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A NEW MIXED INTEGER LINEAR PROGRAMMING FORMULATION FOR THE MAXIMUM DEGREE BOUNDED CONNECTED SUBGRAPH PROBLEM

Zoran Maksimović

Abstract. We give a new mixed integer linear programming (MILP) formulation for Maximum Degree Bounded Connected Subgraph Problem (MDBCSP). The proposed MILP formulation is the first in literature with polynomial number of constraints. Therefore, it will be possible to solve optimally much more instances before in a reasonable time.

1. Introduction

Let $G = (V, E)$ be an undirected graph, $d \geqslant 2$ an integer, and $w : E \to \mathbb{R}^+$ a weight function. The Maximum Degree Bounded Connected Subgraph (MDBCS) problem is to find a subgraph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, such that the subgraph G' is connected, has no vertex with degree exceeding d and $\sum_{e\in E'} w(e)$ has a maximum value. MDBCSP can be illustrated with the following small example.

Example 1.1. Let $|V| = 26$, $|E| = 29$ and graph $G = (V, E)$ with weights is presented on the left side in Fig. 1.

One optimal solution for $d = 2$ is presented on the right-hand side in Fig. 1, with value 99. This result is obtained by using new formulation which is introduced in Section 4.

It is useful to represent discrete optimization problems as integer programming problems in order to use different well-known optimization techniques for their exact solving [**8, 11, 22**]. The main effort is to design integer linear programming models with polynomial number of constraints.

2. Related work

MDBCSP is one of the classical NP-hard problems listed in Garey and Johnson's monograph [**10**] (as problem GT26), and there is no polynomial factor approximation algorithm for this problem [**1**]. However, in [**6, 7**] it is shown that

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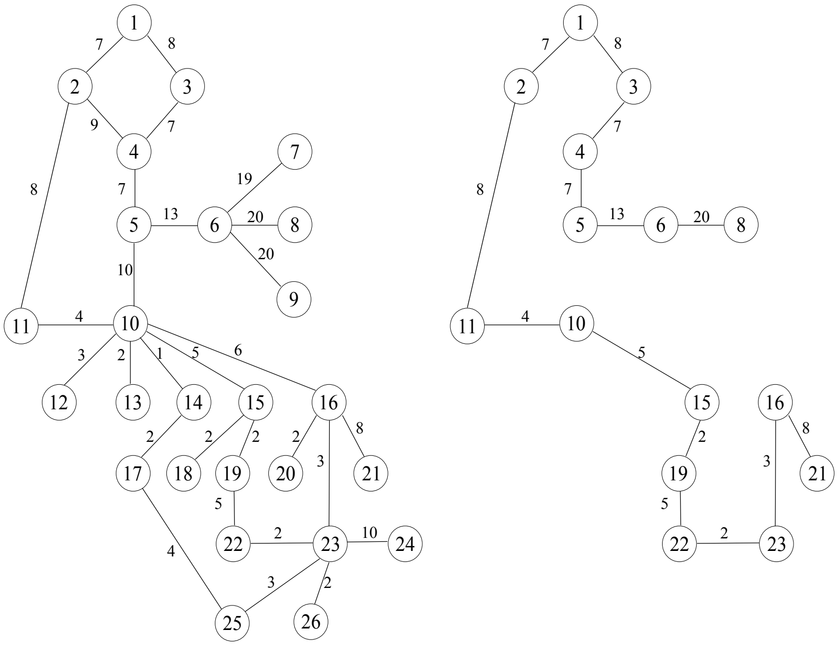

FIGURE 1. Graph G from Example 1 and its optimal solution

when the cost function satisfies triangle inequalities, there are constant factor approximation algorithms for several degree bounded network design problems. The presented approximation algorithm has constant factor $2 + \frac{k-1}{n} + \frac{1}{k}$. The parameterized complexity of some related problems is given in [**2**]. Note that without the connectivity constraint, the corresponding problem is known to be solvable in polynomial time using matching techniques.

There exist graph theoretical results in the literature related to the proposed problem. For example, in [**12**] is proved that every 3-connected $K_{1,d}$-free graph on n vertices contains a 3-connected spanning subgraph of maximum degree at most $2d-1$. In [**13**] a similar statement about 2-connected graphs is proved. It is proved that every 2-connected $K_{1,d}$-free graph has an a 2-connected spanning subgraph of maximum degree at most d.

