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# FINITE DIFFERENCE APPROXIMATION FOR PARABOLIC INTERFACE PROBLEM WITH TIME-DEPENDENT COEFFICIENTS

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ABSTRACT. The convergence of difference scheme for two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficients of the space derivatives, is considered. An estimate of the rate of convergence in a special discrete $\widetilde{W}_2^{1,1/2}$ Sobolev norm, compatible with the smoothness of the coefficients and solution, is proved.

## 1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has a weak global regularity and it is impossible to establish convergence of finite difference schemes using the classical Taylor series expansion. Often, the Bramble–Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [**6**, **8**, **12**].

Following Lazarov et al. [**12**], a convergence rate estimate of the form

$$\|u-v\|_{W_{2,h}^k} \leqslant Ch^{s-k}\|u\|_{W_2^s}, \quad s>k,$$

is called *compatible* with the smoothness (regularity) of the solution $u$ of the boundary-value problem. Here $v$ is the solution of the discrete problem, $h$ is the spatial mesh step, $W_2^s$ and $W_{2,h}^k$ are Sobolev spaces of functions with continuous and discrete argument, respectively, $C$ is a constant which doesn't depend on $u$ and $h$. For the parabolic case typical estimates are of the form

$$\|u-v\|_{W_{2,h\tau}^{k,k/2}} \leqslant C(h+\sqrt{\tau})^{s-k}\|u\|_{W_2^{s,s/2}}, \quad s>k,$$

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where $\tau$ is the time step. In the case of equations with variable coefficients the constant $C$ in the error bounds depends on norms of the coefficients (see, for example, [**1**, **8**, **18**]).

One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow in the singular point is proportional to the time-derivative of the temperature [**14**]. Such problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply to their convergence analysis.

In the present paper a finite-difference scheme, approximating the two-dimensonal initial-boundary value problem for the heat equation with concentrated capacity and time dependent coefficients is derived. Special Sobolev norm (corresponding to the norm $W_2^{1,1/2}$ for a classical heat-conduction problem) is constructed. In this norm, a convergence rate estimate, compatible with the smoothness of the solution of the boundary value problem, is obtained.

Note that the convergence to classical solutions is studied in [**5**, **19**]. One-dimensional parabolic problem with weak solution is studied in [**2**, **3**, **10**]; 2D parabolic problem with variable coefficients (but not time-dependent) is considered in [**3**, **11**].

## 2. Preliminary results

Let $H$ be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and $S$-unbounded self-adjoint positive definite linear operator, with domain $D(S)$ dense in $H$. It is easy to see that the product $(u,v)_S = (Su,v)$, $u,v \in D(S)$ satisfies the inner product axioms. The closure of $D(S)$ in the norm $\|u\|_S = (u,u)_S^{1/2}$ is a Hilbert space $H_S \subset H$. The inner product $(u,v)$ continuously extends to $H_S^* \times H_S$, where $H_S^* = H_S^{-1}$ is the dual space for $H_S$. Spaces $H_S$, $H$ and $H_{S^{-1}}$ represent a Gelfand triple $H_S \subset H \subset H_{S^{-1}}$ with continuous imbeddings. Operator $S$ extends to the map $S : H_S \mapsto H_S^*$. There exist unbounded self-adjoint positive definite linear operator $S^{1/2}$, such that $D(S^{1/2}) = H_S$ and $(u,v)_S = (Su,v) = (S^{1/2}u, S^{1/2}v)$. We also define Sobolev spaces $W_2^s(a,b;H)$, $W_2^0(a,b;H) = L_2(a,b;H)$, of the functions $u = u(t)$ mapping the interval $(a,b) \subset R$ into $H$ (see [**13**, **20**]).

