PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 59–65 DOI: 10.2298/PIM1613059J

EXTENSION THEOREM OF WHITNEY TYPE FOR $\mathcal{S}(\mathbb{R}^d_+)$ BY USE OF THE KERNEL THEOREM

Smiljana Jakšić and Bojan Prangoski

Abstract. We study the expansions of the elements in $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ with respect to the Laguerre orthonormal basis, extending the result of M. Guillemot-Teissier in the one dimensional case. As a consequence, we obtain Kernel theorem for $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ and an extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}^d_+)$.

1. Introduction

We denote by $\mathbb{R}^d_+$ the set $(0,\infty)^d$ and by $\overline{\mathbb{R}^d_+}$ its closure, i.e., $[0,\infty)^d$. We will consider the space $\mathcal{S}(\mathbb{R}^d_+)$ which consists of all $f \in \mathcal{C}^\infty(\mathbb{R}^d_+)$ such that all derivatives $D^p f$, $p \in \mathbb{N}^d_0$, extend to continuous functions on $\overline{\mathbb{R}^d_+}$ and

$$\sup_{x\in\mathbb{R}^d_+} x^k |D^p f(x)| < \infty, \text{ for all } k,p \in \mathbb{N}^d_0.$$

With this system of seminorms, $\mathcal{S}(\mathbb{R}^d_+)$ becomes an (F)-space.

The results concerning the extension of a smooth function or a function of class $\mathcal{C}^k$ out of some region and various reformulation of such problems are called extension theorems of Whitney type. One can see Whitney [**11**], Seeley [**8**] and Hörmander [**3**, Theorem 2.3.6, p. 48]. Here we deal with a problem of extension of a function from $\mathcal{S}(\mathbb{R}^d_+)$ onto $\mathcal{S}(\mathbb{R}^d)$. Theorem 4.2 is the main result of the paper. For the purpose of this theorem we prove the Schwartz kernel theorem for $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$, Theorem 4.1.

Recall, for $n = 0,1,2\dots$ the functions

$$L_n(x) = \frac{e^x}{n!}\Big(\frac{d}{dx}\Big)^n (e^{-x}x^n), \quad x > 0$$

2010 *Mathematics Subject Classification:* Primary 46F05.
Key words and phrases: tempered distributions on $\mathbb{R}^d_+$; kernel theorem for tempered distributions on $\mathbb{R}^d_+$; smooth extensions of smooth rapidly decreasing functions.
Partially supported by the Ministry of Science, Republic of Serbia, project 174024.
Communicated by Stevan Pilipović.

are the Laguerre polynomials and $\mathcal{L}_n(x) = L_n(x)e^{-\frac{x}{2}}$ are Laguerre functions; $\{\mathcal{L}_n(x),\ n = 0, 1, \dots\}$ is an orthonormal basis for $L^2(0,\infty)$ [**10**, p. 108].

The problem of expanding the elements of $\mathcal{S}'(\mathbb{R}_+)$ with respect to the Laguerre orthonormal basis has been treated by Guillemont-Teissier in [**4**] and Duran in [**1**]: If $T \in \mathcal{S}'(\mathbb{R}_+)$ and $a_n = \langle T, \mathcal{L}_n(x)\rangle$, then $T = \sum_{n=0}^{\infty} a_n \mathcal{L}_n(x)$ and $\{a_n\}_{n=0}^{\infty}$ decreases slowly. Conversely, if $\{a_n\}_{n=0}^{\infty}$ decreases slowly, then there exists $T \in \mathcal{S}'(\mathbb{R}_+)$ such that $T = \sum_{n=0}^{\infty} a_n \mathcal{L}_n(x)$.

The papers [**7, 12, 13**] contain expansions of the same kind as in [**4, 1**]. The novelty of this paper is an extension of the results of [**4**] for the d-dimensional case. This leads to the Schwartz kernel theorem (Theorem 4.1) which states that there is one-to-one correspondence between elements from $\mathcal{S}'(\mathbb{R}_+^{m+n})$ in two sets of variables x and y, and the continuous linear mappings of $(\mathcal{S}(\mathbb{R}_+^m))_y$ into $(\mathcal{S}'(\mathbb{R}_+^m))_x$. As a consequence of Theorem 4.2, we explain the convolution in $\mathcal{S}'(\mathbb{R}_+^d)$ in the last remark.

