PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 77–98 DOI: 10.2298/PIM1613077G

ON GENERALIZED NULL MANNHEIM CURVES IN MINKOWSKI SPACE-TIME

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Abstract. We define generalized null Mannheim curves in Minkowski space-time and characterize them and their generalized Mannheim mate curves in terms of curvature functions, and obtain relations between their frames. We provide examples of such curves.

1. Introduction

In the Euclidean space $\mathbb{E}^3$ there are many associated curves (Bertrand mates, Mannheim mates, spherical images, evolutes, the principal-direction curves, etc.) the frame's vector fields of which satisfy some extra conditions. In particular, Mannheim curves in $\mathbb{E}^3$ are defined by the property that their principal normal lines coincide with the binormal lines of their mate curves at the corresponding points [**4, 7, 11**]. The parameter equation of a Mannheim curve α in $\mathbb{E}^3$ is given in [**4**] by $\alpha(t) = (\int h(t)\sin(t)dt, \int h(t)\cos(t)dt, \int h(t)g(t)dt)$, where $g\colon I \to \mathbb{R}$ is any smooth function and the function $h\colon I \to R$ is given by

$$h = \frac{(1+g^2+g'^2)^3 + (1+g^2)^3(g+g'')^2}{(1+g^2)^{3/2}(1+g^2+g'^2)^{5/2}}.$$

Mannheim curves and their partner curves in 3-dimensional space forms are studied in [**2**]. In the Euclidean 4-space, the notion of Mannheim curves is generalized in [**12**] as follows. A special Frenet curve $\alpha\colon I \to \mathbb{E}^4$ is called a *generalized Mannheim curve*, if there exists a special Frenet curve $\alpha^*\colon I^\star \to \mathbb{E}^4$ and a bijection $\phi : \alpha \to \alpha^\star$ such that the principal normal line of α at each point of α lies in the plane spanned by the first binormal and the second binormal line of α^*. The generalized spacelike Mannheim curves in Minkowski space-time, the Frenet frame of which contains only non-null vectors, are characterized in [**5**].

In this paper, we define the generalized null Mannheim curves in Minkowski space-time. We obtain the relations between the curvature functions and the frames

2010 *Mathematics Subject Classification:* Primary 53C50; Secondary 53C40.
Key words and phrases: generalized Mannheim curve, null Cartan curve, Minkowski space-time.
Communicated by Stevan Pilipović.

of the generalized null Mannheim curves and the generalized Mannheim mate curves of theirs. We give the necessary conditions for the null Cartan curve α with the second curvature $\kappa_2 \neq 0$ and its mate curve $\alpha^\star = \alpha + (1/2\kappa_2)N$ to be the generalized null Mannheim curve and the generalized Mannheim mate curve, respectively. In particular, we prove that there are no generalized null Mannheim curves the generalized Mannheim mate curve of which is a partially null or a pseudo null curve. Finally, we characterize the generalized null Mannheim curves in terms of the normal curves and give some examples.

2. Preliminaries

The Minkowski space-time $\mathbb{E}^4_1$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}^4_1$. Recall that a vector $\mathbf{v} \in \mathbb{E}^4_1 \backslash \{0\}$ can be *spacelike* if $g(\mathbf{v}, \mathbf{v}) > 0$, *timelike* if $g(\mathbf{v}, \mathbf{v}) < 0$ and *null (lightlike)* if $g(\mathbf{v}, \mathbf{v}) = 0$. In particular, the vector $\mathbf{v} = 0$ is said to be a spacelike. The *norm* of a vector $\mathbf{v}$ is given by $\|\mathbf{v}\| = \sqrt{|g(\mathbf{v}, \mathbf{v})|}$. Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be *orthogonal*, if $g(\mathbf{v}, \mathbf{w}) = 0$. An arbitrary curve α in $\mathbb{E}^4_1$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [**13**]. A non-null curve α is parametrized by the *arc-length function* s (or has the unit speed), if $g(\alpha'(s), \alpha'(s)) = \pm 1$. A *pseudo-arc length function* (or the canonical parameter) is defined in [**1**] by $s(t) = \int_0^t g(\alpha''(u), \alpha''(u))^{\frac{1}{4}}\, du$. A null curve α is said to be parameterized by the pseudo-arc length function s, if $g(\alpha''(s), \alpha''(s)) = 1$ [**1, 3**].

Definition 2.1. A non-geodesic null curve $\alpha : I \to \mathbb{E}^4_1$ parameterized by the pseudo-arc length function s is called a *Cartan curve*, if there exists a unique positively oriented Cartan frame $\{T, N, B_1, B_2\}$ along α and three smooth functions κ_1, κ_2 and κ_3 satisfying the Cartan equations [**3**].

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2.1}$$

The curvature functions $\kappa_1(s) = 1$, $\kappa_2(s)$ and $\kappa_3(s)$ are respectively called the first, the second and the third *Cartan curvature* of α. Cartan's frame vector fields T, N, B_1 and B_2 are respectively called the tangent, the principal normal, the first binormal and the second binormal vector field, and they satisfy the conditions

$$g(T,T) = g(B_1,B_1) = 0, \quad g(N,N) = g(B_2,B_2) = 1,$$
$$g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0, \quad g(T,B_1) = 1.$$

Definition 2.2. A spacelike or a timelike unit speed smooth curve $\alpha\colon I \to \mathbb{E}^4_1$ is called a *Frenet curve*, if there exists a unique positively oriented orthonormal or pseudo-orthonormal Frenet frame $\{T, N, B_1, B_2\}$ along α and three smooth functions $\kappa_1 \neq 0$, κ_2 and κ_3 satisfying the corresponding Frenet equations.

The smooth functions $\kappa_1 \neq 0$, κ_2 and κ_3 are respectively called the first, the second and the third *Frenet curvatures* of α. Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed Frenet curve $\alpha\colon I \to \mathbb{E}^4_1$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field, respectively. Depending on the causal character of Frenet's vector fields, we have three types of the Frenet equations.

Type 1. If α is a timelike or a spacelike Frenet curve whose orthonormal Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields, the Frenet equations are given by [**10**]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where $g(T,T) = \epsilon_1$, $g(N,N) = \epsilon_2$, $g(B_1,B_1) = \epsilon_3$, $g(B_2,B_2) = \epsilon_4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$, $\epsilon_i \in \{-1,1\}$, $i \in \{1,2,3,4\}$.

Type 2. If α is a pseudo null Frenet curve, i.e. a spacelike Frenet curve with a pseudo-orthonormal frame $\{T, N, B_1, B_2\}$ and null vector fields N and B_2, the Frenet formulae read [**14**]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.3}$$

where the first curvature $k_1(s) = 1$. Moreover, the following conditions are satisfied:

$$g(T,T) = g(B_1,B_1) = 1, \quad g(N,N) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(B_1,B_2) = 0, \quad g(N,B_2) = 1.$$

Type 3. If α is a partially null Frenet curve, i.e. a spacelike Frenet curve with a pseudo-orthonormal frame $\{T, N, B_1, B_2\}$ and null vector fields B_1 and B_2, the Frenet formulae read [**14**]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.4}$$

where the third curvature $\kappa_3(s) = 0$. In particular, there holds

$$g(T,T) = g(N,N) = 1, \quad g(B_1,B_1) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, \quad g(B_1,B_2) = 1.$$

Recall that a *normal curve* in $\mathbb{E}^4_1$ is defined in [**8**] as a curve the position vector of which always lies in its normal space $T^\perp$, which represents the orthogonal complement of the tangent vector field T of the curve.

3. The generalized null Mannheim curves in $\mathbb{E}_1^4$

It is proved in [**6**] that there are no null Mannheim curves in $\mathbb{E}_1^3$. In relation to that, throughout this section we will assume that the null Cartan curve α in $\mathbb{E}_1^4$ has the third curvature $\kappa_3(s) \neq 0$. Note that when $\kappa_3(s) \neq 0$, the second curvature $\kappa_2(s)$ of α can be equal to zero (when α is a null cubic), or different from zero. We firstly define the generalized null Mannheim curves as follows.

Definition 3.1. A null Cartan curve $\alpha\colon I \to \mathbb{E}_1^4$ is called a *generalized null Mannheim curve* if there exists a null Cartan curve or a Frenet curve $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ and a bijection $\phi\colon \alpha \to \alpha^\star$ such that at the corresponding points of the curves, the principal normal line of α is included in the plane spanned by the first binormal line and the second binormal line of $\alpha^\star$, under bijection ϕ.

The curve α^* is called the *generalized Mannheim mate curve* of α.

Remark 3.1. The generalized Mannheim mate curve of the generalized spacelike Mannheim curve in $\mathbb{E}_1^4$, the Frenet frame of which contains only the non-null vector fields, is defined in [**5**] as the spacelike Frenet curve. Unfortunately, such definition is not correct, since it can have arbitrarily causal character.

