contradiction with $\langle x,y\rangle\notin\rho$. Let $d\in x(\kappa)$ and

$d \notin y(\kappa)$. Then there exists some $j \in \kappa$ such that $x(j) = d$ and $y(j) = b$. Thus $b^\kappa = yy_{a\mapsto c}b^\kappa \theta\, xy_{a\mapsto c}b^\kappa$ and since $xy_{a\mapsto c}b^\kappa(\kappa) = \{b, d\}$, we can take $xy_{a\mapsto c}b^\kappa$ for p. The case $d \notin x(\kappa)$ and $d \in y(\kappa)$ is analogous.

Now let $\{x, y\} \cap \{b^\kappa, d^\kappa\} = \emptyset$ and $d \in x(\kappa)$, $d \in y(\kappa)$. Assume that $x(\kappa) \subseteq \{a, d\}$. Then $b^\kappa \rho\, d^\kappa = xb^\kappa \theta\, yb^\kappa$, and since $yb^\kappa(\kappa) = \{b, d\}$, it follows that yb^κ can be taken for p. The case when $y(\kappa) \subseteq \{a, d\}$ is analogous. All remaining cases for $\langle x, y\rangle$ are already in ρ.

Therefore, the groupoid $\mathbf{G}/\rho$ is subdirectly irreducible, and the congruence $\mathrm{Cg}^{\mathbf{G}/\rho}(b^\kappa/\rho, b^\kappa_{j\mapsto d}/\rho)$ is its monolith. $\square$

We turn to the variety $\mathcal{V}_A = HSP(\mathbf{A})$. According to [**4**], more precisely the definition of $\mathcal{V}_A$ and Theorem 7.15, it is the variety generated by

A	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	d	e	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

whose equational base is

(A1) $xx \approx x$ (A2) $xyy \approx xy$

(A3) $x \cdot yz \approx xy$ (A4) $xyzy \approx xyz$

Theorem 3.2. *The variety $\mathcal{V}_A = HSP(\mathbf{A})$ is residually large.*

Proof. Let κ be any cardinal. Consider the groupoid $\mathbf{A}^\kappa$ and the set $G = \{d^\kappa_{i\mapsto u} \mid u \in \{a, b, c\},\ i \in \kappa\} \cup \{d, e\}^\kappa$. Then $\mathbf{G}$ is a subgroupoid of $\mathbf{A}^\kappa$. Define the relation ρ on the groupoid $\mathbf{G}$ by: $x\ \rho\ y$ iff one of the conditions

(1) $x = y$, or
(2) $\{x, y\} \in \{d, e\}^\kappa \setminus \{d^\kappa\}$.

is satisfied. The equivalence relation ρ has κ equivalence classes. Let us show that ρ is a congruence of $\mathbf{G}$. It suffices to prove for all $x, y, z \in G$, if $x\rho y$, then $xz\rho yz$ and $zx\rho zy$. Since d and e are left zeros and right neutral elements of $\mathbf{A}$, it follows that ρ is a congruence of $\mathbf{G}$.

Next we show that $\mathbf{G}/\rho$ is subdirectly irreducible. Let $\theta \in \mathrm{Con}\,\mathbf{G}$ satisfy $\rho \subsetneq \theta$. We aim to show that there exists $p \in \{d, e\}^\kappa \setminus \{d^\kappa\}$ such that $\langle d^\kappa, p\rangle \in \theta$. Note that $\{d^\kappa\}$ and $\{d, e\}^\kappa \setminus \{d^\kappa\}$ are ρ-classes. If we show the claim, then it will follow that the pair $\langle \{d, e\}^\kappa \setminus \{d^\kappa\}, \{d^\kappa\}\rangle$ is contained in any congruence of $\mathbf{G}/\rho$ which is not the equality relation, i.e. it will follow that $\mathbf{G}/\rho$ is subdirectly irreducible.

Let $\langle x, y\rangle \in \theta \setminus \rho$. We consider the following cases for $\langle x, y\rangle$:

Case 1: There exists $j \in \kappa$ such that $\langle x(j), y(j)\rangle \in \{a\}\times\{b, c, d, e\}\cup\{b, c, d, e\}\times\{a\}$. Then $\{d^\kappa_{j\to c}x, d^\kappa_{j\to c}y\} = \{d^\kappa, d^\kappa_{j\to c}\}$ or $\{d^\kappa_{j\to c}x, d^\kappa_{j\to c}y\} = \{d^\kappa, d^\kappa_{j\to e}\}$. Hence,

$$\langle (d^\kappa_{j\mapsto c}x)d^\kappa_{j\mapsto b}, (d^\kappa_{j\mapsto c}y)d^\kappa_{j\mapsto b}\rangle \in \{\langle d^\kappa, d^\kappa_{j\mapsto e}\rangle, \langle d^\kappa_{j\mapsto e}, d^\kappa\rangle\},$$

and thus $\langle d^\kappa, d^\kappa_{j\mapsto e}\rangle \in \mathrm{Cg}^{\mathbf{G}}(x,y) \subseteq \theta$, so we take $p = d^\kappa_{j\mapsto e}$.