The structure of the paper is as follows. We recall in Section 3 some properties of Laguerre series and prove the convergence of the Laguerre series in $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$. In Section 4, we state Schwartz's kernel theorem for $\mathcal{S}(\mathbb{R}_+^d)$ and prove an extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}_+^d)$.

2. Notation

We use the standard multi-index notation. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\alpha| = \sum_{i=1}^{d} \alpha_i$, $x^\alpha = (x_1, \dots, x_d)^{(\alpha_1,\dots,\alpha_d)} = \prod_{i=1}^{d} x_i^{\alpha_i}$, $D^\alpha = \prod_{i=1}^{d} \frac{\partial^{\alpha_i}}{\partial {x_i}^{\alpha_i}}$ for the partial derivative and $X^\alpha f(x) = x^\alpha f(x)$ for the multiplication operator. For $x \in \mathbb{R}^d$, $|x|$ stands for the standard Euclidean norm in $\mathbb{R}^d$.

Let s be the space of rapidly decreasing sequences, i.e.,

$$\{a_n\}_{n\in\mathbb{N}_0^d} \in s \Leftrightarrow \sum_{n\in\mathbb{N}_0^d} |a_n|^2 n^{2k} < \infty, \quad \text{for all } k \in \mathbb{N}.$$

Then s' stands for the strong dual of s, the space of slowly increasing sequences:

$$\{a_n\}_{n\in\mathbb{N}_0^d} \in s' \Leftrightarrow \sum_{n\in\mathbb{N}_0^d} |a_n|^2 n^{-2k} < \infty, \quad \text{for a } k \in \mathbb{N}.$$

3. Laguerre series

The d-dimensional Laguerre functions

$$\mathcal{L}_n(x) = \mathcal{L}_{n_1}(x_1)\cdots\mathcal{L}_{n_d}(x_d) = \prod_{i=1}^{d} \mathcal{L}_{n_i}(x_i)$$

form an orthonormal basis for $L^2(\mathbb{R}_+^d)$ and are the eigenfunctions of the Laguerre operator $E = \big(D_1(x_1D_1) - \frac{x_1}{4}\big)\cdots\big(D_d(x_dD_d) - \frac{x_d}{4}\big)$, $E : \mathcal{S}(\mathbb{R}_+^d) \to \mathcal{S}(\mathbb{R}_+^d)$

$$\mathcal{L}_n(x) \to E(\mathcal{L}_n(x)) = \prod_{i=1}^{d} -\Big(n_i + \frac{1}{2}\Big)\mathcal{L}_n(x).$$

Note that E is a self-adjoint operator, i.e.,

$$\langle Ef, g\rangle = \langle f, Eg\rangle, \quad f, g \in \mathrm{dom}(E) = \{f \in L^2(\mathbb{R}^d_+);\ Ef \in L^2(\mathbb{R}^d_+)\}.$$

For $f \in \mathcal{S}(\mathbb{R}^d_+)$ we define the n-th Laguerre coefficient by $a_n = \int_{\mathbb{R}^d_+} f(x)\mathcal{L}_n(x)dx$. The Laguerre series of the function $f \in \mathcal{S}(\mathbb{R}^d_+)$ is $\sum_{n\in\mathbb{N}_0^d} a_n\mathcal{L}_n(x)$.

In [**4**, p.547], the following bound on the one-dimensional Laguerre functions is obtained:

$$\Big|x^k\Big(\frac{d}{dx}\Big)^p\mathcal{L}_n(x)\Big| \leqslant C_{p,k}(n+1)^{p+k},\ x \geqslant 0,\ n, p, k \geqslant 0.$$

Finding the bound on the d-dimensional Laguerre functions involves not complicated calculation. Hence

$$|x^k D^p\mathcal{L}_n(x)| \leqslant C_{p,k}\prod_{i=1}^d (n_i+1)^{p_i+k_i},\ x \in \mathbb{R}^d_+,\ n, p, k \in \mathbb{N}_0^d. \tag{3.1}$$