By the principal normal (binormal) line, we mean the straight line in direction of the principal normal (binormal) vector field. Throughout this section, let $\{T, N, B_1, B_2\}$ and $\{T^\star, N^\star, B_1^\star, B_2^\star\}$ denote the Cartan frame of the generalized null Mannheim curve α and the corresponding (Cartan or Frenet) frame of the generalized Mannheim mate curve $\alpha^\star$ of α, respectively. Since the principal normal vector $N(s)$ of α lies in the plane spanned by $\{B_1^\star, B_2^\star\}$, it satisfies the equation $N(s) = a(s)B_1^\star(s) + b(s)B_2^\star(s)$, for some differentiable functions $a(s)$ and $b(s)$. Depending on the causal character of the plane $\operatorname{span}\{B_1^\star, B_2^\star\}$, we distinguish three cases: (A) $\operatorname{span}\{B_1^\star, B_2^\star\}$ is a spacelike plane; (B) $\operatorname{span}\{B_1^\star, B_2^\star\}$ is a timelike plane; (C) $\operatorname{span}\{B_1^\star, B_2^\star\}$ is a lightlike plane. In what follows, we consider these three cases separately.

(A) $\operatorname{span}\{B_1^\star, B_2^\star\}$ is a spacelike plane.

Theorem 3.1. *Let $\alpha\colon I \to \mathbb{E}_1^4$ be the generalized null Mannheim curve and $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ the generalized Mannheim mate curve of α such that the principal normal line of α lies in the spacelike plane spanned by $\{B_1^\star, B_2^\star\}$. Then $\alpha^\star$ is the timelike Frenet curve such that the curvatures of α and $\alpha^\star$ satisfy the relations*

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{2\lambda}, \quad |\kappa_1^\star| = |\kappa_2^\star| = |\kappa_3| \neq 0, \quad |\kappa_3^\star| = \frac{1}{\lambda}, \quad \lambda \in \mathbb{R}_0^+, \tag{3.1}$$

and the corresponding frames of α and $\alpha^\star$ are related by

$$\begin{aligned} T^\star &= \frac{1}{2\sqrt{\lambda}}(T - 2\lambda B_1), \\ N^\star &= -\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_3)B_2, \\ B_1^\star &= \operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)\frac{1}{2\sqrt{\lambda}}(T + 2\lambda B_1), \\ B_2^\star &= \operatorname{sgn}(\kappa_2^\star)\operatorname{sgn}(\kappa_3)N. \end{aligned} \tag{3.2}$$

Proof. Since the principal normal line of α lies in the spacelike plane spanned by $\{B_1^\star, B_2^\star\}$, $\alpha^\star$ is a spacelike or a timelike curve the Frenet frame of which satisfies the equations (2.2), where $\epsilon_1^\star = -\epsilon_2^\star$, $\epsilon_3^\star = \epsilon_4^\star = 1$. In particular, the curve $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.3}$$

where s is the pseudo-arc length parameter of α, $s^\star = f(s) = \int_0^s \|\alpha^{\star\prime}(t)\|dt$ is the arc-length parameter of $\alpha^\star$ and $f\colon I \subset \mathbb{R} \to I^\star \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two cases: (A.1) $\kappa_2 = 0$ and (A.2) $\kappa_2 \neq 0$.

$(A.1)$ $\kappa_2 = 0$.

Differentiating relation (3.3) with respect to s and applying (2.1), we find

$$T^\star f' = T + \lambda' N - \lambda B_1. \tag{3.4}$$

By taking the scalar product of (3.4) with $N = aB_1^\star + bB_2^\star$, we find $\lambda' = 0$. Substituting this in (3.4), we get

$$T^\star f' = T - \lambda B_1, \quad \lambda \in R_0, \tag{3.5}$$

where $\mathbb{R}_0$ denotes $\mathbb{R}\backslash\{0\}$. From (3.5) we have $g(T^\star f', T^\star f') = \epsilon_1^\star f'^2 = -2\lambda$ and therefore

$$f'^2 = -2\epsilon_1^\star \lambda = \text{constant} \neq 0. \tag{3.6}$$

Differentiating relation (3.5) with respect to s and using (2.1), (2.2) and (3.6) we find $\epsilon_2^\star \kappa_1^\star N^\star f'^2 = N - \lambda\kappa_3 B_2$. By taking the scalar product of the last equation with $N = aB_1^\star + bB_2^\star$, we obtain a contradiction.

$(A.2)$ $\kappa_2 \neq 0$.

Differentiating relation (3.4) with respect to s and using (2.1), we find

$$T^\star f' = (1 - \lambda\kappa_2)T + \lambda' N - \lambda B_1. \tag{3.7}$$

By taking the scalar product of (3.7) with $N = aB_1^\star + bB_2^\star$ yields

$$\lambda' = 0. \tag{3.8}$$

Substituting (3.8) in (3.7), we get

$$T^\star f' = (1 - \lambda\kappa_2)T - \lambda B_1. \tag{3.9}$$

Differentiating relation (3.9) with respect to s and using (2.1) and (2.2), we obtain

$$\epsilon_2^\star \kappa_1^\star N^\star f'^2 + T^\star f'' = (1 - \lambda\kappa_2)' T + (1 - 2\lambda\kappa_2)N - \lambda\kappa_3 B_2. \tag{3.10}$$

By taking the scalar product of (3.10) with $N = aB_1^\star + bB_2^\star$ and using (3.8), it follows that

$$\kappa_2 = \frac{1}{2\lambda} = \text{constant}, \quad \lambda \in \mathbb{R}_0. \tag{3.11}$$

Moreover, by using (3.9) we obtain

$$g(T^\star f', T^\star f') = \epsilon_1^\star f'^2 = -2\lambda(1 - \lambda\kappa_2). \tag{3.12}$$

Substituting (3.11) in (3.12) yields

$$f'^2 = -\epsilon_1^\star \lambda = \text{constant} \neq 0. \tag{3.13}$$

Relations (3.10), (3.11) and (3.13) imply

$$\epsilon_2^\star \kappa_1^\star N^\star {f'}^2 = -\lambda \kappa_3 B_2. \tag{3.14}$$

Since a spacelike vector B_2 is collinear with $N^\star$, it follows that $T^\star$ is a timelike vector. Consequently, $\alpha^\star$ is a timelike curve. Substituting $\epsilon_1^\star = -1 = -\epsilon_2^\star$ in (3.13) we get

$$f'^2 = \lambda, \quad \lambda \in R_0^+, \tag{3.15}$$

where $\mathbb{R}_0^+$ denotes $\mathbb{R}^+ \backslash \{0\}$. Therefore,

$$f'(s) = \|\alpha^{\star\prime}(s)\| = \sqrt{\lambda}. \tag{3.16}$$

Substituting (3.11) and (3.16) in (3.9), we obtain

$$T^\star = \frac{1}{2\sqrt{\lambda}}(T - 2\lambda B_1). \tag{3.17}$$

Next, substituting $\epsilon_2^\star = 1$ and (3.15) in (3.14), we get $\kappa_1^\star N^\star = -\kappa_3 B_2$. It follows that

$$\kappa_1^\star = -\kappa_3, \quad N^\star = B_2, \tag{3.18}$$

or

$$\kappa_1^\star = \kappa_3, \quad N^\star = -B_2. \tag{3.19}$$

Assume that (3.18) holds. Differentiating the equation $N^\star = B_2$ with respect to s and applying (2.1) and (2.2), it follows that

$$(\kappa_1^\star T^\star + \kappa_2^\star B_1^\star) f' = -\kappa_3 T. \tag{3.20}$$

By taking the scalar product of (3.20) with $N = aB_1^\star + bB_2^\star$, we find $a\kappa_2^\star f' = 0$. If $\kappa_2^\star = 0$, relation (3.20) implies that a timelike vector $T^\star$ is collinear with a null vector T, which is a contradiction. Consequently, $a = 0$ and hence $N = bB_2^\star$. The condition $g(N, N) = 1$ gives $b^2 = 1$ and thus

$$N = B_2^\star, \tag{3.21}$$

or

$$N = -B_2^\star. \tag{3.22}$$

Assume that (3.21) holds. Differentiating relation (3.21) with respect to s and using (2.1) and (2.2), we get $-\kappa_2 T - B_1 = -\kappa_3^\star f' B_1^\star$. The last relation together with (3.11) and (3.15) yield

$$B_1^\star = \frac{1}{\lambda\sqrt{\lambda}\kappa_3^\star}\Big(\frac{1}{2}T + \lambda B_1\Big). \tag{3.23}$$

On the other hand, by using relations (3.16), (3.17), (3.18) and (3.20), we find

$$B_1^\star = \frac{\kappa_1^\star}{\sqrt{\lambda}\kappa_2^\star}\Big(\frac{1}{2}T + \lambda B_1\Big). \tag{3.24}$$

Since $\lambda > 0$, from (3.23) and (3.24), we get $\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star) = \operatorname{sgn}(\kappa_3^\star)$. By using (3.17), (3.18), (3.21), (3.24) and the condition $\det(T^\star, N^\star, B_1^\star, B_2^\star) = 1$, we find

$$\kappa_1^\star = -\kappa_2^\star. \tag{3.25}$$

Therefore, $\operatorname{sgn}(\kappa_3^\star) = -1$. From (3.23) and the condition $g(B_1^\star, B_1^\star) = 1$, we get ${\kappa_3^\star}^2 = \frac{1}{\lambda^2}$. Consequently,

$$\kappa_3^\star = -\frac{1}{\lambda}, \quad \lambda \in \mathbb{R}_0^+. \tag{3.26}$$

Substituting (3.25) in (3.24), we find

$$B_1^\star = -\frac{1}{2\sqrt{\lambda}}(T + 2\lambda B_1). \tag{3.27}$$

Now, relations (3.11), (3.18), (3.25) and (3.26) imply that (3.1) holds, where $\kappa_1^\star = -\kappa_2^\star = -\kappa_3$. By using the last relation and the relations (3.17), (3.18), (3.21) and (3.27), it follows that (3.2) is satisfied. Similarly, when (3.19) and (3.21), or (3.18) and (3.22), or (3.19) and (3.22) hold, we also obtain that (3.1) and (3.2) are satisfied. This completes the proof of the theorem. □

In the following theorem we give the necessary conditions for the null Cartan curve α and its associated curve $\alpha^\star = \alpha + (1/2\kappa_2)N$ to be the generalized null Mannheim curve and the generalized Mannheim mate curve, respectively.