Let $A$ and $B$ be unbounded self-adjoint positive definite linear operators, $A = A(t)$, $B \neq B(t)$, in Hilbert space $H$, in general noncommutative, with $D(A)$ dense in $H$ and $H_A \subset H_B$. We consider the following abstract Cauchy problems:

$$B\frac{du}{dt} + Au = f(t), \quad 0 < t < T, \quad u(0) = 0, \tag{2.1}$$

$$B\frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T, \quad u(0) = 0, \tag{2.2}$$

where $f(t)$ and $g(t)$ are given and $u(t)$ is the unknown function with values in $H$. Let also assume that $A_0 \leqslant A(t) \leqslant kA_0$ where $k = \text{const} > 1$ and $A_0 \neq A_0(t)$ is a constant self-adjoint positive definite linear operator in $H$. The following propositions are proved in [**2**].

LEMMA 2.1. *Let $f \in L_2(0,T;H_{A_0^{-1}})$. Then the solution to problem* (2.1) *satisfies a priori estmate*

$$\int_0^T \|u(t)\|_{A_0}^2 dt + \int_0^T \int_0^T \frac{\|u(t)-u(t')\|_B^2}{|t-t'|^2}\,dt\,dt' \leqslant C\int_0^T \|f(t)\|_{A_0^{-1}}^2 dt. \tag{2.3}$$

LEMMA 2.2. *Let $g \in W_2^{1/2}(0,T;H_{B^{-1}})$. Then the solution to problem* (2.2) *satisfies a priori estimate*

$$\int_0^T \|u(t)\|_{A_0}^2 dt + \int_0^T \int_0^T \frac{\|u(t)-u(t')\|_B^2}{|t-t'|^2}\,dt\,dt'$$
$$\leqslant C\bigg[\int_0^T \int_0^T \frac{\|g(t)-g(t')\|_{B^{-1}}^2}{|t-t'|^2}\,dt\,dt' + \int_0^T \Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|g(t)\|_{B^{-1}}^2 dt\bigg].$$

An analogous result hold for the operator-difference schemes. Let $H_h$ be a finite-dimensional real Hilbert space with the inner product $(\cdot,\cdot)_h$ and the norm $\|\cdot\|_h$. For a self-adjoint positive linear operator $S_h$ in $H_h$, by $H_{S_h}$ we denote the space $H_{S_h}=H_h$ with the inner product $(v,w)_{S_h}=(S_hv,w)_h$ and the norm $\|v\|_{S_h}=(S_hv,v)_h^{1/2}$.

Let $\omega_\tau$ be a uniform mesh on $(0,T)$ with the step size $\tau=T/m$, $\omega_\tau^+=\omega_\tau\cup\{T\}$ and $\bar\omega_\tau=\omega_\tau\cup\{0,T\}$. Further, we shall use standard notation from the theory of the difference schemes [**17**]. In particular, we set

$$v_{\bar t}=v_{\bar t}(t)=\frac{v(t)-v(t-\tau)}{\tau}.$$

We consider the implicit operator-difference scheme

$$B_hv_{\bar t}+A_hv=\varphi(t),\quad t\in\omega_\tau^+,\quad v(0)=0, \tag{2.4}$$

where $A_h=A_h(t)$ and $B_h\neq B_h(t)$ are linear positive definite self-adjoint operators in $H_h$, in general case noncommutative, $\varphi(t)$ is given and $v(t)$ is an unknown function with values in $H_h$. Let us also consider the scheme

$$B_hv_{\bar t}+A_hv=\psi_{\bar t},\quad t\in\omega_\tau^+,\quad v(0)=0, \tag{2.5}$$

where $\psi(t)$ is a given mesh function with values in $H_h$. Analogously, as in the previous case, we assume that $A_{h0}\leqslant A_h(t)\leqslant kA_{h0}$ where $k=\text{const}>1$ and $A_{h0}\neq A_{h0}(t)$ is a self-adjoint positive linear operator in $H_h$. The following analogs of Lemmas 2.1 and 2.2 are proved in [**11**].