3.1. Convergence of the Laguerre series in $\mathcal{S}(\mathbb{R}^d_+)$.

THEOREM 3.1. *For $f \in \mathcal{S}(\mathbb{R}^d_+)$, let $a_n(f) = \int_{\mathbb{R}^d_+} f(x)\mathcal{L}_n(x)\,dx$. Then $f = \sum_{n\in\mathbb{N}_0^d} a_n(f)\mathcal{L}_n$ and the series converges absolutely in $\mathcal{S}(\mathbb{R}^d_+)$. Moreover the mapping $\iota : \mathcal{S}(\mathbb{R}^d_+) \to s$, $\iota(f) = \{a_n(f)\}_{n\in\mathbb{N}_0^d}$, is a topological isomorphism.*

PROOF. For $f \in \mathcal{S}(\mathbb{R}^d_+)$ we have

$$a_n(Ef) = \langle Ef, \mathcal{L}_n\rangle = \langle f, E(\mathcal{L}_n)\rangle = a_n(f)(-1)^d\prod_{i=1}^d\Big(n_i+\frac{1}{2}\Big).$$

Moreover,

$$a_n(E^p f) = a_n(f)\prod_{i=1}^d(-1)^{p_i}\Big(n_i+\frac{1}{2}\Big)^{p_i}$$

for any $p \in \mathbb{N}^d$. As $E^p f \in \mathcal{S}(\mathbb{R}^d_+) \subset L^2(\mathbb{R}^d_+)$, we have

$$\sum_{n\in\mathbb{N}_0^d}|a_n(f)|^2\prod_{i=1}^d\Big(n_i+\frac{1}{2}\Big)^{2p_i} < \infty, \quad \text{for every } p \in \mathbb{N}_0^d,$$

i.e., $\{a_n(f)\}_{n\in\mathbb{N}_0^d} \in s$. Clearly $f = \sum_{n\in\mathbb{N}_0^d} a_n(f)\mathcal{L}_n$ as elements of $L^2(\mathbb{R}^d_+)$. By (3.1), we obtain

$$\sum_{n\in\mathbb{N}_0^d}|x^k D^p(a_n(f)\mathcal{L}_n(x))| \leqslant C_{p,k}\sum_{n\in\mathbb{N}_0^d}|a_n(f)|\prod_{i=1}^d(n_i+1)^{p_i+k_i} < \infty \tag{3.2}$$

which yields the absolute convergence of the series in $\mathcal{S}(\mathbb{R}^d_+)$.

To prove that ι is a topological isomorphism, first observe that by the above consideration it is well defined and it is clearly an injection. Let $\{a_n\}_{n\in\mathbb{N}_0^d} \in s$. Define $f = \sum_{n\in\mathbb{N}_0^d} a_n\mathcal{L}_n \in L^2(\mathbb{R}^d_+)$. Now (3.2) proves that this series converges in $\mathcal{S}(\mathbb{R}^d_+)$, hence $f \in \mathcal{S}(\mathbb{R}^d_+)$. Thus ι is bijective. Observe that, (3.2) proves that ι^{-1} is continuous. Since $\mathcal{S}(\mathbb{R}^d_+)$ and s are (F)-spaces, the open mapping theorem proves that ι is topological isomorphism. $\square$

3.2. Convergence of the Laguerre series in $\mathcal{S}'(\mathbb{R}^d_+)$.

THEOREM 3.2. *For $T \in \mathcal{S}'(\mathbb{R}^d_+)$, let $b_n(T) = \langle T, \mathcal{L}_n\rangle$. Then $\{b_n(T)\}_{n\in\mathbb{N}^d_0} \in s'$ and $T = \sum_{n\in\mathbb{N}^d_0} b_n(T)\mathcal{L}_n$. The series converges absolutely in $\mathcal{S}'(\mathbb{R}^d_+)$. Conversely, if $\{b_n\}_{n\in\mathbb{N}^d_0} \in s'$, then there exists a $T \in \mathcal{S}'(\mathbb{R}^d_+)$ such that $T = \sum_{n\in\mathbb{N}^d_0} b_n\mathcal{L}_n$. As a consequence, $\mathcal{S}'(\mathbb{R}^d_+)$ is topologically isomorphic to s'.*