Theorem 3.2. *Let $\alpha\colon I \to \mathbb{E}_1^4$ be a null Cartan curve with a non-zero constant second curvature $\kappa_2 \in \mathbb{R}_0^+$ and the Cartan frame $\{T, N, B_1, B_2\}$. If the curve $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ defined by $\alpha^\star = \alpha + (1/2\kappa_2)N$ is the Frenet curve, then α is the generalized null Mannheim curve and $\alpha^\star$ the generalized timelike Mannheim mate curve of α.*

Proof. Assume that the curve $\alpha^\star$ defined by

$$\alpha^\star(s) = \alpha(s) + \frac{1}{2\kappa_2}N(s), \quad \kappa_2 \in \mathbb{R}_0^+, \tag{3.28}$$

is the Frenet curve, where s is the pseudo-arc length parameter of α and $s^\star = f(s) = \int_0^s \|\alpha^{\star\prime}(t)\|dt$ is the arc-length parameter of $\alpha^\star$. Putting $\lambda = 1/\kappa_2$, $\kappa_2 \in \mathbb{R}_0^+$, it can be easily checked that $g(\alpha^{\star\prime}(s), \alpha^{\star\prime}(s)) = -\lambda$. This means that $\alpha^\star$ is a timelike curve. Consequently, $f(s) = \sqrt{\lambda}s$. Differentiating relation (3.28) with respect to s, and using (2.1) and $f' = \sqrt{\lambda}$, we obtain

$$T^\star = \frac{1}{2\sqrt{\lambda}}(T - 2\lambda B_1). \tag{3.29}$$

Differentiating the last relation with respect to s and using (2.1) and (2.2), we find $\kappa_1^\star N^\star = -\kappa_3 B_2$. It follows that

$$\kappa_1^\star = -\kappa_3, \quad N^\star = B_2, \tag{3.30}$$

or

$$\kappa_1^\star = \kappa_3, \quad N^\star = -B_2. \tag{3.31}$$

Assume that relation (3.30) holds. Differentiating the relation $N^\star = B_2$ with respect to s and using (2.1) and (2.2), we get

$$(\kappa_1^\star T^\star + \kappa_2^\star B_1^\star)f' = -\kappa_3 T. \tag{3.32}$$

The last relation implies $(-\kappa_1^{\star 2} + \kappa_2^{\star 2})f'^2 = 0$. It follows that

$$|\kappa_1^\star| = |\kappa_2^\star|. \tag{3.33}$$

Substituting (3.29) in (3.32) and using (3.30), (3.33) and $f' = \sqrt{\lambda}$, we get

$$B_1^\star = \frac{\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)}{2\sqrt{\lambda}}(T + 2\lambda B_1). \tag{3.34}$$

Differentiating the last relation with respect to s and using (2.1), (2.2), (3.30) and $\lambda = 1/2\kappa_2$, we get

$$B_2^\star = \frac{\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)}{\lambda\kappa_3^\star}N. \tag{3.35}$$

Relations (3.29), (3.30), (3.34), (3.35) and the condition $\det(T^\star, N^\star, B_1^\star, B_2^\star) = 1$ imply $\kappa_3^\star = -1/\lambda$, $\lambda \in \mathbb{R}_0^+$. Substituting the last relation in (3.35), it follows that

$$B_2^\star = -\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)N. \tag{3.36}$$

By using (3.29), (3.30), (3.34) and (3.36), we obtain that the frames of α and $\alpha^\star$ are related by

$$\begin{aligned}
T^\star &= \frac{1}{2\sqrt{\lambda}}(T - 2\lambda B_1),\\
N^\star &= B_2,\\
B_1^\star &= \frac{\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)}{2\sqrt{\lambda}}(T + 2\lambda B_1),\\
B_2^\star &= -\operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)N.
\end{aligned}$$

Since the principal normal N of α lies in the spacelike plane spanned by $\{B_1^\star, B_2^\star\}$, it follows that α is the generalized null Mannheim curve and $\alpha^\star$ the generalized timelike Mannheim mate curve of α. Assuming that relation (3.31) holds, we also obtain that N lies in the spacelike plane spanned by $\{B_1^\star, B_2^\star\}$. This completes the proof of the theorem. □

(B) $\operatorname{span}\{B_1^\star, B_2^\star\}$ is a timelike plane.

In this case, we obtain two theorems depending on the causal character of the basis vectors $B_1^\star$ and $B_2^\star$ of a timelike plane $\operatorname{span}\{B_1^\star, B_2^\star\}$. It is known that a timelike plane can be spanned by the spacelike and the timelike mutually orthogonal unit vectors or else by two linearly independent null vectors. Theorems 3.3 and 3.4 can be proved in a similar way as Theorems 3.1 and 3.2 respectively, so we omit their proofs.

Theorem 3.3. *Let $\alpha\colon I \to \mathbb{E}_1^4$ be the generalized null Mannheim curve and $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ the generalized Mannheim mate curve of α such that the principal normal line of α lies in the timelike plane spanned by non-null vectors $\{B_1^\star, B_2^\star\}$. Then $\alpha^\star$ is the spacelike Frenet curve such that the curvatures of α and $\alpha^\star$ satisfy* *the relations* $\kappa_1 = 1$, $\kappa_2 = 1/2\lambda$, $|\kappa_1^\star| = |\kappa_2^\star| = |\kappa_3|$, $|\kappa_3^\star| = -\frac{1}{\lambda}$, $\lambda \in \mathbb{R}_0^-$, *and the corresponding frames of* α *and* $\alpha^\star$ *are related by*

$$\begin{aligned}
T^\star &= \frac{1}{2\sqrt{|\lambda|}}(T - 2\lambda B_1),\\
N^\star &= \operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_3)B_2,\\
B_1^\star &= \operatorname{sgn}(\kappa_1^\star)\operatorname{sgn}(\kappa_2^\star)\frac{1}{2\sqrt{|\lambda|}}(T + 2\lambda B_1),\\
B_2^\star &= \operatorname{sgn}(\kappa_2^\star)\operatorname{sgn}(\kappa_3)N.
\end{aligned}$$

Theorem 3.4. *Let* $\alpha\colon I \to \mathbb{E}_1^4$ *be a null Cartan curve with a non-zero constant second curvature* $\kappa_2 \in \mathbb{R}_0^-$ *and the Cartan frame* $\{T, N, B_1, B_2\}$. *If the curve* $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ *defined by* $\alpha^\star = \alpha + (1/2\kappa_2)N$ *is the Frenet curve, then* α *is the generalized null Mannheim curve and* $\alpha^\star$ *the generalized spacelike Mannheim mate curve of* α.

In the case when a timelike plane $\operatorname{span}\{B_1^\star, B_2^\star\}$ is spanned by two linearly independent null vectors $B_1^\star$ and $B_2^\star$, we obtain the following theorem.

Theorem 3.5. *There are no generalized null Mannheim curves in* $\mathbb{E}_1^4$ *the generalized Mannheim mate curve of which is a partially null Frenet curve.*

Proof. Assume that there exists the generalized null Mannheim curve $\alpha\colon I \to \mathbb{E}_1^4$ the generalized Mannheim mate curve $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ of which is a partially null Frenet curve. Then the principal normal line of α lies in the timelike plane spanned by null vector fields $B_1^\star$ and $B_2^\star$. In particular, the curve $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.37}$$

where s is the pseudo-arc length parameter of α, $s^\star = f(s) = \int_0^s \|\alpha^{\star\prime}(t)\|dt$ the arc-length parameter of $\alpha^\star$ and $f\colon I \subset \mathbb{R} \to I^\star \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two cases: (B.1) $\kappa_2 = 0$ and (B.2) $\kappa_2 \neq 0$.

(B.1) $\kappa_2 = 0$.

Differentiating relation (3.37) with respect to s and applying (2.1), we obtain

$$T^\star f' = T + \lambda' N - \lambda B_1. \tag{3.38}$$

By taking the scalar product of (3.38) with $N = aB_1^\star + bB_2^\star$, we find $\lambda' = 0$. Substituting this in (3.38) it follows that

$$T^\star f' = T - \lambda B_1, \quad \lambda \in R_0. \tag{3.39}$$

From (3.39) we have

$$g(T^\star f', T^\star f') = f'^2 = -2\lambda = \text{constant} \neq 0. \tag{3.40}$$

Differentiating relation (3.39) with respect to s and using (2.1), (2.4) and (3.40), it follows that

$$\kappa_1^\star N^\star f'^2 = N - \lambda\kappa_3 B_2. \tag{3.41}$$

By taking the scalar product of (3.41) with $N = aB_1^\star + bB_2^\star$, we obtain a contradiction.