In [**23, 24**] a research about MDBCSP for chemical compounds is given. Although the problem is NP-hard in a general case, graphs associated with chemical compounds have polynomial number of spanning trees, so a polynomial time algorithm is proposed.

Subexponential parametrized algorithms on planar graph for finding a connected subgraph with bounded maximum degree and maximal number of edges (or vertices) is presented in [**18, 19**]. Their approach uses bidimensionality theory combined with dynamic programming techniques over branch decompositions of the input graph, but it is applicable only for small graphs.

Rahman and Kaykobad in [**17**] proved that a connected graph having independence number d, for $d \geqslant 2$, has a degree-d-bounded spanning tree. They also presented an $O(n^2)$ algorithm for constructing such a spanning tree, but independent number of graphs is also an NP-hard problem.

In [**5**] is presented an $O(n)$-time algorithm that constructs a connected subgraph of a given triangulation whose maximum degree is at most $14+\lceil 2\pi/\gamma \rceil$, for a given real number γ, where $0 < \gamma < \pi$. Sufficient conditions for a connected graph to have degree bounded trees are presented in [**16, 15**].

Generalizations of the problem of independent transversals are introduced and discussed in [**21**]. It is discussed whether any graph of maximum degree at most d with a vertex partition into classes of size at least p admits a acyclic H-free transversal having connected components of order at most r. It is proven that if the vertex set of a d-regular graph is partitioned into classes of size $d + \lfloor d/r \rfloor$, then it is possible to select a transversal inducing vertex disjoin trees on at most r vertices. Their approach applies appropriate triangulations of the simplex and SpernerâĂŹs Lemma.

The MDBCSP with maximum diameter is considered in [**9**]. It discussed some applications in security, network design and parallel processing, and it derived some bounds for the order of the largest subgraph in host graphs of practical interest: the mesh and the hypercube. A heuristic strategy for solving the problem is proposed, and an approximation ratio for the algorithm is proved. As usual, experimental results with a variety of host networks showed that the algorithm performs better in practice than the prediction provided by theoretical approximation ratio.

In [**3**] is introduced Integer Linear Programming (ILP) formulation with exponential number of constraints, consequently, the standard ILP solvers were able to solve only very small instances. Additionally, for solving larger instances one metaheuristic approach based on genetic algorithm was proposed in [**3, 4**]. It is based on binary representation and used fine-grained tournament selection, one-point crossover, simple mutation with frozen genes and caching technique. The genetic algorithm give approximation solutions on several graphs with up to 212 vertices and 229 edges.

3. An existing ILP formulation

The first integer linear programming (ILP) formulation was proposed in [**3**]. There are great benefits for representing discrete optimization problems as integer linear programming models [**14, 3**] or their generalization [**20**]. Since it has exponential number of constraints, its ability to solve MDBCSP instances is limited, so it is unable to solve even the instance presented in Example 1.

ILP formulation uses variables

$$x_e = \begin{cases} 1, & e \in E' \\ 0, & e \notin E' \end{cases} \tag{3.1}$$

$$y_e = \begin{cases} 1, & e \in T' \\ 0, & e \notin T' \end{cases} \tag{3.2}$$

$$z_i = \begin{cases} 1, & i \in V' \\ 0, & i \notin V' \end{cases} \tag{3.3}$$

where $e \in E$, and T' is the spanning tree for the subgraph $G' = (V', E')$.

Then the MDBCSP was formulated as:

$$\max \sum_{e \in E} w(e) \cdot x_e \tag{3.4}$$

subject to:

$$\sum_{e: e \ni i} x_e \leqslant d, \quad i \in V, \tag{3.5}$$

$$y_e \leqslant x_e, \quad e \in E, \tag{3.6}$$

$$\sum_{e \in E} y_e = -1 + \sum_{i \in V} z_i \tag{3.7}$$

$$x_e \leqslant z_{i_e}, \quad e \in E, \tag{3.8}$$

$$x_e \leqslant z_{j_e}, \quad e \in E, \tag{3.9}$$

$$\sum_{i_e, j_e \in S} y_e \leqslant |S| - 1, \quad S \subseteq V, |S| \geqslant 3 \tag{3.10}$$

$$x_e, y_e \in \{0,1\}, z_i \in \{0,1\}, \quad e \in E, i \in V \tag{3.11}$$

The i_e and j_e denote the vertices that are incident to edge e. As can be seen, objective function (3.4) maximizes the sum of overall weights, constraints (3.5) bounded the degree of vertices, (3.6) means that subgraph G' is connected since it contains spanning tree given by constraints (3.7)–(3.10). Constraints (3.11) denote binary nature of decision variables. It is obvious that the number of constraints (3.10) rise exponentially with the increase of the number of vertices. In order to produce integer linear programming formulation with polynomial number of constraints, constraints (3.10) should be replaced, because they are the only constraints from (3.4)–(3.11) whose number is exponential.