LEMMA 2.3. *The solution $v$ of operator-difference scheme* (2.4) *satisfies a priori estimate*

$$\tau\sum_{t\in\omega_\tau^+}\|v(t)\|_{A_{h0}}^2+\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,t'\neq t}\frac{\|v(t)-v(t')\|_{B_h}^2}{|t-t'|^2}\leqslant C\tau\sum_{t\in\omega_\tau^+}\|\varphi(t)\|_{A_{h0}^{-1}}^2.$$

LEMMA 2.4. *The solution $v$ of operator-difference scheme* (2.5) *satisfies a priori estimate*

$$\tau\sum_{t\in\omega_\tau^+}\|v(t)\|_{A_{h0}}^2+\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,t'\neq t}\frac{\|v(t)-v(t')\|_{B_h}^2}{|t-t'|^2}$$

$$\leqslant C\Big[\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,t'\neq t}\frac{\|\psi(t)-\psi(t')\|^2_{B_h^{-1}}}{|t-t'|^2}+\tau\sum_{t\in\bar\omega_\tau}\Big(\frac{1}{t+\tau}+\frac{1}{T-t+\tau}\Big)\|\psi(t)\|^2_{B_h^{-1}}\Big]$$

## 3. Differential problem and its approximation

Let us consider the 2D initial-boundary-value problem for the heat equation in the presence of a concentrated capacity at the line $x_2=\xi$, $0<\xi<1$:

$$\begin{aligned}
&(1+K\delta_\Sigma(x))\frac{\partial u}{\partial t}-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\Big(a_i(x,t)\frac{\partial u}{\partial x_i}\Big)=f,\ \text{ on } Q,\\
&u=0,\ \text{ on } \partial\Omega\times(0,T),\\
&u(x,0)=u_0(x),\ \text{ on } \Omega,
\end{aligned}\tag{3.1}$$

where $\delta_\Sigma(x)=\delta(x_2-\xi)$ is the Dirac delta function, $K=\text{const}>0$ and $\Omega=(0,1)^2$, $Q=\Omega\times(0,T)$. We shall assume that

$$\begin{aligned}
&a_i\in W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)\cap W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2),\quad \varepsilon>0,\\
&f\in W_2^{1+\varepsilon,1/2+\varepsilon/2}(Q),\\
&u_0\in W_2^2(\Omega_1)\cap W_2^2(\Omega_2),
\end{aligned}\tag{3.2}$$

$$u\in W_2^{3,3/2}(Q_1)\cap W_2^{3,3/2}(Q_2)\cap W_2^{3,3/2}(\Sigma\times(0,T)),\tag{3.3}$$

where $\Omega_1=(0,1)\times(0,\xi)$, $\Omega_2=(0,1)\times(\xi,1)$, $Q_1=\Omega_1\times(0,T)$, $Q_2=\Omega_2\times(0,T)$, $\Sigma=\{(x_1,\xi)|x_1\in(0,1)\}$. Note that conditions (3.2) express the minimal smoothness requirements on the data under which the solution $u$ of (3.1) may belong to the function space stated in (3.3). To guarantee that such $u$ really exists, we also need some additional compatibility conditions at the corners of $\Omega$ (see [**7**]). We also assume that and $0<c_1\leqslant a_i(x,t)\leqslant c_2$, on $Q$.

Let $\bar\omega_h$–uniform mesh with step size $h$ in $\bar\Omega$, $\omega_h=\bar\omega_h\cap\Omega$, $\omega_{1h}=\bar\omega_h\cap([0,1)\times(0,1))$, $\omega_{2h}=\bar\omega_h\cap((0,1)\times[0,1))$, $\sigma_h=\omega_h\cap\Sigma$. Suppose that $\xi$ is a rational number. Then one can choose step $h$ so that $\sigma_h\neq\emptyset$. Also we assume that the condition $c_1h^2\leqslant\tau\leqslant c_2h^2$ is satisfied. Define the finite differences in the usual way:

$$v_{\bar x_i}(x,t)=\frac{v-v^{-i}}{h},\quad v_{x_i}(x,t)=\frac{v^{+i}-v}{h},$$

where $v^{\pm i}(x,t)=v(x\pm e_ih,t)$, $e_1=(1,0)$, $e_2=(0,1)$. Problem (3.1) can be approximated on the mesh $\overline{Q}_{h\tau}=\bar\omega_h\times\bar\omega_\tau$ by the following difference scheme with averaged right-hand side:

$$\begin{aligned}
&(1+K\delta_{\sigma_h})v_{\bar t}+L_hv=T_1^2T_2^2T_t^-f,\ \text{ on } Q_{h\tau},\\
&v=0,\ \text{ on } \gamma_h\times\omega_\tau^+,\quad v(x,0)=u_0(x),\ \text{ on } \omega_h,
\end{aligned}\tag{3.4}$$

where $L_hv=-\frac12\sum_{i=1}^2((a_iv_{x_i})_{\bar x_i}+(a_iv_{\bar x_i})_{x_i})$,

$$\delta_{\sigma_h}(x)=\begin{cases}0, & x\notin\sigma_h\\ 1/h, & x\in\sigma_h\end{cases}$$

is the mesh Dirac function, and $T_1, T_2, T_t^-$ are Steklov averaging operators defined by

$$T_1 f(x_1,x_2) = T_1^{\pm} f(x_1 \mp h/2, x_2) = \frac{1}{h}\int_{x_1-h/2}^{x_1+h/2} f(x_1', x_2)\, dx_1',$$

$$T_2 f(x_1,x_2) = T_2^{\pm} f(x_1, x_2 \mp h/2) = \frac{1}{h}\int_{x_2-h/2}^{x_2+h/2} f(x_1, x_2')\, dx_2',$$

$$T_t^- f(x,t) = T_t^+ f(x, t-\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,t')\, dt'.$$

Note that these operators are commutative and transform the derivatives to divided differences, for example:

$$T_i^- \frac{\partial u}{\partial x_i} = u_{\bar{x}_i}, \quad T_i^+ \frac{\partial u}{\partial x_i} = u_{x_i}, \quad T_i^2 \frac{\partial^2 u}{\partial x_i^2} = u_{x_i \bar{x}_i}, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}}.$$

We also define

$$T_2^{2-} f(x_1,x_2) = \frac{1}{h}\int_{x_2-h}^{x_2} \Big(1 + \frac{x_2' - x_2}{h}\Big) f(x_1, x_2')\, dx_2',$$

$$T_2^{2+} f(x_1,x_2) = \frac{1}{h}\int_{x_2}^{x_2+h} \Big(1 - \frac{x_2' - x_2}{h}\Big) f(x_1, x_2')\, dx_2'.$$

We define the following discrete norms and seminorms:

$$\|v\|^2_{L_2(Q_{h\tau})} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{L_2(\omega_h)}, \quad \|v\|^2_{L_2(\sigma_h\times\omega_\tau)} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{L_2(\sigma_h)}$$

$$|v|^2_{L_2(\omega_\tau;\, W_2^{1/2}(\sigma_h))} = \tau \sum_{t\in\omega_\tau^+} |v(\cdot,t)|^2_{W_2^{1/2}(\sigma_h)},$$

$$|v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} = \tau \sum_{t\in\bar{\omega}_\tau} \tau \sum_{t'\in\bar{\omega}_\tau,\, t'\neq t} \frac{\|v(\cdot,t) - v(\cdot,t')\|^2_{L_2(\omega_h)}}{|t-t'|^2},$$

$$|v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} = \tau \sum_{t\in\bar{\omega}_\tau} \tau \sum_{t'\in\bar{\omega}_\tau,\, t'\neq t} \frac{\|v(\cdot,t) - v(\cdot,t')\|^2_{L_2(\sigma_h)}}{|t-t'|^2},$$

$$\|v\|^2_{\widetilde{W}_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} = |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} + \tau \sum_{t\in\bar{\omega}_\tau} \Big(\frac{1}{t+\tau} + \frac{1}{T-t+\tau}\Big) \|v(\cdot,t)\|^2_{L_2(\omega_h)},$$

$$\|v\|^2_{\widetilde{W}_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} = |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} + \tau \sum_{t\in\bar{\omega}_\tau} \Big(\frac{1}{t+\tau} + \frac{1}{T-t+\tau}\Big) \|v(\cdot,t)\|^2_{L_2(\sigma_h)},$$

$$\|v\|^2_{\widetilde{W}_2^{1,\,1/2}(Q_{h\tau})} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{W_2^1(\omega_h)} + |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} + |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))}.$$