(B.2) $\kappa_2 \neq 0$.

Differentiating relation (3.37) with respect to s and using (2.1), we find

$$T^\star f' = (1-\lambda\kappa_2)T + \lambda' N - \lambda B_1. \tag{3.42}$$

By taking the scalar product of (3.42) with N yields

$$\lambda' = 0. \tag{3.43}$$

Substituting (3.43) in (3.42) we get

$$T^\star f' = (1-\lambda\kappa_2)T - \lambda B_1, \quad \lambda \in \mathbb{R}_0. \tag{3.44}$$

Differentiating relation (3.44) with respect to s and using (2.1) and (2.4), we obtain

$$\kappa_1^\star N^\star f'^2 + T^\star f'' = (1-\lambda\kappa_2)'T + (1-2\lambda\kappa_2)N - \lambda\kappa_3 B_2. \tag{3.45}$$

By taking the scalar product of (3.45) with $N = aB_1^\star + bB_2^\star$, it follows that

$$\kappa_2 = \frac{1}{2\lambda}, \quad \lambda \in \mathbb{R}_0. \tag{3.46}$$

By using (3.44) we obtain

$$g(T^\star f', T^\star f') = f'^2 = -2\lambda(1-\lambda\kappa_2). \tag{3.47}$$

Substituting (3.46) in (3.47) it follows that

$$f'^2 = -\lambda = \text{constant}, \quad \lambda \in \mathbb{R}_0^-. \tag{3.48}$$

From (3.45), (3.46) and (3.48) we obtain

$$\kappa_1^\star N^\star = \kappa_3 B_2. \tag{3.49}$$

Relation (3.49) implies $\kappa_1^\star = \kappa_3$, $N^\star = B_2$ or $\kappa_1^\star = -\kappa_3$, $N^\star = -B_2$. Differentiating the relation $N^\star = \pm B_2$ with respect to s and applying (2.1) and (2.4), it follows that

$$(-\kappa_1^\star T^\star + \kappa_2^\star B_1^\star)f' = \mp\kappa_3 T. \tag{3.50}$$

By taking the scalar product of (3.50) with $N = aB_1^\star + bB_2^\star$ we find $b\kappa_2^\star f' = 0$. If $\kappa_2^\star = 0$, relation (3.50) implies that a spacelike vector $T^\star$ is collinear with a null vector T, which is a contradiction. It follows that $b = 0$, so $N = aB_1^\star$. This means that a spacelike vector N is collinear with a null vector $B_1^\star$, which is impossible. □

(C) span$\{B_1^\star, B_2^\star\}$ is a lightlike plane.

In this case, we obtain two theorems depending on the causal character of the basis vectors of a lightlike plane span$\{B_1^\star, B_2^\star\}$, which can be spanned by a null vector $B_1^\star$ and a spacelike vector $B_2^\star$, or else by a spacelike vector $B_1^\star$ and a null vector $B_2^\star$.

THEOREM 3.6. *Let $\alpha\colon I \to \mathbb{E}_1^4$ be the generalized null Mannheim curve and $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ the generalized Mannheim mate curve of α, such that the principal normal line of α lies in the lightlike plane spanned by a null vector $B_1^\star$ and a spacelike vector $B_2^\star$. Then $\alpha^\star$ is a null Cartan curve such that one of the two statements hold:*

(i) *the curvatures of* α *and* $\alpha^\star$ *satisfy the relations*

$$\kappa_2 = \frac{1-\sinh^2\big(\frac{\sqrt{2}}{2}s\big)}{\cosh^2\big(\frac{\sqrt{2}}{2}s\big)},\ \kappa_2^\star = 0,\ |\kappa_3| = \cosh^6\Big(\frac{\sqrt{2}}{2}s\Big),\ |\kappa_3^\star| = \frac{1}{\cosh^2\big(\frac{\sqrt{2}}{2}s\big)}, \tag{3.51}$$

and the corresponding Cartan frames of α *and* $\alpha^\star$ *are related by*

$$\begin{aligned}
T^\star &= \frac{\sinh^2(\frac{\sqrt{2}}{2}s)}{\cosh^4(\frac{\sqrt{2}}{2}s)}T + \frac{\sqrt{2}\sinh(\frac{\sqrt{2}}{2}s)}{\cosh^3(\frac{\sqrt{2}}{2}s)}N - \frac{1}{\cosh^2(\frac{\sqrt{2}}{2}s)}B_1,\\
N^\star &= -\operatorname{sgn}(\kappa_3^\star)B_2,\\
B_1^\star &= -\cosh^2\Big(\frac{\sqrt{2}}{2}s\Big)T,\\
B_2^\star &= -\operatorname{sgn}(\kappa_3^\star)\bigg(\frac{\sqrt{2}\sinh\big(\frac{\sqrt{2}}{2}s\big)}{\cosh\big(\frac{\sqrt{2}}{2}s\big)}T + N\bigg);
\end{aligned} \tag{3.52}$$

(ii) *the curvatures of* α *and* $\alpha^\star$ *satisfy the relations*

$$\begin{aligned}
&\kappa_2 = \frac{2\lambda-\lambda'^2}{2\lambda^2} \neq 0, \quad \kappa_2^\star = \frac{X}{\lambda\lambda'^2 f'^2} \neq 0, \quad |\kappa_3| = \frac{\sqrt{\lambda^2 f'^4 - X^2}}{\lambda^2},\\
&|\kappa_3^\star| = \frac{\big(\frac{\lambda' X}{2\lambda^2 f'^2}\big)' + \frac{2(\lambda^2 f'^4 - X^2)-\lambda X}{2\lambda^3 f'^2} + \big(\big(\frac{X}{\lambda' f'^2}\big)' + \frac{X}{\lambda'^2 f'^2}\big)\big(\frac{2\lambda-\lambda'^2}{2\lambda^2}\big)}{f'^2}.
\end{aligned} \tag{3.53}$$

where $X(s) = \lambda\lambda'' - \lambda'^2 - \lambda$, $f' = e^{\int \frac{\lambda\lambda''+\lambda'^2-\lambda}{\lambda\lambda'}ds}$ *and* $\lambda(s) \neq \text{constant}$ *satisfy the differential equation*

$$\begin{aligned}
&\frac{X^2[\lambda^2 f'^4 + \lambda\lambda' X' - (3\lambda\lambda'' + 2\lambda'^2 - 3\lambda)X]^2}{\lambda^4\lambda'^2 f''^4(\lambda^2 f'^4 - X^2)}\\
&\qquad + 2\bigg[\Big(\frac{\lambda' X}{2\lambda^2 f'^2}\Big)' + \frac{2(\lambda^2 f'^4 - X^2) - \lambda X}{2\lambda^3 f'^2}\bigg]\bigg[\Big(\frac{X}{\lambda' f'^2}\Big)' + \frac{X}{\lambda'^2 f'^2}\bigg] = 0,
\end{aligned}$$

while the corresponding Cartan frames of α *and* $\alpha^\star$ *are related by*

$$\begin{aligned}
T^\star &= \frac{\lambda'^2}{2\lambda f'}T + \frac{\lambda'}{f'}N - \frac{\lambda}{f'}B_1,\\
N^\star &= -\operatorname{sgn}(\kappa_3^\star)\Big(\frac{\lambda'(\lambda\lambda''-\lambda'^2-\lambda)}{2\lambda^2 f'^2}T + \Big(\frac{\lambda\lambda''-\lambda'^2-\lambda}{\lambda' f'^2}\Big)B_1\\
&\qquad - \frac{\sqrt{\lambda^2 f'^4 - (\lambda\lambda''-\lambda'^2-\lambda)^2}}{\lambda f'^2}\Big)B_2,\\
B_1^\star &= xT + yB_1 + zB_2,\\
B_2^\star &= -\frac{\operatorname{sgn}(\kappa_3^\star)}{f'\kappa_3^\star}[(x' - z\kappa_3 - \kappa_2^\star m f')T + (x + y\kappa_2)N\\
&\qquad + (y' - \kappa_2^\star n f')B_1 + (y\kappa_3 + z' - \kappa_2^\star p f')B_2],
\end{aligned} \tag{3.54}$$

where x, y, z are given by

$$x = -\frac{1}{f'}\Big(\Big(\frac{\lambda' X}{2\lambda^2 f'^2}\Big)' + \frac{2(\lambda^2 f'^4 - X^2) - \lambda X}{2\lambda^3 f'^2}\Big),$$
$$y = -\frac{1}{f'}\Big(\Big(\frac{X}{\lambda' f'^2}\Big)' + \frac{X}{\lambda'^2 f'^2}\Big),$$
$$z = -\frac{1}{f'}\Big(\frac{X[\lambda^2 f'^4 + \lambda\lambda' X' - (3\lambda\lambda'' + 2\lambda'^2 - 3\lambda)X]}{\lambda^2\lambda' f'^2\sqrt{\lambda^2 f'^4 - X^2}}\Big).$$

PROOF. By assumption the principal normal line of α lies in the lightlike plane spanned by a null vector $B_1^\star$ and a spacelike vector $B_2^\star$. Therefore, $\alpha^\star$ is a null Cartan curve the Cartan frame of which satisfies the relation (2.1). The curve $\alpha^\star$ has parametrization of the form

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.55}$$

where s and $s^\star = f(s)$ are the pseudo-arc length parameters of α and $\alpha^\star$ respectively and $f\colon I \subset \mathbb{R} \to I^\star \subset \mathbb{R}$ and λ are some smooth functions. We distinguish two subcases: (C.1) $\kappa_2 = 0$ and (C.2) $\kappa_2 \neq 0$.