PROOF. Assume that $\{b_n\}_{n\in\mathbb{N}^d_0} \in s'$. Then there exists a $k \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}^d_0} |b_n|^2(|n|+1)^{-2k} < \infty$. For a bounded subset B of $\mathcal{S}(\mathbb{R}^d_+)$, Theorem 3.1 implies that there exists $C > 0$ such that $\sum_{n\in\mathbb{N}^d_0} |a_n(f)|^2(|n|+1)^{2k} \leqslant C$, for all $f \in B$, where we denote $\{a_n(f)\}_{n\in\mathbb{N}^d_0} = \iota(f)$. Observe that for an arbitrary $q \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{|n|\leqslant q} \sup_{f\in B} |\langle b_n\mathcal{L}_n, f\rangle| &\leqslant \sup_{f\in B} \sum_{n\in\mathbb{N}^d_0}\sum_{m\in\mathbb{N}^d_0} |\langle b_n\mathcal{L}_n, a_m(f)\mathcal{L}_m\rangle| \\ &= \sup_{f\in B}\sum_{n\in\mathbb{N}^d_0} |b_n||a_n(f)| \leqslant C',\end{aligned}$$

i.e.,

$$\sum_{n\in\mathbb{N}^d_0} \sup_{f\in B} |\langle b_n\mathcal{L}_n, f\rangle| < \infty,$$

hence $\sum_{n\in\mathbb{N}^d_0} b_n\mathcal{L}_n$ converges absolutely in $\mathcal{S}'(\mathbb{R}^d_+)$.

Let $T \in \mathcal{S}'(\mathbb{R}^d_+)$. Theorem 3.1 implies that ${}^t\iota : s' \to \mathcal{S}'(\mathbb{R}^d_+)$ is an isomorphism (${}^t\iota$ denotes the transpose of ι). Now, one easily verifies that $({}^t\iota)^{-1}T = \{b_n\}_{n\in\mathbb{N}^d_0}$, where $b_n(T) = \langle T, \mathcal{L}_n\rangle$. Observe that for $f \in \mathcal{S}(\mathbb{R}^d_+)$

$$\langle T, f\rangle = \sum_{n\in\mathbb{N}^d_0} a_n(f)\langle T, \mathcal{L}_n\rangle = \sum_{n\in\mathbb{N}^d_0} a_n(f)b_n(T) = \left\langle \sum_{n\in\mathbb{N}^d_0} b_n(T)\mathcal{L}_n, f\right\rangle,$$

i.e., $T = \sum_{n\in\mathbb{N}^d_0} b_n(T)\mathcal{L}_n$. $\square$

4. Kernel theorem

The completions of the tensor product are denoted by $\hat{\otimes}_\epsilon$ and $\hat{\otimes}_\pi$ with respect to ϵ and π topologies. If they are equal, we drop the subindex.

PROPOSITION 4.1. *The spaces $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ are nuclear.*

PROOF. Since s is nuclear, Theorem 3.1 implies that $\mathcal{S}(\mathbb{R}^d_+)$ is also nuclear. Now $\mathcal{S}'(\mathbb{R}^d_+)$ is nuclear as the strong dual of a nuclear (F)-space. $\square$

THEOREM 4.1. *The following canonical isomorphisms hold:*

$$\mathcal{S}(\mathbb{R}^m_+)\hat{\otimes}\mathcal{S}(\mathbb{R}^n_+) \cong \mathcal{S}(\mathbb{R}^{m+n}_+), \quad \mathcal{S}'(\mathbb{R}^m_+)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_+) \cong \mathcal{S}'(\mathbb{R}^{m+n}_+).$$

PROOF. The second isomorphism follows from the first one since $\mathcal{S}(\mathbb{R}^d_+)$ is a nuclear (F)-space. Thus, it is enough to prove the first isomorphism.