Case 2: There exists $j \in \kappa$ such that $\langle x(j), y(j)\rangle \in \{b\}\times\{c,d,e\}\cup\{c,d,e\}\times\{b\}$. Then $\{d^\kappa_{j\to c}x, d^\kappa_{j\to c}y\} = \{d^\kappa_{j\to e}, d^\kappa_{j\to c}\}$. Hence,

$$\langle (d^\kappa_{j\mapsto c}x)d^\kappa_{j\mapsto a}, (d^\kappa_{j\mapsto c}y)d^\kappa_{j\mapsto a}\rangle \in \{\langle d^\kappa, d^\kappa_{j\mapsto e}\rangle, \langle d^\kappa_{j\mapsto e}, d^\kappa\rangle\}.$$

Therefore, $\langle d^\kappa, d^\kappa_{j\mapsto e}\rangle \in \mathrm{Cg}^{\mathbf{G}}(x,y) \subseteq \theta$, so we take $p = d^\kappa_{j\mapsto e}$.

Case 3: There exists $j \in \kappa$ such that $\{x,y\} = \{d^\kappa_{j\mapsto c}, p\}$ for some $p \in \{d,e\}^\kappa \smallsetminus \{d^\kappa\}$. Without loss of generality, let $y = p$. Then $\langle d^\kappa, p\rangle = \langle x d^\kappa_{j\mapsto a}, y d^\kappa_{j\mapsto a}\rangle \in \mathrm{Cg}^{\mathbf{G}}(x,y) \subseteq \theta$, and since $p \in \{d,e\}^\kappa \smallsetminus \{d^\kappa\}$, the case is done.

Case 4: There exist $j, l \in \kappa$ so that $\{x,y\} = \{d^\kappa_{j\to c}, d^\kappa_{l\to c}\}$. Then $\{x d^\kappa_{l\to b}, y d^\kappa_{l\to b}\} = \{d^\kappa_{j\to c}, d^\kappa_{l\to e}\}$, and since $\langle x d^\kappa_{l\to b}, y d^\kappa_{l\to b}\rangle \in \mathrm{Cg}^{\mathbf{G}}(x,y) \subseteq \theta$, θ is also in Case 3.

Case 5: There exists $j \in \kappa$ such that $x = d^\kappa_{j\mapsto c}$ and $y = d^\kappa$ (or vice versa, but it is analogous). Then $\langle d^\kappa_{j\mapsto e}, d^\kappa\rangle = \langle x d^\kappa_{j\mapsto b}, y d^\kappa_{j\mapsto b}\rangle \in \mathrm{Cg}^{\mathbf{G}}(x,y) \subseteq \theta$, so we take $p = d^\kappa_{j\mapsto e}$. □

We turn our attention to the variety $\mathcal{V}_B = HSP(\mathbf{B})$. According to [**4**], more precisely the definition of $\mathcal{V}_B$ and Theorem 7.15, it is the variety generated by

$\mathbf{B}$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	d	c	c	c	c
d	d	e	d	d	d
e	d	e	e	e	e

(actually, it is also generated by the subalgebra with universe $\{a,b,d,e\}$). The equational base of $\mathcal{V}_B$ is

(B1) $xx \approx x$ (B2) $xyy \approx xy$

(B3) $x \cdot yz \approx xy$ (B4) $xyzy \approx xzy$

THEOREM 3.3. *The variety $\mathcal{V}_B$ is residually large.*

PROOF. Let κ be any cardinal. For any $x \in \kappa$, x' and x'' are short notation for $\langle 0, x\rangle$ and $\langle 1, x\rangle$, respectively. Consider the groupoid $\mathbf{G} = (\{0,1\}\times\kappa, \cdot)$ where the operation $\cdot$ is defined like this:

$$x \cdot y = \begin{cases} 1'', & \text{if } x = 0'',\ y = 0'; \\ 0'', & \text{if } x = k'',\ y = l',\ l \neq 0 \text{ and } k \leqslant l; \\ x, & \text{else.} \end{cases}$$

The operation is given in Table 2. Notice that the elements of $\{0\}\times\kappa$ are left zeros, while the elements of $\{1\}\times\kappa$ are right neutral. Besides, we note that $1''y = 0''y$ for all $y \in \{0\}\times\kappa$.

We show that $\mathbf{G} \in \mathcal{V}_B$, by showing $\mathbf{G}$ satisfies (B1)–(B4).

Identity (B1): $\mathbf{G}$ is idempotent since it consists of left zeros and right units.

Table 2. The Cayley table of the subdirectly irreducible groupoid in the variety $\mathcal{V}_B$ of size 2κ.