(C.1) $\kappa_2 = 0$.

Differentiating relation (3.55) with respect to s and using (2.1) we obtain

$$T^\star f' = T + \lambda' N - \lambda B_1. \tag{3.56}$$

Since $g(T^\star f', T^\star f') = \lambda'^2 - 2\lambda = 0$, it follows that

$$\lambda(s) = \frac{(s+c)^2}{2}, \quad c \in \mathbb{R}. \tag{3.57}$$

Differentiating relation (3.56) with respect to s and using (2.1) and (3.57), we obtain

$$N^\star f'^2 + T^\star f'' = 2N - 2(s+c)B_1 - \frac{1}{2}(s+c)^2\kappa_3 B_2. \tag{3.58}$$

By taking the scalar products of (3.56) and (3.58) with $N = aB_1^\star \pm B_2^\star$, we respectively find

$$af' = s + c, \quad c \in \mathbb{R}, \tag{3.59}$$
$$af'' = 2. \tag{3.60}$$

Relations (3.59) and (3.60) imply

$$a = \frac{1}{c_1(s+c)}, \quad f' = c_1(s+c)^2, \quad c_1 \in \mathbb{R}_0^+, \quad c \in \mathbb{R}. \tag{3.61}$$

Substituting (3.57) and (3.61) in (3.56), we obtain

$$T^\star = \frac{1}{c_1(s+c)^2}T + \frac{1}{c_1(s+c)}N - \frac{1}{2c_1}B_1. \tag{3.62}$$

Substituting (3.61) in (3.58) and using (3.62), we get

$$N^\star = -\frac{2}{c_1^2(s+c)^5}T - \frac{1}{c_1^2(s+c)^3}B_1 - \frac{\kappa_3}{2c_1^2(s+c)^2}B_2. \tag{3.63}$$

Now we distinguish two subcases: (C.1.1) $\kappa_2^\star = 0$ and (C.1.2) $\kappa_2^\star \neq 0$.

(C.1.1) $\kappa_2^\star = 0$;

Differentiating relation (3.58) with respect to s and using (2.1), we obtain

$$f'''T^\star + 3f'f''N^\star - f'^3 B_1^\star = \frac{\kappa_3^2}{2}(s+c)^2 T - 4B_1 - (s+c)\Big(3\kappa_3 + \frac{(s+c)}{2}\kappa_3'\Big)B_2.$$

By taking the scalar product of the last relation with $N = aB_1^\star \pm B_2^\star$, we get $af''' = 0$. This implies $a = 0$ or $f''' = 0$, which is a contradiction with (3.61).

(C.1.2) $\kappa_2^\star \neq 0$; Differentiating relation (3.63) with respect to s and using (2.1), we find

$$(3.64)\qquad (-\kappa_2^\star T^\star - B_1^\star)f' = \frac{20 + \kappa_3^2(s+c)^4}{2c_1^2(s+c)^6}T - \frac{2}{c_1^2(s+c)^5}N + \frac{3}{c_1^2(s+c)^4}B_1 - \frac{\kappa_3'}{2c_1^2(s+c)^2}B_2.$$

By taking the scalar product of the last relation with $N = aB_1^\star \pm B_2^\star$ and using (3.61), it follows that

$$(3.65)\qquad \kappa_2^\star = \frac{2}{c_1^2(s+c)^6}, \quad c_1 \in \mathbb{R}_0^+, \quad c \in \mathbb{R}.$$

Substituting (3.62) and (3.65) in (3.64), we get

$$-B_1^\star f' = \frac{24 + \kappa_3^2(s+c)^4}{2c_1^2(s+c)^6}T + \frac{2}{c_1^2(s+c)^4}B_1 - \frac{\kappa_3'}{2c_1^2(s+c)^2}B_2.$$

The last relation and the condition $g(B_1^\star, B_1^\star) = 0$ imply $[2(24 + \kappa_3^2(s+c)^4)/c_1^4(s+c)^{10}] + [\kappa_3'^2/4c_1^4(s+c)^4] = 0$, which is a contradiction.

(C.2) $\kappa_2 \neq 0$;

Differentiating relation (3.55) with respect to s and using (2.1), we obtain

$$(3.66)\qquad T^\star = \Big(\frac{1-\lambda\kappa_2}{f'}\Big)T + \frac{\lambda'}{f'}N - \frac{\lambda}{f'}B_1.$$

Since $g(T^\star, T^\star) = (\lambda'^2 - 2\lambda(1-\lambda\kappa_2))/f'^2 = 0$, it follows that

$$(3.67)\qquad \kappa_2 = \frac{2\lambda - \lambda'^2}{2\lambda^2}.$$

Next we prove that $\lambda \neq$ constant. If $\lambda =$ constant $\neq 0$, relation (3.67) gives $\kappa_2 = \frac{1}{\lambda}$. Substituting this in (3.66) we obtain $T^\star f' = -\lambda B_1$. By taking the scalar product of the last equation with $N = aB_1^\star \pm B_2^\star$, we find $af' = 0$. Therefore, $a = 0$ and thus $N = \pm B_2^\star$. Differentiating the last relation with respect to s and using (2.1), we get $-\kappa_2 T - B_1 = \pm\kappa_3^\star f' T^\star$. The last relation implies $g(-\kappa_2 T - B_1, -\kappa_2 T - B_1) = 2\kappa_2 = 0$, which is a contradiction. Consequently, $\lambda \neq$ constant.

Putting

$$(3.68)\qquad u = \frac{1-\lambda\kappa_2}{f'} \quad v = \frac{\lambda'}{f'} \quad w = -\frac{\lambda}{f'},$$

relation (3.66) becomes

$$(3.69)\qquad T^\star = uT + vN + wB_1.$$

Differentiating the last relation with respect to s and using (2.1), we find

$$N^\star = \Big(\frac{u' - v\kappa_2}{f'}\Big)T + \Big(\frac{u + v' + w\kappa_2}{f'}\Big)N + \Big(\frac{w' - v}{f'}\Big)B_1 + \frac{w\kappa_3}{f'}B_2. \tag{3.70}$$

By taking the scalar product of (3.70) with $N = aB_1^\star \pm B_2^\star$, we get

$$u + v' + w\kappa_2 = 0. \tag{3.71}$$

Substituting (3.67) and (3.68) in (3.71), we obtain

$$\frac{f''}{f'} = \frac{\lambda\lambda'' + \lambda'^2 - \lambda}{\lambda\lambda'}. \tag{3.72}$$

Consequently, $f' = ce^{\int \frac{\lambda\lambda''+\lambda'^2-\lambda}{\lambda\lambda'}\,ds}$, $c \in \mathbb{R}_0^+$. Taking $c = 1$, we find

$$f' = e^{\int \frac{\lambda\lambda''+\lambda'^2-\lambda}{\lambda\lambda'}\,ds}. \tag{3.73}$$

Substituting (3.71) in (3.70) yields

$$N^\star = \Big(\frac{u' - v\kappa_2}{f'}\Big)T + \Big(\frac{w' - v}{f'}\Big)B_1 + \frac{w\kappa_3}{f'}B_2. \tag{3.74}$$

Next, the condition $g(N^\star, N^\star) = 1$ and the relation (3.74) imply $\kappa_3^2 = [f'^2 - 2(u' - v\kappa_2)(w' - v)]/w^2$. Substituting (3.67) and (3.68) in the last relation, we get

$$|\kappa_3| = \frac{\sqrt{\lambda^2 f'^4 - (\lambda\lambda'' - \lambda'^2 - \lambda)^2}}{\lambda^2}. \tag{3.75}$$

Putting

$$m = \frac{u' - v\kappa_2}{f'}, \quad n = \frac{w' - v}{f'}, \quad p = \frac{w\kappa_3}{f'}, \tag{3.76}$$

relation (3.74) becomes

$$N^\star = mT + nB_1 + pB_2. \tag{3.77}$$

Differentiating the last relation with respect to s and using (2.1), we get

$$(-\kappa_2^\star T^\star - B_1^\star)f' = (m' - p\kappa_3)T + (m + n\kappa_2)N + n'B_1 + (n\kappa_3 + p')B_2. \tag{3.78}$$

Now we distinguish two subcases: (C.2.1) $\kappa_2^\star = 0$ and (C.2.2) $\kappa_2^\star \neq 0$.

(C.2.1) $\kappa_2^\star = 0$.