Step 1: From Theorem 3.1 it follows that $\mathcal{S}(\mathbb{R}^m_+)\otimes\mathcal{S}(\mathbb{R}^n_+)$ is dense in $\mathcal{S}(\mathbb{R}^{m+n}_+)$. It suffices to show that the latter induces on the former the topology $\pi=\epsilon$ (the π and the ϵ topologies are the same because $\mathcal{S}(\mathbb{R}^d_+)$ is nuclear). Since the bilinear mapping $(f,g)\mapsto f\otimes g$ of $\mathcal{S}(\mathbb{R}^m_+)\times\mathcal{S}(\mathbb{R}^n_+)$ into $\mathcal{S}(\mathbb{R}^{m+n}_+)$ is separately continuous, it follows that it is continuous ($\mathcal{S}(\mathbb{R}^m_+)$ and $\mathcal{S}(\mathbb{R}^n_+)$ are (F)-spaces). The continuity of this bilinear mapping proves that the inclusion $\mathcal{S}(\mathbb{R}^m_+)\otimes_\pi\mathcal{S}(\mathbb{R}^n_+)\to\mathcal{S}(\mathbb{R}^{m+n}_+)$ is continuous, hence the topology π is stronger than the induced one from $\mathcal{S}(\mathbb{R}^{m+n}_+)$ onto $\mathcal{S}(\mathbb{R}^m_+)\otimes\mathcal{S}(\mathbb{R}^n_+)$.

Step 2: Let A' and B' be equicontinuous subsets of $\mathcal{S}'(\mathbb{R}^m_+)$ and $\mathcal{S}'(\mathbb{R}^n_+)$, respectively. There exist $C>0$ and $j,l\in\mathbb{N}$ such that

$$\sup_{T\in A'}|\langle T,\varphi\rangle|\leqslant C\|\varphi\|_{j,l}\quad\text{and}\quad\sup_{F\in B'}|\langle F,\psi\rangle|\leqslant C\|\psi\|_{j,l},$$

where

$$\|f\|_{j,l}=\sup_{\substack{|k|\leqslant j\\|p|\leqslant l}}\sup_{x\in\mathbb{R}^d_+}|x^kD^pf(x)|<\infty.$$

For all $T\in A'$ and $F\in B'$ we have

$$\begin{aligned}|\langle T_x\otimes F_y,\chi(x,y)\rangle|&=|\langle F_y,\langle T_x,\chi(x,y)\rangle\rangle|\leqslant C\sup_{\substack{|k|\leqslant j\\|p|\leqslant l}}\sup_{y\in\mathbb{R}^n_+}|y^k\langle T_x,D^p_y\chi(x,y)\rangle|\\&\leqslant C^2\sup_{\substack{|k|\leqslant j\\|p|\leqslant l}}\sup_{\substack{|k'|\leqslant j\\|p'|\leqslant l}}\sup_{\substack{x\in\mathbb{R}^m_+\\y\in\mathbb{R}^n_+}}|x^{k'}y^kD^{p'}_xD^p_y\chi(x,y)|\\&\leqslant C^2\|\chi(x,y)\|_{(k',k),(p',p)},\qquad\text{for all }\chi\in\mathcal{S}(\mathbb{R}^m_+)\otimes\mathcal{S}(\mathbb{R}^n_+).\end{aligned}$$

It follows that the ϵ topology on $\mathcal{S}(\mathbb{R}^m_+)\otimes\mathcal{S}(\mathbb{R}^n_+)$ is weaker than the induced one from $\mathcal{S}(\mathbb{R}^{m+n}_+)$. □

As a consequence of this theorem we have the following important

THEOREM 4.2. *The restriction mapping $f\mapsto f|_{\mathbb{R}^d_+}$, $\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d_+)$ is a topological homomorphism onto.*

The space $\mathcal{S}(\mathbb{R}^d_+)$ is topologically isomorphic to the quotient space $\mathcal{S}(\mathbb{R}^d)/N$, where $N=\{f\in\mathcal{S}(\mathbb{R}^d)\mid\operatorname{supp}f\subseteq\mathbb{R}^d\setminus\mathbb{R}^d_+\}$. Consequently, $\mathcal{S}'(\mathbb{R}^d_+)$ can be identified with the closed subspace of $\mathcal{S}'(\mathbb{R}^d)$ which consists of all tempered distributions with support in $\overline{\mathbb{R}^d_+}$.