	$0'$	$1'$	$2'$	$3'$	$4'$	$\dots$	$4''$	$3''$	$2''$	$1''$	$0''$
$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$\dots$	$0'$	$0'$	$0'$	$0'$	$0'$
$1'$	$1'$	$1'$	$1'$	$1'$	$1'$	$\dots$	$1'$	$1'$	$1'$	$1'$	$1'$
$2'$	$2'$	$2'$	$2'$	$2'$	$2'$	$\dots$	$2'$	$2'$	$2'$	$2'$	$2'$
$3'$	$3'$	$3'$	$3'$	$3'$	$3'$	$\dots$	$3'$	$3'$	$3'$	$3'$	$3'$
$4'$	$4'$	$4'$	$4'$	$4'$	$4'$	$\dots$	$4'$	$4'$	$4'$	$4'$	$4'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4''$	$4''$	$4''$	$4''$	$4''$	$4''$	$\dots$	$4''$	$4''$	$4''$	$4''$	$4''$
$3''$	$3''$	$3''$	$3''$	$3''$	$3''$	$\dots$	$3''$	$3''$	$3''$	$3''$	$3''$
$2''$	$2''$	$2''$	$2''$	$2''$	$2''$	$\dots$	$2''$	$2''$	$2''$	$2''$	$2''$
$1''$	$1''$	$1''$	$1''$	$1''$	$1''$	$\dots$	$1''$	$1''$	$1''$	$1''$	$1''$
$0''$	$0''$	$0''$	$0''$	$0''$	$0''$	$\dots$	$0''$	$0''$	$0''$	$0''$	$0''$

Identity (B2): If $x = k'$ is a left zero or y a right unit, (B2) holds trivially. Assume that $x = k''$ and $y = l'$. In the case $x = 0''$ and $y = 0'$ we verify (B2) directly, since both sides equal $1''$. If $l \neq 0$ and $k \leqslant l$, then $xyy = 0''y = 0'' = xy$. If $k > l$, then $xy = x$ and $xyy = xy$ $(= x)$.

Identity (B3): If x is a left zero, y a left zero, or z a right unit, the equality trivially holds. Assume that $x = k''$, $y = l''$ and $z = m'$. Then $yz \in \{1\} \times \kappa$, say $yz = s''$, and hence $x \cdot yz = k''s'' = k'' = k''l'' = xy$.

Identity (B4): If x is a left zero, or y a right unit, (B4) trivially holds. If z is a right unit, (B4) reduces to (B2). So we assume that $x = k''$, $y = l'$ and $z = m'$.

Case $x = 0''$: If $y = 0'$, then $xyzy = 1''zy = 0''zy = xzy$, and if $y \neq 0'$, then $xy = x$, so $xyzy = xzy$.

Case $k > 0$: Then $xy, xz \in \{x, 0''\}$. If $xy = x$, then $xyzy = xzy$. If $xy = 0''$ and $xz = x$, then $xyzy = 0''zy \in \{0''y, 1''y\} = \{1''y\}$ since $0''y = 1''y$. Thus $xyzy = 1''y = xyy =_{(B2)} xy = xzy$. Finally, if $xy = xz = 0''$, then $xyzy = 0''zy = xzzy =_{(B2)} xzy$.

To complete the proof, we are proving that $\mathbf{G}$ is subdirectly irreducible with the monolith $\mathrm{Cg}^{\mathbf{G}}(0'', 1'')$. Let $x, y \in G$ be distinct. We want to prove that $\langle 0'', 1'' \rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y)$.

Case $x = k''$, $y = l''$: Without loss of generality, assume $k < l$. Then

$$\langle 0'', y \rangle = \langle xk', yk' \rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y),$$
$$\langle 1'', y \rangle = \langle 0''0', y0' \rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y),$$

so transitivity implies that $\langle 0'', 1'' \rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y)$.

Case $x = k'$, $y = l'$: Without loss of generality, assume $k < l$. Then

$$\langle l'', 0'' \rangle = \langle l''x, l''y \rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y).$$

From the previous case we obtain $\langle 0'', 1'' \rangle \in \mathrm{Cg}^{\mathbf{G}}(0'', l'', y) \subseteq \mathrm{Cg}^{\mathbf{G}}(x, y)$.

Case $x = k'$, $y = l''$: If $k = 0$, then $\langle 1'', 0''\rangle = \langle 0''x, 0''y\rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y)$. If $k \neq 0$, then $\langle 0'', 1''\rangle = \langle 1''x, 1''y\rangle \in \mathrm{Cg}^{\mathbf{G}}(x, y)$. ☐

4. Residual bound of $\mathcal{V}_C$

We turn to the variety $\mathcal{V}_C$. According to [**4**], more precisely the definition of $\mathcal{V}_C$ and Lemma 7.12, $\mathcal{V}_C$ is the variety generated by

$\mathbf{G}_5$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

which is also its 2-generated free groupoid. The equational base of $\mathcal{V}_C$ is

$(C1)$ $xx \approx x$ $\qquad$ $(C2)$ $xyy \approx xy$

$(C3)$ $x \cdot yz \approx xy$ $\qquad$ $(C4)$ $xyz \approx xzy$

Lemma 4.1. *Let $\mathbf{G} \in \mathcal{V}_C$, and assume that $ab = b$ for some $a, b \in G$. Then $a = b$.*