Then relation (3.78) reads

$$B_1^\star = -\frac{1}{f'}[(m' - p\kappa_3)T + (m + n\kappa_2)N + n'B_1 + (n\kappa_3 + p')B_2]. \tag{3.79}$$

By taking the scalar product of (3.79) with $N = aB_1^\star \pm B_2^\star$, we get $m + n\kappa_2 = 0$. Substituting (3.76) in the last relation, we find $u' - 2v\kappa_2 + w'\kappa_2 = 0$. Next, substituting (3.67), (3.68) and (3.72) in the last equation, we obtain the second order differential equation in terms of λ, which reads $\lambda\lambda'' - \lambda'^2 - \lambda = 0$. The general solution of the last differential equation is given by $\lambda = (2/c_1)\cosh^2\big(\frac{\sqrt{c_1}}{2}(s + c_2)\big)$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Taking $c_1 = 2$ and $c_2 = 0$, we get

$$\lambda = \cosh^2(\frac{\sqrt{2}}{2}s). \tag{3.80}$$

Substituting (3.80) in (3.67) and (3.75), we respectively get

$$\kappa_2 = \frac{1-\sinh^2(\frac{\sqrt{2}}{2}s)}{\cosh^2(\frac{\sqrt{2}}{2}s)}, \quad |\kappa_3| = \cosh^6(\frac{\sqrt{2}}{2}s). \tag{3.81}$$

Next, substituting (3.80) in (3.73), we obtain

$$f' = \cosh^4(\frac{\sqrt{2}}{2}s). \tag{3.82}$$

From (3.80), (3.81), (3.82) and (3.66) we have

$$T^\star = \frac{\sinh^2\left(\frac{\sqrt{2}}{2}s\right)}{\cosh^4\left(\frac{\sqrt{2}}{2}s\right)}T + \frac{\sqrt{2}\sinh\left(\frac{\sqrt{2}}{2}s\right)}{\cosh^3\left(\frac{\sqrt{2}}{2}s\right)}N - \frac{1}{\cosh^2\left(\frac{\sqrt{2}}{2}s\right)}B_1. \tag{3.83}$$

Differentiating the previous equation with respect to s, using (2.1) and (3.82), we obtain $N^\star = -\kappa_3/\cosh^6\left(\frac{\sqrt{2}}{2}s\right)B_2$. It follows that

$$N^\star = B_2, \quad \kappa_3 = -\cosh^6\left(\frac{\sqrt{2}}{2}s\right), \tag{3.84}$$

or

$$N^\star = -B_2, \quad \kappa_3 = \cosh^6\left(\frac{\sqrt{2}}{2}s\right). \tag{3.85}$$

Assume that (3.84) holds. Differentiating the relation $N^\star = B_2$ with respect to s, using (2.1), (3.82) and (3.84), we find

$$B_1^\star = -\cosh^2\left(\frac{\sqrt{2}}{2}s\right)T. \tag{3.86}$$

Differentiating relation (3.86) with respect to s and using (2.1) and (3.82), we get

$$\kappa_3^\star B_2^\star = -\frac{\sqrt{2}\sinh\left(\frac{\sqrt{2}}{2}s\right)}{\cosh^3\left(\frac{\sqrt{2}}{2}s\right)}T - \frac{1}{\cosh^2\left(\frac{\sqrt{2}}{2}s\right)}N. \tag{3.87}$$

Relation (3.87) and the condition $g(B_2^\star, B_2^\star) = 1$ imply

$$|\kappa_3^\star| = \frac{1}{\cosh^2\left(\frac{\sqrt{2}}{2}s\right)}. \tag{3.88}$$

Next, by using (3.83), (3.84), (3.86), (3.87) and the condition $\det(T^\star, N^\star, B_1^\star, B_2^\star) = 1$, we obtain

$$\kappa_3^\star = -\frac{1}{\cosh^2\left(\frac{\sqrt{2}}{2}s\right)}. \tag{3.89}$$

Substituting (3.89) in (3.87) yields

$$B_2^\star = \frac{\sqrt{2}\sinh\left(\frac{\sqrt{2}}{2}s\right)}{\cosh\left(\frac{\sqrt{2}}{2}s\right)}T + N. \tag{3.90}$$

Finally, relations (3.81) and (3.88) imply that (3.51) holds. By using relations (3.83), (3.84), (3.86) and (3.90), we obtain that (3.52) holds. Assuming that (3.85) holds, in a similar way it can be proved that (3.51) and (3.52) are satisfied. This proves statement (i).

(C.2.2) $\kappa_2^\star \neq 0$;

Since relation (3.73) holds, it follows that the curvatures κ_2 and κ_3 in relations (3.67) and (3.75) are expressed only in terms of λ. Next, we show that another two curvatures $\kappa_2^\star$ and $\kappa_3^\star$ can also be expressed only in terms of λ, such that $\lambda \neq$ *constant* satisfies the corresponding differential equation. Substituting (3.69) in (3.78), we obtain

$$\begin{aligned} B_1^\star = &-\frac{1}{f'}\big[(m' - p\kappa_3 + uf'\kappa_2^\star)T + (m + n\kappa_2 + vf'\kappa_2^\star)N \\ &+ (n' + wf'\kappa_2^\star)B_1 + (n\kappa_3 + p')B_2\big]. \end{aligned} \tag{3.91}$$

By taking the scalar product of (3.91) with $N = aB_1^\star \pm B_2^\star$, we find

$$\kappa_2^\star = -\frac{m + n\kappa_2}{vf'}. \tag{3.92}$$

Substituting (3.67), (3.68), (3.73) and (3.76) in (3.92), it follows that

$$\kappa_2^\star = \frac{\lambda + \lambda'^2 - \lambda\lambda''}{\lambda\lambda'^2 e^{2\int \frac{\lambda\lambda'' + \lambda'^2 - \lambda}{\lambda\lambda'}\, ds}}. \tag{3.93}$$

Substituting (3.92) in (3.91), we find

$$B_1^\star = -\frac{1}{f'}\big[(m' - p\kappa_3 + uf'\kappa_2^\star)T + (n' + wf'\kappa_2^\star)B_1 + (n\kappa_3 + p')B_2\big]. \tag{3.94}$$

By using relation (3.94) and the condition $g(B_1^\star, B_1^\star) = 0$, we get

$$(n\kappa_3 + p')^2 + 2(m' - p\kappa_3 + uf'\kappa_2^\star)(n' + wf'\kappa_2^\star) = 0. \tag{3.95}$$

Assuming that $\operatorname{sgn}(\kappa_3) = 1$ and putting

$$X = \lambda\lambda'' - \lambda'^2 - \lambda, \tag{3.96}$$

relation (3.75) becomes $\kappa_3 = \sqrt{\lambda^2 f'^4 - X^2}/\lambda^2$. By using the last relation and relations (3.68), (3.75), (3.76), (3.93), a straightforward calculation yields

$$n\kappa_3 + p' = \frac{X[\lambda^2 f'^4 + \lambda\lambda' X' - (3\lambda\lambda'' + 2\lambda'^2 - 3\lambda)X]}{\lambda^2\lambda' f'^2\sqrt{\lambda^2 f'^4 - X^2}}, \tag{3.97}$$

$$m' - p\kappa_3 + uf'\kappa_2^\star = \Big(\frac{\lambda' X}{2\lambda^2 f'^2}\Big)' + \frac{2(\lambda^2 f'^4 - X^2) - \lambda X}{2\lambda^3 f'^2}, \tag{3.98}$$

$$n' + wf'\kappa_2^\star = \Big(\frac{X}{\lambda' f'^2}\Big)' + \frac{X}{\lambda'^2 f'^2}, \tag{3.99}$$

where f'' and X are given by (3.73) and (3.96), respectively. Substituting the relations (3.97), (3.98) and (3.99) in (3.95), we obtain the third order differential equation only in terms of λ, which reads

$$\frac{X^2[\lambda^2 f'^4 + \lambda\lambda' X' - (3\lambda\lambda'' + 2\lambda'^2 - 3\lambda)X]^2}{\lambda^4\lambda'^2 f'^4(\lambda^2 f'^4 - X^2)}$$

$$+2\Big[\Big(\frac{\lambda' X}{2\lambda^2 f'^2}\Big)' + \frac{2(\lambda^2 f'^4 - X^2) - \lambda X}{2\lambda^3 f'^2}\Big]\Big[\Big(\frac{X}{\lambda' f'^2}\Big)' + \frac{X}{\lambda'^2 f'^2}\Big] = 0.$$

Putting

$$\begin{aligned} x &= -\frac{m' - p\kappa_3 + uf'\kappa_2^\star}{f'}, \\ y &= -\frac{n' + wf'\kappa_2^\star}{f'}, \\ z &= -\frac{n\kappa_3 + p'}{f'}, \end{aligned} \tag{3.100}$$

relation (3.94) becomes

$$B_1^\star = xT + yB_1 + zB_2. \tag{3.101}$$

Differentiating relation (3.101) with respect to s and using (2.1), we obtain

$$f'(\kappa_2^\star N^\star + \kappa_3^\star B_2^\star) = (x' - z\kappa_3)T + (x + y\kappa_2)N + y'B_1 + (y\kappa_3 + z')B_2. \tag{3.102}$$