## 4. Convergence of the difference scheme

In this section we prove the convergence of difference scheme (3.4) in the $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$ norm. Let $u$ be the solution to boundary-value problem (3.1) and

$v$ the solution of difference problem (3.4). The error $z = u - v$ satisfies the finite difference scheme

$$(4.1)\qquad \begin{aligned} &(1 + K\delta_{\sigma_h})z_{\bar t} + L_h z = \varphi, \ \text{on } \omega_h \times \omega_\tau^+, \\ &z = 0, \ \text{on } \ \gamma_h \times \omega_\tau^+, \quad z(x,0) = 0, \ \text{on } \omega_h. \end{aligned}$$

where

$$\begin{aligned} &\varphi = \sum_{i=1}^{2} \eta_{i,\bar x_i} + \chi_{bart} + \delta_{\sigma_h}\mu_{\bar t}, \\ &\chi = u - T_1^2 T_2^2 u, \\ &\eta_i = T_i^+ T_{3-i}^2 T_t^- \Big(a_i \frac{\partial u}{\partial x_i}\Big) - \frac{1}{2}(a_i + a_i^{+i})u_{x_i}, \\ &\mu = Ku - T_1^2(Ku). \end{aligned}$$

Let us set $\eta_1 = \tilde\eta_1 + \delta_{\sigma_h}\hat\eta_1$, $\chi = \tilde\chi + \delta_{\sigma_h}\hat\chi$, where

$$\hat\eta_1 = \frac{h^2}{6} T_1^+ T_t^- \Big(\Big[a_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial a_1}{\partial x_2}\frac{\partial u}{\partial x_1}\Big]_\Sigma\Big), \quad \hat\chi = \frac{h^2}{6}\Big[T_1^2 \frac{\partial u}{\partial x_2}\Big]_\Sigma,$$

and $[u]_\Sigma = u(x_1, \xi + 0, t) - u(x_1, \xi - 0, t)$.

Using Lemmas 2.3 and 2.4, we directly obtain the following a priori estimate for the solution of difference scheme (4.1):

$$(4.2)\qquad \begin{aligned} \|z\|_{\widetilde W_2^{1,1/2}(Q_{h\tau})} \leqslant C\big[&\|\eta_2\|_{L_2(Q_{h\tau})} + \|\tilde\eta_1\|_{L_2(Q_{h\tau})} + |\hat\eta_1|_{L_2(\omega_\tau;\, W_2^{1/2}(\sigma_h))} \\ &+ \|\tilde\chi\|_{\widetilde W_2^{1/2}(\omega_\tau, L_2(\omega_h))} + \|\hat\chi\|_{\widetilde W_2^{1/2}(\omega_\tau, L_2(\sigma_h))} + \|\mu\|_{\widetilde W_2^{1/2}(\omega_\tau, L_2(\sigma_h))}\big]. \end{aligned}$$

Therefore, in order to estimate the rate of convergence of difference scheme (3.4), it is sufficient to estimate the right-hand side of inequality (4.2).

We decompose $\eta_i = \eta_{i1} + \eta_{i2} + \eta_{i3}$, where

$$\begin{aligned} &\eta_{i1} = T_i^+ T_{3-i}^2 T_t^- \Big(a_i \frac{\partial u}{\partial x_i}\Big) - (T_i^+ T_{3-i}^2 T_t^- a_i)\Big(T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i}\Big), \\ &\eta_{i2} = \big[T_i^+ T_{3-i}^2 T_t^- a_i - 0,5(a_i + a_i^{+i})\big]\Big(T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i}\Big), \\ &\eta_{i3} = -0,5(a_i + a_i^{+i})\Big\{T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i} - u_{x_i}\Big\}. \end{aligned}$$