Proof. $a = aa =_{(C3)} a \cdot ab = ab = b$. ☐

Let $\mathbf{G}$ be an n-element groupoid in $\mathcal{V}_C$, and define for any $a \in G$ the set C_a as

$$C_a = a \cdot G \cup a \cdot G \cdot G \cup \cdots \cup a \cdot \underbrace{G \cdot \ldots \cdot G}_{n-1}.$$

Here, $a \cdot \underbrace{G \cdot \ldots \cdot G}_{n-1} = \{ag_1g_2 \ldots g_{n-1} : (\forall i < n) g_i \in G\}$. Note that $a \cdot \underbrace{G \cdot \ldots \cdot G}_{k-1} \subseteq a \cdot \underbrace{G \cdot \ldots \cdot G}_{k}$ by idempotence. Also, note that the left-associated products need only go as far as $n = |G|$ terms since any repetition is cancelled by applying $(C4)$ several times, and then $(C1)$ or $(C2)$. It follows that $C_a \cdot G = C_a$. The relation ρ_a on G is defined by:

$$b \,\rho_a\, c \text{ iff } b = c \text{ or } b, c \in C_a.$$

Lemma 4.2. *Let $\mathbf{G} \in \mathcal{V}_C$ and $|G| = n$. Then ρ_a is a congruence on $\mathbf{G}$.*

Proof. Let $b \,\rho_a\, c$ for distinct elements b, c and let $d \in G$. Then there exist $i, j \leqslant n$ such that $b \in a \cdot \underbrace{G \cdot \ldots \cdot G}_{i}$ and $c \in a \cdot \underbrace{G \cdot \ldots \cdot G}_{j}$. According to $(C3)$ and transitivity of equality, it follows that $da = db$ and $da = dc$. Therefore, $\langle db, dc\rangle \in \rho_a$. On the other hand, $bd, cd \in C_a$, hence $\langle bd, cd\rangle \in \rho_a$. ☐

Corollary 4.1. *Let $\mathbf{G} \in \mathcal{V}_C$ be a finite groupoid, and let $a \in G$. Then for all $b, c \in C_a$, and all $x \in G$, we have $xb = xc$.*

Proof. By a repeated use of the identity $(C3)$, we obtain $xa = xd$ for any $d \in C_a$ and any $x \in G$. ☐

Let $b \in G$, $\mathbf{G} \in \mathcal{V}_C$. Denote by $F_b = \{g \in G : bg = b\}$.

Lemma 4.3. *Let* $\mathbf{G} \in \mathcal{V}_C$ *be a finite groupoid, and let* $a \in G$, $b \in a \cdot G$, $b \neq a$. *For all* $c \in G$ *we have*

$$(ac = a) \vee (ac = b) \Rightarrow bc = b.$$

Consequently, $|F_b| > |F_a|$ *and* $C_b \subseteq C_a$.

Proof. From $b \in a \cdot G$ we know that there exists $d \in G$ such that $ad = b$. Moreover, $bc = adc =_{(C4)} acd = ad = b$ if $ac = a$, or $bc = acc =_{(C2)} ac = b$ if $ac = b$. From the first part we get $F_a \subseteq F_b$, but we also know that $ad = b \neq a$ and $bd = b$. Finally, $b \cdot \underbrace{G \cdot \ldots \cdot G}_{i} \subseteq a \cdot \underbrace{G \cdot \ldots \cdot G}_{i+1}$ since $b \in a \cdot G$, hence $C_b \subseteq C_a$. □

Lemma 4.4. *Let* $\mathbf{G} \in \mathcal{V}_C$ *be a finite groupoid,* $|G| = n$, $c \in C_a$ *so that* $c = ab_1 \ldots b_i$ *for some* $i < n$ *and* $b_1, \ldots, b_i \in G$. *If* $c \cdot G = c$ *(c is a left zero), then for every* $d \in C_a$, $db_1 \ldots b_i = c$.

Proof. Let $d = ab'_1 \ldots b'_j$ for some $j \leqslant n$ and $b'_1, \ldots, b'_j \in G$. Then

$$db_1 \ldots b_i = ab'_1 \ldots b'_j b_1 \ldots b_i =_{(C4)} \ldots =_{(C4)} ab_1 \ldots b_i b'_1 \ldots b'_j = cb'_1 \ldots b'_j = c. \quad \square$$

Lemma 4.5. *Let* $\mathbf{G} \in \mathcal{V}_C$ *be a finite groupoid,* $|G| = n$, *and let* $a \in G$. *Then there exists a unique* $c \in C_a$ *such that* $c \cdot G = c$.

Proof. Assume the opposite. Let $b \in C_a$ be such that $|F_b|$ is maximal. Since $\{b\} \neq b \cdot G$, then there exists $c \in b \cdot G$, $c \neq b$, so by Lemma 4.3 we get $|F_c| > |F_b|$. Moreover, the conditions $c \in b \cdot G$ and $b \in C_a$ mean that we can select some $i < n$ and $b_1, \ldots, b_i, d \in G$ such that $b = ab_1b_2 \ldots b_i$ and $c = bd$. Thus $c \in C_a$, which contradicts the choice of b. Uniqueness of c in C_a follows from Lemma 4.4. □

Lemma 4.6. *Let* $\mathbf{G} \in \mathcal{V}_C$ *be a finite groupoid,* $a \in G$, *and let* $|C_a| = 3$. *Then* $\mathbf{G}$ *is not subdirectly irreducible.*

Proof. Assume that $C_a = \{a, b, c\}$ and according to Lemma 4.5, one of b, c is a left zero, say $c \cdot G = c$.