Next, substituting (3.77) in (3.102) we get

$$\begin{aligned} B_2^\star = \frac{1}{f'\kappa_3^\star}\big[&(x' - z\kappa_3 - \kappa_2^\star m f')T + (x + y\kappa_2)N \\ &+ (y' - \kappa_2^\star n f')B_1 + (y\kappa_3 + z' - \kappa_2^\star f' p)B_2\big], \end{aligned} \tag{3.103}$$

where x, y, z are given by (3.100). By taking the scalar product of (3.103) with $N = aB_1^\star \pm B_2^\star$, we find

$$|\kappa_3^\star| = \frac{x + y\kappa_2}{f'}.$$

By using (3.69), (3.77), (3.101), (3.103) and the condition $det(T^\star, N^\star, B_1^\star, B_2^\star) = 1$, we get

$$\kappa_3^\star = \frac{x + y\kappa_2}{f'}. \tag{3.104}$$

Substituting (3.67) and (3.100) in (3.104), we get

$$\kappa_3^\star = -\frac{\big(\frac{\lambda' X}{2\lambda^2 f'^2}\big)' + \frac{2(\lambda^2 f'^4 - X^2) - \lambda X}{2\lambda^3 f'^2} + \big(\big(\frac{X}{\lambda' f'^2}\big)' + \frac{X}{\lambda'^2 f'^2}\big)\big(\frac{2\lambda - \lambda'^2}{2\lambda^2}\big)}{f'^2}. \tag{3.105}$$

Finally, relations (3.67), (3.75), (3.93) and (3.105) imply that (3.53) holds. By using (3.67), (3.68), (3.69), (3.76), (3.77), (3.101) and (3.103), we obtain that the Cartan frames of α and $\alpha^\star$ are related by (3.54). Moreover, a straightforward calculation shows that $N = (\lambda'/f')B_1^\star + B_2^\star$. Assuming that $\text{sgn}(\kappa_3) = -1$, we also obtain that (3.53) and (3.54) hold. This proves statement (ii) of the theorem. □

Remark 3.2. Note that the statement (i) of Theorem 3.6 is the special case of statement (ii), when $X = 0$.

When a lightlike plane $\text{span}\{B_1^\star, B_2^\star\}$ is spanned by a spacelike vector $B_1^\star$ and a null vector $B_2^\star$, we get the following theorem.

THEOREM 3.7. *There are no generalized null Mannheim curves in $\mathbb{E}_1^4$ the generalized Mannheim mate curve of which is a pseudo null Frenet curve.*

PROOF. Assume that there exists the generalized null Mannheim curve α in E_1^4 the generalized Mannheim mate curve $\alpha^\star\colon I^\star \to \mathbb{E}_1^4$ of which is a pseudo null Frenet curve. The curve $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.106}$$

where s is the pseudo-arc length parameter of α, $s^\star = f(s) = \int_0^s \|\alpha^{\star\prime}(t)\|dt$ is the arc-length parameter of $\alpha^\star$ and $f\colon I \subset \mathbb{R} \to I^\star \subset \mathbb{R}$ and λ are some smooth functions on I. We distinguish two cases: (C*.1) $\kappa_2 = 0$ and (C*.2) $\kappa_2 \neq 0$.

(C*.1) $\kappa_2 = 0$.

Differentiating relation (3.106) with respect to s and applying (2.1), we obtain

$$T^\star f' = T + \lambda' N - \lambda B_1. \tag{3.107}$$

By taking the scalar product of (3.107) with $N = \pm B_1^\star + bB_2^\star$, we find $\lambda' = 0$. Substituting this in (3.107), we get

$$T^\star f' = T - \lambda B_1, \quad \lambda \in \mathbb{R}_0, \tag{3.108}$$

where $\mathbb{R}_0$ denotes $\mathbb{R}\backslash\{0\}$. Relation (3.108) implies

$$g(T^\star f', T^\star f') = f'^2 = -2\lambda = \text{constant}\,. \tag{3.109}$$

Differentiating relation (3.108) with respect to s and using (2.1), (2.3) and (3.109), it follows that

$$N^\star f'^2 = N - \lambda\kappa_3 B_2.$$

The last relation gives $g(N^\star f'^2, N^\star f'^2) = 1 + \lambda^2\kappa_3^2 = 0$, which is a contradiction.

(C*.2) $\kappa_2 \neq 0$.

Differentiating relation (3.106) with respect to s and using (2.1), we obtain $T^\star f' = (1-\lambda\kappa_2)T + \lambda' N - \lambda B_1$. By taking the scalar product of the last relation with $N = \pm B_1^\star + bB_2^\star$, we get $\lambda' = 0$. Consequently,

$$T^\star = \Big(\frac{1-\lambda\kappa_2}{f'}\Big)T - \frac{\lambda}{f'}B_1, \quad \lambda \in \mathbb{R}_0. \tag{3.110}$$

By using the condition $g(T^\star, T^\star) = 1$ and relation (3.110), we find

$$f'^2 = -2\lambda(1-\lambda\kappa_2). \tag{3.111}$$

Differentiating relation (3.110) with respect to s, using (2.1), (2.3) and (3.111), we obtain

$$N^\star = -\frac{\lambda\kappa_2'}{2f'^2}T + \Big(\frac{1-2\lambda\kappa_2}{f'^2}\Big)N + \frac{\lambda^3\kappa_2'}{f'^4}B_1 - \frac{\lambda\kappa_3}{f'^2}B_2. \tag{3.112}$$

Assuming that $\kappa_2 = \mathit{constant} \neq 0$, relation (3.112) implies that a null vector $N^\star$ is a linear combination of two mutually orthogonal spacelike vectors N i B_2, which is impossible. Therefore,

$$\kappa_2 \neq \text{constant}\,. \tag{3.113}$$

Relation (3.112) and the condition $g(N^\star, N^\star) = 0$ imply

$$\kappa_3^2 = \frac{\lambda^4 \kappa_2'^2 - f'^2(1-2\lambda\kappa_2)^2}{\lambda^2 f'^2} \tag{3.114}$$

Substituting (3.111) in (3.114), we get

$$\kappa_3^2 = \frac{\lambda^3 \kappa_2'^2 + 2(1-\lambda\kappa_2)(1-2\lambda\kappa_2)^2}{-2\lambda^2(1-\lambda\kappa_2)} \tag{3.115}$$

Putting

$$m = -\frac{\lambda\kappa_2'}{2f'^2}, \quad n = \frac{1-2\lambda\kappa_2}{f'^2}, \quad p = \frac{\lambda^3\kappa_2'}{f'^4}, \quad q = -\frac{\lambda\kappa_3}{f'^2}, \tag{3.116}$$

relation (3.112) becomes $N^\star = mT + nN + pB_1 + qB_2$. Differentiating the last relation with respect to s and using (2.1) and (2.3), we find

$$\kappa_2^\star B_1^\star f' = (m' - n\kappa_2 - q\kappa_3)T + (m + n' + p\kappa_2)N + (p' - n)B_1 + (p\kappa_3 + q')B_2.$$

By taking the scalar product of the last relation with $N = \pm B_1^\star + bB_2^\star$ it follows that $\pm\kappa_2^\star f' = m + n' + p\kappa_2$. Assuming that $\kappa_2^\star f' = m + n' + p\kappa_2$, we obtain

$$B_1^\star = \Big(\frac{m' - n\kappa_2 - q\kappa_3}{\kappa_2^\star f'}\Big)T + N + \Big(\frac{p' - n}{\kappa_2^\star f'}\Big)B_1 + \Big(\frac{p\kappa_3 + q'}{\kappa_2^\star f'}\Big)B_2. \tag{3.117}$$

The condition $g(B_1^\star, B_1^\star) = 1$ and relation (3.117) yield

$$2(m' - n\kappa_2 - q\kappa_3)(p' - n) + (p\kappa_3 + q')^2 = 0. \tag{3.118}$$

By using (3.115) and (3.116), we find

$$m' - n\kappa_2 - q\kappa_3 = \frac{\kappa_2''\lambda^2(1-\lambda\kappa_2) + 2\lambda^3\kappa_2'^2 + 2(1-\lambda\kappa_2)^2(1-2\lambda\kappa_2)}{4\lambda^2(1-\lambda\kappa_2)^2},$$

$$p' - n = \frac{\kappa_2''\lambda^2(1-\lambda\kappa_2) + 2\lambda^3\kappa_2'^2 + 2(1-\lambda\kappa_2)^2(1-2\lambda\kappa_2)}{4\lambda(1-\lambda\kappa_2)^3}. \tag{3.119}$$

From the last two relations we get

$$m' - n\kappa_2 - q\kappa_3 = \frac{(1-\lambda\kappa_2)}{\lambda}(p' - n). \tag{3.120}$$

Substituting (3.120) in (3.118), we obtain

$$2\Big(\frac{1-\lambda\kappa_2}{\lambda}\Big)(p' - n)^2 + (p\kappa_3 + q')^2 = 0. \tag{3.121}$$

Since (3.113) holds, relation (3.121) implies

$$p' - n = 0, \tag{3.122}$$

$$p\kappa_3 + q' = 0. \tag{3.123}$$

By using (3.114) and (3.116), a straightforward calculation yields

$$p\kappa_3 + q' = \frac{\kappa_2'\big[\kappa_2''\lambda^2(1-\lambda\kappa_2) + 2\lambda^3\kappa_2'^2 - (1-\lambda\kappa_2)(1-2\lambda\kappa_2)(1+2\lambda\kappa_2)\big]}{-4\lambda\kappa_3(1-\lambda\kappa_2)^3} \tag{3.124}$$

By using (3.119) and (3.122), we have

$$\kappa_2''\lambda^2(1-\lambda\kappa_2)+2\lambda^3\kappa_2'^2+2(1-\lambda\kappa_2)^2(1-2\lambda\kappa_2)=0. \tag{3.125}$$

Similarly, by using (3.123) and (3.124), we get

$$\kappa_2''\lambda^2(1-\lambda\kappa_2)+2\lambda^3\kappa_2'^2+(1-\lambda\kappa_2)(1-2\lambda\kappa_2)(-1-2\lambda\kappa_2)=0. \tag{3.126}$$

Finally, relations (3.125) and (3.126) imply $\kappa_2 = 1/2\lambda =$ constant or $\kappa_2 = 1/\lambda =$ constant, which is a contradiction with (3.113). □

In order to characterize the generalized null Mannheim curves in $\mathbb{E}^4_1$ in terms of the normal curves, we recall the next theorem given in [**9**].