The term $\eta_{i1}$ is a bounded bilinear functional of the argument $(a_i, u) \in W_4^{1,1/2}(e) \times W_4^{2,1}(e)$,

$$e = e(x,t) = \{(x_1', x_2', t') : x_i < x_i' < x_i + h,\ |x_{3-i}' - x_{3-i}| < h,\ t' \in (t - \tau, t)\},$$

and for $i = 1$, $x_2 \neq \xi$. Further, $\eta_{i1} = 0$ whenever $a_i$ is a constant or $u$ is a polynomial of degree one in $x_1$ or $x_2$ or a constant. Applying the Bramble–Hilbert lemma [**6**] we get:

$$(4.3)\qquad |\eta_{i1}(x,t)| \leqslant C|a_i|_{W_4^{1,1/2}(e)} |u|_{W_4^{2,1}(e)}.$$

The term $\eta_{i2}$ is a bounded bilinear functional of the argument $(a_i, u) \in W_q^{2,1}(e) \times W_{2q/(q-2)}^{1,1/2}(e)$, $q = 2+\varepsilon$. Further, $\eta_{i2} = 0$ whenever $a_i$ is a polynomial of degree one in $x_1$ or $x_2$ or constant or $u$ is constant. Applying the Bramble–Hilbert lemma we get the following estimate:

$$|\eta_{i2}(x,t)| \leqslant C|a_i|_{W_q^{2,\,1}(e)}|u|_{W_{2q/(q-2)}^{1,\,1/2}(e)}. \tag{4.4}$$

The term $\eta_{i3}$ is a bounded bilinear functional of the argument $(a_i, u) \in \mathbb{C}(\overline{Q_k}) \times W_2^{3,3/2}(e)$, $k = 1, 2$, Further, $\eta_{i3} = 0$ whenever $u$ is a polynomial of degree two in $x_1$ or $x_2$ and a polynomial of arbitrary degree in $t$. Applying the Bramble–Hilbert lemma, we get the estimate

$$|\eta_{i3}(x,t)| \leqslant C\|a_i\|_{\mathbb{C}(\overline{Q_k})}|u|_{W_2^{3,\,3/2}(e)}. \tag{4.5}$$

From estimates (4.3)–(4.5), choosing $i = 2$, after summation and using the imbeddings

$$\begin{aligned} &W_2^{2+\varepsilon,1+\varepsilon/2} \subset W_4^{1,1/2}, \quad W_2^{3,3/2} \subset W_4^{2,1},\\ &W_2^{2+\varepsilon,1+\varepsilon/2} \subset W_q^{2,1}, \quad\;\; W_2^{3,3/2} \subset W_{2q/(q-2)}^{1,1/2},\\ &W_2^{2+\varepsilon,1+\varepsilon/2} \subset \mathbb{C}, \qquad \text{for } q = 2+\varepsilon, \end{aligned} \tag{$*$}$$

we get

$$\begin{aligned} \|\eta_2\|_{L_2(Q_{h\tau})} \leqslant Ch^2\big(&\|a_2\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)}\\ &+ \|a_2\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}\big). \end{aligned} \tag{4.6}$$

Let us estimate the term $\tilde{\eta}_1$. At the point $x \notin \sigma_h$, we have $\tilde{\eta}_1 = \eta_1$ and estimates (4.3)–(4.5) are valid. At the point $x \in \sigma_h$, we decompose