First we show that $C_a = a \cdot G$. We know that $\{a\} \subseteq a \cdot G \subseteq a \cdot G \cdot G \subseteq \ldots$, and that once this sequence stabilizes, it stays constant and equal to C_a. Therefore, either $a \cdot G = \{a, b, c\} = C_a$, or $\{a\} \subsetneq a \cdot G \subsetneq a \cdot G \cdot G = \{a, b, c\}$. If $a \cdot G = \{a, c\}$, then $\{a, b, c\} = a \cdot G \cdot G = a \cdot G \cup c \cdot G = \{a, c\} \cup \{c\}$, a contradiction. If $a \cdot G = \{a, b\}$, then $ade = c$ for some $d, e \in G$. We know $ad \in a \cdot G = \{a, b\}$. If $ad = a$, then $c = ade = ae \in a \cdot G$. So, we may assume that $ad = b$. According to $(C4)$, $c = ade = aed$, so an analogous argument proves that we may assume $ae = b$. However, from Lemma 4.3 it follows that $be = b$, and this leads to a contradiction: $c = ade = be = b$.

Let $ad = c$. According to Lemma 4.4 we have $\{a, b, c\} \cdot d = c$. Lemma 4.3 implies that $b \cdot G \subseteq \{b, c\}$, and Lemma 4.4 implies $c \in b \cdot G$, while $(C1)$ implies $b \in b \cdot G$, so $b \cdot G = \{b, c\}$. Thus $C_b = \{b, c\}$, as c is a left zero. Lemma 4.2 implies that $\alpha = \Delta_G \cup \{\langle b, c\rangle, \langle c, b\rangle\}$ is a congruence of $\mathbf{G}$. Let us show that $\beta = \Delta_G \cup \{\langle a, b\rangle, \langle b, a\rangle\}$ is a congruence of $\mathbf{G}$. Let $ae = b$. If $g \in G$, then $gb = g \cdot ae =_{(C3)} ga$. On the other hand, we know $\{ag, bg\} \subseteq \{a, b, c\}$. If $ag \in \{a, b\}$, Lemma 4.3 implies $bg = b$. If $ag = c$, then Lemma 4.4 implies $bg = c$. So, $\alpha, \beta \in \operatorname{Con}\mathbf{G} \smallsetminus \{\Delta_G\}$, but $\alpha \cap \beta = \Delta_G$, hence $\mathbf{G}$ is not subdirectly irreducible. □

LEMMA 4.7. *Let $\mathbf{G} \in \mathcal{V}_C$ be a finite groupoid, $a, b \in G$, and let $|C_a| = |C_b| = 2$. Then $\mathbf{G}$ is not subdirectly irreducible.*

PROOF. Lemma 4.2 implies that ρ_a, ρ_b are atoms in $\mathbf{Con}\,\mathbf{G}$. Assume that they are equal. Then from $a \in C_a$ and $b \in C_b$ we get $C_a = C_b = \{a, b\}$. According to Lemma 4.5, one of a, b must be a left zero, but this contradicts the assumption that $|C_a| = |C_b| = 2$. Thus ρ_a and ρ_b are distinct atoms in $\mathbf{Con}\,\mathbf{G}$, so $\mathbf{G}$ is not subdirectly irreducible. □

THEOREM 4.1. $\operatorname{resb}(\mathcal{V}_C) \leqslant 4$.

PROOF. Let $\mathbf{G} \in \mathcal{V}_C$ be a subdirectly irreducible groupoid with at least 4 elements. $\mathcal{V}_C$ is finitely generated, thus it is locally finite. Recall the Quackenbush Lemma from [**8**]: if a locally finite variety has an infinite subdirectly irreducible algebra, it has no bound on sizes of finite subdirectly irreducibles. So we may assume that $\mathbf{G}$ is finite.

According to Lemma 4.6, for all $a \in G$, $|C_a| \neq 3$. Assume that $a \in G$ is such that $|C_a| > 3$. Lemma 4.5 implies that there exists a unique $c \in C_a$ such that $c \cdot G = c$. Lemma 4.4 implies that for all $d \in C_a$, $c \in C_d$. Let $b \in C_a \smallsetminus \{c\}$ be such that C_b is minimal, and such that $|F_b|$ is maximal among $|F_d|$ which satisfy $d \in C_a \smallsetminus \{c\}$ and $C_d = C_b$. If $C_b \neq \{b, c\}$, then $b \cdot G \neq \{b, c\}$ (otherwise we would get $b \cdot G \cdot G = b \cdot G \cup c \cdot G = \{b, c\}$). Let $d \in b \cdot G \smallsetminus \{b, c\}$. Then $d \in C_b$ and $b \in C_a$ imply $d \in C_a$. Moreover, Lemma 4.3 implies that $C_d \subseteq C_b$ and $|F_b| < |F_d|$, contradicting the choice of b. Therefore, $C_b = \{c, b\}$. Next we select $d \in C_a \smallsetminus \{b, c\}$ such that C_d is minimal, and such that $|F_d|$ is maximal among $|F_e|$ which satisfy $e \in C_a \smallsetminus \{b, c\}$ and $C_e = C_d$. The analogous argument proves that either $C_d = \{c, d\}$, or $C_d = \{b, c, d\}$. But this is a contradiction with Lemma 4.7, or with Lemma 4.6.