THEOREM 3.8. *Let $\alpha(s)$ be a null curve in $\mathbb{E}^4_1$ parameterized by pseudo-arc s and with curvatures $\kappa_1(s)=1$, $\kappa_2(s)\neq 0$, $\kappa_3(s)\neq 0$. Then α is a normal curve if and only if one of the following statements holds:*

(i) *α lies in pseudosphere $\mathbb{S}^3_1(r)$, $r\in\mathbb{R}^+_0$;*
(ii) *the third curvature $\kappa_3(s)$ is a non-zero constant;*
(iii) *the second binormal component of the position vector α is a non-zero constant, i.e. $g(\alpha,B_2)=c_0$, $c_0\in\mathbb{R}_0$.*

By using Theorems 3.1–3.4, 3.6 and 3.8, we get the following corollaries.

COROLLARY 3.1. *Every null Cartan helix in $\mathbb{E}^4_1$ is a normal generalized null Mannheim curve the generalized Mannheim mate curve of which is a timelike or a spacelike helix.*

COROLLARY 3.2. *There are no normal generalized null Mannheim curves the generalized Mannheim mate curve of which is a null Cartan curve.*

4. Some examples

EXAMPLE 4.1. Consider the null Cartan curve α in $\mathbb{E}^4_1$ with parameter equation

$$\alpha(s)=\Big(\int e^s\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big)ds,-\int e^s\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big)ds,\\ -\int e^s\sin\Big(\frac{\sqrt{2}}{2e^s}\Big)ds,\int e^s\cos\Big(\frac{\sqrt{2}}{2e^s}\Big)ds\Big)$$

where s is the pseudo-arc length parameter. The curvatures of α are given by $\kappa_1(s)=1$, $\kappa_2(s)=-1/2$, $\kappa_3(s)=1/2e^{2s}$. The principal normal vector $N(s)$ of α is given by

$$N(s)=\alpha''(s)=e^s\Big(\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big),-\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big),-\sin\Big(\frac{\sqrt{2}}{2e^s}\Big),\cos\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big)\\ +\frac{\sqrt{2}}{2}\Big(-\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big),\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big),\cos\Big(\frac{\sqrt{2}}{2e^s}\Big),\sin\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big).$$

Define the curve $\alpha^\star\colon I\to\mathbb{E}^4_1$ by $\alpha^\star(s)=\alpha(s)-N(s)$. Since $g(\alpha^{\star\prime}(s),\alpha^{\star\prime}(s))=1$, $\alpha^\star$ is a unit speed spacelike curve. The curvatures of $\alpha^\star$ are given by $\kappa^\star_1(s)=$

$\kappa_2^\star(s) = 1/2e^{2s}$, $\kappa_3^\star(s) = 1$. Moreover, by using a straightforward calculation, it follows that the Frenet frame of $\alpha^\star$ is given by

$$\begin{aligned}
T^\star(s) &= \frac{\sqrt{2}}{2}\Big(\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cos\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sin\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big)\\
&\quad - \frac{e^{-s}}{2}\Big(\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), \sin\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cos\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big),\\
N^\star(s) &= \frac{\sqrt{2}}{2}\Big(\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), \cos\Big(\frac{\sqrt{2}}{2e^s}\Big), \sin\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big),\\
B_1^\star(s) &= -\frac{\sqrt{2}}{2}\Big(\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cos\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sin\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big)\\
&\quad + \frac{e^{-s}}{2}\Big(\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), \sin\Big(\frac{\sqrt{2}}{2e^s}\Big), -\cos\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big)\\
&\quad + e^s\Big(\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sin\Big(\frac{\sqrt{2}}{2e^s}\Big), \cos\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big),\\
B_2^\star(s) &= e^s\Big(\cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), -\sin\Big(\frac{\sqrt{2}}{2e^s}\Big), \cos\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big)\\
&\quad + \frac{\sqrt{2}}{2}\Big(-\sinh\Big(\frac{\sqrt{2}}{2e^s}\Big), \cosh\Big(\frac{\sqrt{2}}{2e^s}\Big), \cos\Big(\frac{\sqrt{2}}{2e^s}\Big), \sin\Big(\frac{\sqrt{2}}{2e^s}\Big)\Big).
\end{aligned}$$

According to Theorem 3.4, α is the generalized null Mannheim curve and $\alpha^\star$ the generalized spacelike Mannheim mate curve of α. Since $N(s) = B_2^\star(s)$, the vector $N(s)$ lies in the timelike plane $\operatorname{span}\{B_1^\star, B_2^\star\}$.

Example 4.2. Consider the null Cartan helix α in $\mathbb{E}_1^4$, parameterized by the pseudo-arc length function s, with parameter equation $\alpha(s) = (\sinh(s/\sqrt{2}), (1/\sqrt{3})\sin(s\sqrt{3}/\sqrt{2}), \cosh(s/\sqrt{2}), (-1/\sqrt{3})\cos(s\sqrt{3}/\sqrt{2}))$ and the curvature functions $\kappa_1(s) = 1$, $\kappa_2(s) = 1/2$, $\kappa_3(s) = \sqrt{3}/2$. The principal normal vector $N(s)$ of α is given by

$$N(s) = \alpha''(s) = \Big(\frac{1}{2}\sinh\Big(\frac{s}{\sqrt{2}}\Big), -\frac{\sqrt{3}}{2}\sin\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big), \frac{1}{2}\cosh\Big(\frac{s}{\sqrt{2}}\Big), \frac{\sqrt{3}}{2}\cos\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big)\Big).$$

Define the curve $\alpha^\star\colon I \to \mathbb{E}_1^4$ by $\alpha^\star(s) = \alpha(s) + N(s)$. Since $g(\alpha^{\star\prime}(s), \alpha^{\star\prime}(s)) = -1$, $\alpha^\star$ is a unit speed timelike curve. A straightforward calculation shows that the curvatures of $\alpha^\star$ are given by $\kappa_1^\star(s) = \kappa_2^\star(s) = \sqrt{3}/2$, $\kappa_3^\star(s) = 1$. Consequently, $\alpha^\star$ is a timelike helix. Moreover, the Frenet frame of $\alpha^\star$ has the form

$$\begin{aligned}
T^\star(s) &= \Big(\frac{3}{2\sqrt{2}}\cosh\Big(\frac{s}{\sqrt{2}}\Big), -\frac{1}{2\sqrt{2}}\cos\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big), \frac{3}{2\sqrt{2}}\sinh\Big(\frac{s}{\sqrt{2}}\Big), -\frac{1}{2\sqrt{2}}\sin\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big)\Big),\\
N^\star(s) &= \Big(\frac{\sqrt{3}}{2}\sinh\Big(\frac{s}{\sqrt{2}}\Big), \frac{1}{2}\sin\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big), \frac{\sqrt{3}}{2}\cosh\Big(\frac{s}{\sqrt{2}}\Big), -\frac{1}{2}\cos\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big)\Big),\\
B_1^\star(s) &= \Big(-\frac{1}{2\sqrt{2}}\cosh\Big(\frac{s}{\sqrt{2}}\Big), \frac{3}{2\sqrt{2}}\cos\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big), -\frac{1}{2\sqrt{2}}\sinh\Big(\frac{s}{\sqrt{2}}\Big), \frac{3}{2\sqrt{2}}\sin\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big)\Big),\\
B_2^\star(s) &= \Big(\frac{1}{2}\sinh\Big(\frac{s}{\sqrt{2}}\Big), -\frac{\sqrt{3}}{2}\sin\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big), \frac{1}{2}\cosh\Big(\frac{s}{\sqrt{2}}\Big), \frac{\sqrt{3}}{2}\cos\Big(\frac{s\sqrt{3}}{\sqrt{2}}\Big)\Big).
\end{aligned}$$

According to Corollary 3.1, α is the normal generalized null Mannheim curve and $\alpha^\star$ the generalized timelike Mannheim mate curve of α, such that $N(s) = B_2^\star(s)$.

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(Received 17 08 2014)

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