PROOF. Obviously, the restriction mapping $f\mapsto f|_{\mathbb{R}^d_+}$, $\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d_+)$ is continuous. We prove its surjectivity by induction on d. For clarity, denote the d-dimensional restriction by R_d. For $d=1$, the surjectivity of R_1 is proved in [**1**, p. 168]. Assume that R_d is surjective. By the open mapping theorem, R_d and R_1 are topological homomorphisms onto since all the underlying spaces are (F)-spaces. By the above theorem, $R_d\hat{\otimes}_\pi R_1$ is continuous mapping from $\mathcal{S}(\mathbb{R}^{d+1})$ to $\mathcal{S}(\mathbb{R}^{d+1}_+)$ ($\mathcal{S}(\mathbb{R}^d)\hat{\otimes}\mathcal{S}(\mathbb{R})\cong\mathcal{S}(\mathbb{R}^{d+1})$ by the Schwartz kernel theorem). Clearly

$R_d\hat{\otimes}_\pi R_1 = R_{d+1}$. As $\mathcal{S}(\mathbb{R}^{d+1})$ and $\mathcal{S}(\mathbb{R}^{d+1}_+)$ are (F)-spaces, [**6**, Theorem 7, p. 189] implies that R_{d+1} is also surjective.

The surjectivity of the restriction mapping together with the open mapping theorem implies that it is homomorphism. Clearly N is a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ and $\ker R_d = N$. Thus R_d induces natural topological isomorphism between $\mathcal{S}(\mathbb{R}^d)/N$ and $\mathcal{S}(\mathbb{R}^d_+)$. Hence $\big(\mathcal{S}(\mathbb{R}^d)/N\big)'_b$ is topologically isomorphic to $\mathcal{S}'(\mathbb{R}^d_+)$ (the index b stands for the strong dual topology). Since $\mathcal{S}(\mathbb{R}^d)$ is an (FS)-space, [**5**, Theorem A.6.5, p. 255] implies that $\big(\mathcal{S}(\mathbb{R}^d)/N\big)'_b$ is topologically isomorphic to the closed subspace $N^\perp = \{T \in \mathcal{S}'(\mathbb{R}^d) \mid \langle T, f\rangle = 0, \text{ for all } f \in N\}$ of $\mathcal{S}'(\mathbb{R}^d)$ which is exactly the subspace of all tempered distributions with support in $\overline{\mathbb{R}^d_+}$. $\square$

Given $f, g \in \mathcal{S}'(\mathbb{R}^d_+)$, Theorem 4.2 implies that we can consider them as elements of $\mathcal{S}'(\mathbb{R}^d)$ with support in $\overline{\mathbb{R}^d_+}$. Now, one easily verifies that for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have $(f(x)\otimes g(y))\varphi(x+y) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$, hence the $\mathcal{S}'$-convolution of f and g exists (see [**9**, p. 26]). Also, if $\operatorname{supp}\varphi \cap \overline{\mathbb{R}^d_+} = \emptyset$, then $(f(x)\otimes g(y))\varphi(x+y) = 0$, hence $\operatorname{supp} f * g \subseteq \overline{\mathbb{R}^d_+}$, i.e., $f * g \in \mathcal{S}'(\mathbb{R}^d_+)$. Thus

$$\langle f * g, \varphi\rangle = \langle f(x) \otimes g(y), \varphi(x + y)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d_+)$$

(observe that the function $\varphi^\Delta(x, y) = \varphi(x + y)$ is an element of $\mathcal{S}(\mathbb{R}^{2d}_+)$).

Remark 4.1. [**1**, Remark 3.7 for $d = 1$] Let us show that $\mathcal{S}'(\mathbb{R}^d_+)$ is a convolution algebra. Given $f, g \in \mathcal{S}'(\mathbb{R}^d_+)$, we compute the n-th Laguerre coefficient of $f * g$. If $a_n = \langle f, \mathcal{L}_n\rangle$ and $b_n = \langle g, \mathcal{L}_n\rangle$, then

$$\langle f * g, \mathcal{L}_n(t)\rangle = \langle f(x) \otimes g(y), \mathcal{L}_n(x + y)\rangle.$$