Assume now that $|C_a| \leqslant 2$ for all $a \in G$. Lemma 4.7 implies that at most one $a \in G$ is such that $|C_a| = 2$. If $|C_a| = 1$ for all $a \in G$, then $\mathbf{G}$ is a left zero semigroup, and the only subdirectly irreducible left zero semigroup has two elements. Let $a \in G$ be the only element such that $|C_a| = 2$, say $C_a = \{a, b\}$, and all other elements of G are left zeros. Since $|G| \geqslant 4$, then there exist left zeros $u, v \in G \smallsetminus \{a\}$ such that $au = av$ (since G contains at least three left zeros, and $a \cdot G = \{a, b\}$). But then $\operatorname{Cg}^{\mathbf{G}}(a, b) = \rho_a = \{a, b\}^2 \cup \Delta_G$ and $\operatorname{Cg}^{\mathbf{G}}(u, v) = \{u, v\}^2 \cup \Delta_G$ are distinct atoms in $\mathbf{Con}\,\mathbf{G}$, which is a contradiction. □

THEOREM 4.2. *All subdirectly irreducible members of $\mathcal{V}_C$ are depicted in Table 3. Thus,* $\operatorname{resb}(\mathcal{V}_C) = 4$.

PROOF. Let $\mathbf{G} \in \mathcal{V}_C$ be subdirectly irreducible. By Theorem 4.1, $|G| < 4$.

emphCase $|G| = 3$: Let $G = \{a, b, c\}$. $\mathbf{G}$ is not a left zero semigroup, since the three-element left zero semigroup is not subdirectly irreducible. From Lemma 4.1 it follows that we may assume $ab = c$, from Lemmas 4.6 and 4.7 it follows that there is at most one element of G which is not a left zero, and it must be a, so b and c are left zeros. Finally, $ac = a$ since $ac \in C_a = \{a, c\}$ and $ac = c$ would imply

TABLE 3. Cayley tables of subdirectly irreducible groupoids in $\mathcal{V}_C$.

	a	b	c
a	a	c	a
b	b	b	b
c	c	c	c

	a	b
a	a	a
b	b	b

$a = c$ by Lemma 4.1. This means that $\mathbf{G}$ is the left-hand side groupoid depicted in Table 3, which is subdirectly irreducible with the monolith $\mathrm{Cg}^{\mathbf{g}}(a, c)$.

Case $|G| = 2$: Let $G = \{a, b\}$. We know that at least one of $\{a, b\}$ is a left zero. If a is not a left zero, then idempotence implies that $ab = b$, but from Lemma 4.1 it follows that $a = b$. So $\mathbf{G}$ is a left zero semigroup. $\square$

5. Residual bound of $\mathcal{W}$

We turn to the variety $\mathcal{W}$. According to [**4**], more precisely Lemma 6.1 and Theorem 6.3, $\mathcal{W}$ is the variety generated by

$\mathbf{G}_4$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

which is also its 2-generated free groupoid. The equational base of $\mathcal{W}$ is

(W1) $xx \approx x$, (W2) $xyy \approx x$,

(W3) $x \cdot yz \approx xy$, (W4) $xyz \approx xzy$,

LEMMA 5.1. *Let $\mathbf{G} \in \mathcal{W}$. The groupoid $\mathbf{G}$ is right cancellative, i.e. for all $a, b, c \in G$, $ba = ca \Rightarrow b = c$. If $a, b \in G$ are distinct, then $ab \neq b$.*

PROOF. The proof follows from $(W2)$ and idempotence $(W1)$. $\square$

LEMMA 5.2. *Let $\mathbf{G} \in \mathcal{W}$ and assume $a, b, c \in G$ are pairwise distinct elements which satisfy $ab = c$. Then for every pair $\langle d, e\rangle \in \mathrm{Cg}^{\mathbf{G}}(a, c)$ and all $z \in G$, $zd = ze$. The congruence $\alpha = \mathrm{Cg}^{\mathbf{G}}(a, c)$ is an atom in the lattice $\boldsymbol{Con}\,\mathbf{G}$ and all α-classes have at most two elements.*

PROOF. From $(W1)$–$(W4)$ we get that $ac = a \cdot ab = aa = a$, $ca = aba = aab = ab = c$, $cb = abb = a$, i.e. the following equalities hold in $\mathbf{G}$:

$\mathbf{G}$	a	b	c	$\dots$
a	a	c	a	$\dots$
b	.	b	.	$\dots$
c	c	a	c	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Recall that the congruences are generated by Mal'cev chains, so it suffices to prove that for any $z \in G$ and translation $p(x) \in \mathrm{Tr}\, G$, $zp(a) = zp(c)$. We prove this by an induction on the complexity of the shortest composition of basic translations equal to $p(x)$. Since $a = cb$, it follows that for all $z \in G$, $za = zc$ ($za = z(cb) =_{(W3)} zc$), which proves the inductive base. Assume that the shortest composition of basic translations which equals $p(x)$ uses any left translation λ_u, so $p(x) = (r \circ \lambda_u \circ q)(x)$, where $r, q \in \operatorname{Tr} G$, then $(\lambda_u \circ q)(a) = uq(a) = uq(c) = (\lambda_u \circ q)(c)$ by the inductive assumption on $q(x)$. Therefore, $p(a) = p(c)$, and also $zp(a) = zp(c)$. The only case we need to consider is when p is a composition of right translations.

For each $z \in G$ we have

$$\begin{aligned} z \cdot ad_1 \dots d_n &=_{(W3)} z \cdot ad_1 \dots d_{n-1} =_{(W3)} \dots =_{(W3)} za = zc \\ &=_{(W3)} z \cdot cd_1 =_{(W3)} \dots =_{(W3)} z \cdot cd_1 \dots d_n. \end{aligned}$$

We obtain that all translations we need to consider for generating links in the Mal'cev chains in $\operatorname{Cg}^{\mathbf{G}}(a, c)$ are of the form $p(x) = xd_1 \dots d_n$. Moreover,

$$ad_1 \dots d_n b =_{(W4)} ad_1 \dots bd_n =_{(W4)} \dots =_{(W4)} abd_1 \dots d_n = cd_1 \dots d_n,$$

and equivalently, $cd_1 \dots d_n b = ad_1 \dots d_n$, since $cb = a$ can replace $ab = c$ and then the proof is analogous to above derivation. We have proved for any $u, v \in G$ that there exists a translation $p(x)$ such that $u = p(a)$ and $v = p(c)$ iff there exists a translation $q(x)$ such that $u = q(c)$ and $v = q(a)$.

Let $d_1, \dots, d_n, e_1, \dots, e_m \in G$ be parameters such that $ad_1 \dots d_n \neq cd_1 \dots d_n$ and $ae_1 \dots e_m \neq ce_1 \dots e_m$. We claim that

$$|\{ad_1 \dots d_n, cd_1 \dots d_n\} \cap \{ae_1 \dots e_m, ce_1 \dots e_m\}| \neq 1.$$

Assume the opposite. By the above considerations, it suffices to consider the case when $ad_1 \dots d_n = ae_1 \dots e_m$ and $cd_1 \dots d_n \neq ce_1 \dots e_m$ and also the case when $ad_1 \dots d_n \neq ae_1 \dots e_m$, $cd_1 \dots d_n = ce_1 \dots e_m$. The two cases are symmetric, thus we may assume without loss of generality that $ad_1 \dots d_n = ae_1 \dots e_m$ and $cd_1 \dots d_n \neq ce_1 \dots e_m$. We get

$$cd_1 \dots d_n = ad_1 \dots d_n b = ae_1 \dots e_m b = ce_1 \dots e_m \neq cd_1 \dots d_n,$$

a contradiction (the third equality was proved above). We conclude that all classes of the congruence $\operatorname{Cg}^{\mathbf{G}}(a, c)$ have at most two elements. Since the identity $(W2)$ implies that

$$\operatorname{Cg}^{\mathbf{G}}(a, c) = \operatorname{Cg}^{\mathbf{G}}(ad_1 \dots d_n, cd_1 \dots d_n),$$

it follows that $\operatorname{Cg}^{\mathbf{G}}(a, c)$ is an atom in the lattice $\mathbf{Con}\,\mathbf{G}$. □

LEMMA 5.3. *Let $\mathbf{G} \in \mathcal{W}$ be subdirectly irreducible and for three distinct elements $a, b, c \in G$ let $ab = c$ hold. Then $\{a, c\} \cdot G = \{a, c\}$ and all elements in $G \setminus \{a, c\}$ are left zeros.*

PROOF. The table in the proof of Lemma 5.2 proves that $\{a, c\} \cdot \{a, b, c\} = \{a, c\}$.