$$\tilde{\eta}_1 = \sum_{k=1}^{3}(\eta_{1,k}^- + \eta_{1,k}^+),$$

where $\eta_{1,k}^\pm$ are defined at the point $x_2 = \xi \pm 0$

$$\begin{aligned} \eta_{1,1}^\pm = {}& T_1^+T_2^{2\pm}T_t^-\Big(a_1\frac{\partial u}{\partial x_1}\Big) - 2(T_1^+T_2^{2\pm}T_t^-a_1)\Big(T_1^+T_2^{2\pm}T_t^-\frac{\partial u}{\partial x_1}\Big)\\ &\pm \frac{h}{6}\Big(T_1^+T_t^-\frac{\partial a_1}{\partial x_2}\Big)\Big[2\Big(T_1^+T_2^{2\pm}T_t^-\frac{\partial u}{\partial x_1}\Big) - \Big(T_1^+T_t^-\frac{\partial u}{\partial x_1}\Big)\Big]\\ &\pm \frac{h}{6}\Big[\frac{a_1 + a_1^{+1}}{2}\Big(T_1^+T_t^-\frac{\partial^2 u}{\partial x_1\partial x_2}\Big) - \Big(T_1^+T_t^-a_1\frac{\partial^2 u}{\partial x_1\partial x_2}\Big)\Big]\\ &\pm \frac{h}{6}\Big[\Big(T_1^+T_t^-\frac{\partial a_1}{\partial x_2}\Big)\Big(T_1^+T_t^-\frac{\partial u}{\partial x_1}\Big) - \Big(T_1^+T_t^-\frac{\partial a_1}{\partial x_2}\frac{\partial u}{\partial x_1}\Big)\Big],\\ \eta_{1,2}^\pm = {}& \Big[2(T_1^+T_2^{2\pm}T_t^-a_1) - \frac{a_1 + a_1^{+1}}{2} \mp \frac{h}{3}\Big(T_1^+T_t^-\frac{\partial a_1}{\partial x_2}\Big)\Big] \times \Big(T_1^+T_2^{2\pm}T_t^-\frac{\partial u}{\partial x_1}\Big),\\ \eta_{1,3}^\pm = {}& \frac{a_1 + a_1^{+1}}{4}\Big[2\Big(T_1^+T_2^{2\pm}T_t^-\frac{\partial u}{\partial x_1}\Big) - u_{x_1} \mp \frac{h}{3}\Big(T_1^+T_t^-\frac{\partial^2 u}{\partial x_1\partial x_2}\Big)\Big]. \end{aligned}$$

The term $\eta_{1,1}^{\pm}$ is a bounded bilinear functional of the argument $(a_1,u)\in W_4^{1,1/2}(e_1^{\pm})\times W_4^{2,1}(e_1^{\pm})$ where

$$e_1^+=(x_1,x_1+h)\times(\xi,\xi+h)\times(t-\tau,t),$$
$$e_1^-=(x_1,x_1+h)\times(\xi-h,\xi)\times(t-\tau,t).$$

Further, $\eta_{1,1}^{\pm}=0$ whenever $a_1$ is a constant or $u$ is a polynomial of degree one in $x_1$ or $x_2$ or a constant. Applying the Bramble–Hilbert lemma we get

$$|\eta_{1,1}^{\pm}(x,t)|\leqslant C|a_1|_{W_4^{1,1/2}(e_1^{\pm})}|u|_{W_4^{2,1}(e_1^{\pm})}. \tag{4.7}$$

The term $\eta_{1,2}^{\pm}$ is a bounded bilinear functional of the argument $(a_1,u)\in W_q^{2,1}(e_1^{\pm})\times W_{2q/(q-2)}^{1,1/2}(e_1^{\pm})$, $q=2+\varepsilon$. Further, $\eta_{1,2}^{\pm}=0$ whenever $a_1$ is a polynomial of degree one in $x_1$ or $x_2$ or constant or $u$ is a constant. Applying the Bramble–Hilbert lemma we get the estimate

$$|\eta_{1,2}^{\pm}(x,t)|\leqslant C|a_1|_{W_q^{2,1}(e_1^{\pm})}|u|_{W_{2q/(q-2)}^{1,1/2}(e_1^{\pm})}. \tag{4.8}$$

The term $\eta_{1,3}^{\pm}$ is a bounded bilinear functional of the argument $(a_1,u)\in \mathbb{C}(\overline{Q_k})\times W_2^{3,3/2}(e_1^{\pm})$, $k=1,2$. Further, $\eta_{1,3}^{\pm}=0$ whenever $u$ is a polynomial of degree two in $x_1$ or $x_2$ and polynomial of arbitraty degree in $t$. Applying the Bramble–Hilbert lemma we get

$$|\eta_{1,3}^{\pm}(x,t)|\leqslant C\|a_1\|_{\mathbb{C}(\overline{Q_k})}|u|_{W_2^{3,\,3/2}(e_1^{\pm})}. \tag{4.9}$$