Now, $L^1_n(x+y) = \sum_{k=0}^n L_{n-k}(x)L_k(y)$ and $L_n(t) = L^1_n(t) - L^1_{n-1}(t)$ (see [**2**, p. 192]) where $L^1_n(x) = \sum_{k=0}^n \binom{n+1}{n-k}((-x)^k/k!)$. In order to simplify the proof, we consider the case $d = 2$. Then

$$\begin{aligned}
\langle f * g, \mathcal{L}_n(t)\rangle &= \Big\langle f(x) \otimes g(y), \prod_{i=1}^{2} (\mathcal{L}^1_{n_i}(x_i + y_i) - \mathcal{L}^1_{n_i-1}(x_i + y_i))\Big\rangle \\
&= \Big\langle f(x) \otimes g(y), \prod_{i=1}^{2}\Big(\sum_{k_i=0}^{n_i} \mathcal{L}_{n_i-k_i}(x_i)\mathcal{L}_{k_i}(y_i) - \sum_{k_i=0}^{n_i-1} \mathcal{L}_{n_i-k_i-1}(x_i)\mathcal{L}_{k_i}(y_i)\Big)\Big\rangle \\
&= \Big\langle f(x)g(y), \sum_{k\leqslant(n_1,n_2)} \mathcal{L}_{(n_1,n_2)-k}(x)\mathcal{L}_k(y) - \sum_{k\leqslant(n_1-1,n_2)} \mathcal{L}_{(n_1-1,n_2)-k}(x)\mathcal{L}_k(y) \\
&\qquad - \sum_{k\leqslant(n_1,n_2-1)} \mathcal{L}_{(n_1,n_2-1)-k}(x)\mathcal{L}_k(y) + \sum_{k\leqslant(n_1-1,n_2-1)} \mathcal{L}_{(n_1-1,n_2-1)-k}(x)\mathcal{L}_k(y)\Big\rangle \\
&= \sum_{k\leqslant(n_1,n_2)} a_{(n_1,n_2)-k}b_k - \sum_{k\leqslant(n_1-1,n_2)} a_{(n_1-1,n_2)-k}b_k \\
&\quad - \sum_{k\leqslant(n_1,n_2-1)} a_{(n_1,n_2-1)-k}b_k + \sum_{k\leqslant(n_1-1,n_2-1)} a_{(n_1-1,n_2-1)-k}b_k,
\end{aligned}$$

where a_n or b_n equals zero if some component of the subindex n is less than zero. It is easy to verify that if $(a_n)_{n\in\mathbb{N}^2} \in s'$ and $(b_n)_{n\in\mathbb{N}^2} \in s'$, then $\langle f * g, \mathcal{L}_n(t)\rangle \in s'$.

References

1. A. J. Duran, *Laguerre expansions of Tempered Distributions and Generalized Functions*, J. Math. Anal. Appl. **150** (1990), 166–180.
2. A. Erdelyi, *Higher Transcedentals Function, Vol. 2*, McGraw-Hill, New York, 1953.
3. L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Springer-Verlag, 1990.
4. M. Guillemot-Teissier, *Développements des distributions en séries de fonctions orthogonales. Séries de Legendre et de Laguerre*, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. **25**(3) (1971), 519–573.
5. M. Morimoto, *An Introduction to Sato's Hyperfunctions*, 129, American Mathematical Soc., 1993.
6. G. Köthe, *Topological vector spaces II, Vol. II*, Springer-Verlag, New York, 1979.
7. S. Pilipović, *On the Laguerre expansions of generalized functions*, C. R. Math. Rep. Acad. Sci. Canada **11** (1) (1989), 23–27.
8. R. T. Seeley, *Extension of C8 functions defined in a half space*, Proc. Am. Math. Soc. **15** (1964), 625–626.
9. R. Shiraishi, *On the definition of convolutions for distributions*, J. Sci. Hiroshima Univ. Ser. A **23** (1959), 19–32.
10. G. Szego, *Orthogonal Polynomials*, Am. Math. Soc. Collequium, 1959.
11. H. Whitney, *Analytic extensions of functions defined in closed sets*, Trans. Am. Math. Soc. **36** (1934), 63–89.
12. A. I. Zayed, *Laguerre series as boundary values*, SIAM J. Math. Anal. **13** (2) (1982), 263–279.
13. A. H. Zemanian, *Generalized Integral Transformations*, Interscience, New York, 1968.

Faculty of Forestry
Belgrade University
Belgrade, Serbia
smiljana.jaksic@sfb.bg.ac.rs

(Received 26 08 2015)

Faculty of Mechanical Engineering
University Ss. Cyril and Methodius
Skopje, Macedonia
bprangoski@yahoo.com