Assume that there exists an element $d \in G \setminus \{a, b, c\}$ such that $ad \notin \{a, c\}$. Then $ad \neq d$ by Lemma 5.1, so $ad \notin \{a, d\}$ and Lemma 5.2 proves that $\operatorname{Cg}^{\mathbf{G}}(a, ad)$ is an atom in the lattice $\mathbf{Con}\,\mathbf{G}$, which is incomparable with $\operatorname{Cg}^{\mathbf{G}}(a, c)$. This contradicts the assumption that $\mathbf{G}$ is subdirectly irreducible. The proof that $cd \in$

TABLE 4. Cayley tables of all subdirectly irreducible groupoids in $\mathcal{W}$.

	a	b	c	d
a	a	c	a	a
b	b	b	b	b
c	c	a	c	c
d	d	d	d	d

	a	b	c
a	a	c	a
b	b	b	b
c	c	a	c

	a	b
a	a	a
b	b	b

$\{a, c\}$ is analogous, obtained by interchanging a and c above. Therefore, $\{a, c\} \cdot G = \{a, c\}$. Thus $\mathrm{Cg}^{\mathbf{G}}(a, c) = \Delta_G \cup \{a, c\}^2$, according to Lemma 5.2.

Assume that there exist $d \in G \smallsetminus \{a, c\}$ and $e \in G \smallsetminus \{d\}$ such that $d \neq de$. Then $de \neq e$ by Lemma 5.1, so $de \notin \{d, e\}$. The argument from the preceding paragraph proves that $\mathrm{Cg}^{\mathbf{G}}(d, de) = \Delta_G \cup \{d, de\}^2$ and $\mathrm{Cg}^{\mathbf{G}}(a, c) \cap \mathrm{Cg}^{\mathbf{G}}(d, de) = \Delta_G$, which is impossible since $\mathbf{G}$ is subdirectly irreducible. Thus all $d \in G \smallsetminus \{a, c\}$ are left zeros. $\square$

THEOREM 5.1. $\mathrm{resb}(\mathcal{W}) \leqslant 5$.

PROOF. Assume the opposite, i.e. let $\mathbf{G} \in \mathcal{W}$ be a subdirectly irreducible groupoid with $|G| \geqslant 5$. $\mathbf{G}$ can not be a left zero semigroup, since the only subdirectly irreducible left zero semigroup has two elements. From Lemma 5.1 and idempotence it follows that there exist $a, b, c \in G$ such that $ab = c$. Lemma 5.2 proves that $\mathrm{Cg}^{\mathbf{G}}(a, c)$ is an atom in the congruence lattice $\mathbf{Con}\,\mathbf{G}$ and $\mathrm{Cg}^{\mathbf{G}}(a, c) \neq \nabla_G$ since $|G| \geqslant 5$ and all equivalence classes of $\mathrm{Cg}^{\mathbf{G}}(a, c)$ have at most two elements.

Lemma 5.3 implies that $\{a, c\} \cdot G = \{a, c\}$ and all elements of $G \smallsetminus \{a, c\}$ are left zeros. Since $|G| \geqslant 5$, there exist elements $d, e \in G \smallsetminus \{a, c\}$ such that $ad = ae$. Thus for each $z \in G \smallsetminus \{c\}$ we have $zd = ze$, for $z \in G \smallsetminus \{a, c\}$ are left zeros. On the other hand, if $z = c$, we get $cd = abd =_{(W4)} adb = aeb =_{(W4)} abe = ce$. Therefore, $\mathrm{Cg}^{\mathbf{G}}(d, e) = \Delta_G \cup \{d, e\}^2$, and $\mathrm{Cg}^{\mathbf{G}}(a, c) \cap \mathrm{Cg}^{\mathbf{G}}(d, e) = \Delta_G$, a contradiction. $\square$

THEOREM 5.2. $\mathrm{resb}(\mathcal{W}) = 5$ *and all subdirectly irreducible groupoids in* $\mathcal{W}$ *are given in Table 4.*

PROOF. Let $\mathbf{G} \in \mathcal{W}$ be a subdirectly irreducible groupoid. From Theorem 5.1 we know that $|G| < 5$.

Case $|G| = 4$: Let $G = \{a, b, c, d\}$ and we may assume that $ab = c$ since $\mathbf{G}$ can't be a left zero semigroup and $ab \neq b$ by Lemma 5.1. In the proof of Lemma 5.2 we showed that $ac = cb = a$ and $ab = ca = c$ in this case. From Lemma 5.3 it follows that b and d are left zeros and that $\{a, c\} \cdot G = \{a, c\}$. If $ab = ad$, it would follow that $cd = ab \cdot d = ad \cdot b = ab \cdot b = cb$, and $\Delta_G \cup \{b, d\}^2$ would have been a congruence, a contradiction. Similarly we conclude $cb \neq cd$, so $ad \neq ab = c$ forces $ad = a$, and we also get $cd = c$. This is the leftmost groupoid in Table 4 for which we verify that it is subdirectly irreducible and in $\mathcal{W}$ by checking directly the identities and the congruence lattice.

Case $|G| = 3$: Let $G = \{a, b, c\}$. We may assume that $ab = c$ like in the previous case, and the proofs that b is a left zero, that $ac = cb = a$ and $ab = ca = c$

are also just as above. Thus we obtain the middle groupoid in 4 which is subdirectly irreducible with the monolith $\mathrm{Cg}^{\mathbf{g}}(a, c)$.

Case $|G| = 2$: Let $G = \{a, b\}$. By idempotence and Lemma 5.1 we get that $\mathbf{G}$ is the left zero semigroup. $\square$

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