From estimates (4.3)–(4.5) and (4.7)–(4.9), after summation and using imbeddings $(*)$, we get

$$\begin{aligned}\|\widetilde{\eta}_1\|_{L_2(Q_{h\tau})}\leqslant Ch^2\big(&\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)}\|u\|_{W_2^{3,\,3/2}(Q_1)}\\&+\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}\big).\end{aligned} \tag{4.10}$$

Let us estimate the term $\hat{\eta}_1$. For $\phi\in W_2^{1/2}(\Sigma)$, the following estimate is valid

$$|T_1^+\phi|_{W_2^{1/2}(\sigma_h)}\leqslant C|\phi|_{W_2^{1/2}(\Sigma)}\leqslant C\|\phi\|_{W_2^1(\Omega_k)},\quad k=1,2,$$

wherefrom

$$|\widehat{\eta}_1(\cdot,t)|_{W_2^{1/2}(\sigma_h)}\leqslant Ch^2\big(\|T_t^-\nu(\cdot,t)\|_{W_2^1(\Omega_1)}+\|T_t^-\nu(\cdot,t)\|_{W_2^1(\Omega_2)}\big),$$

where $\nu=\nu_1+\nu_2$ , and

$$\nu_1=a_1\frac{\partial^2 u}{\partial x_1\partial x_2},\quad \nu_2=\frac{\partial a_1}{\partial x_2}\frac{\partial u}{\partial x_1}.$$

After summation, we have

$$|\hat{\eta}_1|_{L_2(\omega_\tau,W_2^{1/2}(\sigma_h))}\leqslant Ch^2\big(\|\nu\|_{W_2^{1,0}(Q_1)}+\|\nu\|_{W_2^{1,0}(Q_2)}\big). \tag{4.11}$$

Using the Hölder inequality and the imbeddings $(*)$, we get

$$\begin{aligned}\|\nu_1\|_{W_2^{1,0}(Q_k)}&\leqslant\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_k)}\|u\|_{W_2^{3,3/2}(Q_k)},\\ \|\nu_2\|_{W_2^{1,0}(Q_k)}&\leqslant\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_k)}\|u\|_{W_2^{3,3/2}(Q_k)},\end{aligned}\qquad k=1,2. \tag{4.12}$$

From (4.11)–(4.12) we obtain

$$|\hat{\eta}_1|_{L_2(\omega_\tau, W_2^{1/2}(\sigma_h))} \leqslant Ch^2 \big( \|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)} \|u\|_{W_2^{3,3/2}(Q_1)} + \|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)} \|u\|_{W_2^{3,3/2}(Q_2)} \big). \tag{4.13}$$

The estimates of terms $\tilde{\chi}$, $\mu$ and $\hat{\chi}$ are obtained in [**11**]:

$$\|\tilde{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} \leqslant Ch^2 \sqrt{1/h} \big( \|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)} \big), \tag{4.14}$$

$$\|\mu\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \leqslant Ch^2 \sqrt{1/h} \, \|u\|_{W_2^{3,3/2}(\Sigma \times (0,T))}, \tag{4.15}$$

$$\|\widehat{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \leqslant Ch^2 \sqrt{1/h} \, \|u\|_{W_2^{2,1}(\Sigma \times (0,T))}. \tag{4.16}$$

Finally, from (4.2)–(4.16) we obtain the following result.

THEOREM 4.1. *The solution of problem* (3.4) *converges in* $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$ *to the solution of differential problem* (3.1), *provided* $c_1 h^2 \leqslant \tau \leqslant c_2 h^2$. *Furthermore,*

$$\|u-v\|_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})} \leqslant Ch^2 \big( \max_i \|a_i\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)} + \max_i \|a_i\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)} + l(h) \big) \times \big( \|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)} + \|u\|_{W_2^{3,3/2}(\Sigma\times(0,T))} \big),$$

*where* $l(h) = \sqrt{\log 1/h}$.

REMARK 4.1. The previous estimate is "almost" compatible with the smoothness of the coefficients and solution of differential problem (3.1). The compatibility is spoiled only by the term $l(h)$, which slowly increases when $h \to 0$.

REMARK 4.2. Convergence in $\widetilde{W}_2^{2,1}(Q_{h\tau})$ is proved in [**4**].

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