Therefore, decision variables y, which, participate in (3.10), should be omitted together with all constraints referring to these variables.

4. A new MILP formulation

Let us a now construct new ILP formulation based on network flow. The general idea is to model graph $G = (V, E)$ as network, where each vertex from V' consumes amount of flow equal to 1, and the flow can stream only through edges from E'. This idea is completely different from well-known network flow problem, since V' and E' and origin of flow are unknown in advance, but the network flow is only used as general idea to model explanation. Furthermore, the network flow preservation principle is used, having in mind, that each vertex consumes amount of flow equal to 1, then the quantity of external flow is equal to cardinality of V'.

To accomplish this, we try to input external flow through single vertex (origin of external network flow), in amount as much as possible, into the network represented by the graph $G = (V, E)$.

To achieve this goal additional decision variables will be introduced. Binary decision variables t represent the origin of external network flow, i.e. $t_i = 1, i \in V$ if vertex i is the origin of external network flow, and 0 otherwise. The amount of external network flow is represented by the variables u. Note that $u_i = 0$ for all i, which are not origin of external network flow. Continuous variables v_e denote the amount of the flow which streams through edge e. Since the graph is undirected, but network flow must be defined on directed networks, direction of each edge will be defined in advance. For example, one way of directing graph is to use lexicographical order ($i_e < j_e$). If network flow streams in the direction of edge e, then v_e is positive, otherwise it is negative. Network flow could stream only through edges from E', while $v_e = 0, e \in E \setminus E'$. Since flow network is completely independent with weights of edges (it only guarantees the connectivity of G'), then v_e could be equal to 0 even for some e from E'.

Let the the binary variables x_e and z_i be defined as in (3.1) and (3.3), then the new MILP formulation for the solution of MDBCSP can be formulated as follows

$$\max \sum_{e \in E} w(e) \cdot x_e \tag{4.1}$$

subject to:

$$\sum_{e : e \ni i} x_e \leqslant d, \quad i \in V, \tag{4.2}$$

$$x_e \leqslant \frac{1}{2} z_{i_e} + \frac{1}{2} z_{j_e}, \quad e \in E \tag{4.3}$$

$$\sum_{i=1}^{n} t_i = 1 \tag{4.4}$$

$$u_i \leqslant n \cdot t_i, \quad i \in V \tag{4.5}$$

$$v_e \leqslant n \cdot x_e, \quad e \in E \tag{4.6}$$

$$v_e \geqslant -n \cdot x_e, \quad e \in E \tag{4.7}$$

$$u_i + \sum_{e : j_e = i} v_e - \sum_{e : i_e = i} v_e = z_i, \quad i \in V \tag{4.8}$$

$$z_i \leqslant \sum_{e : e \ni i} x_e \quad i \in V \tag{4.9}$$

$$\begin{gathered} x_e \in \{0,1\}, z_i \in \{0,1\}, \quad e \in E,\ i \in V \\ t_i \in [0, +\infty), u_i \in \mathbb{N} \cup \{0\}, \quad v_e \in [-n, n], \quad e \in E,\ i \in V \end{gathered} \tag{4.10}$$

Constraints (4.4) and (4.5) implies that external flow in the G' is obtained from one origin, with quantity at most n. Constraints (4.6) and (4.7) ensure that the flow is distributed only by edges in G'. Network preservation principle is represented by (4.8). Every vertex from V' must have at least one established edge, which is ensured by (4.9).

Note that, in order to give compact formulation, constraints (3.8) and (3.9) were substituted with (4.3).

Let us now prove the correctness of the new formulation.

THEOREM 4.1. *If a subgraph $G' = (V', E')$ of a given graph $G = (V, E)$ is the solution of the Maximum Degree-Bounded Connected Subgraph problem, then the values of the appropriate variables x, z, t, v, u give the optimal solution of the new MILP model.*

PROOF. Let $G' = (V', E')$ be an optimal solution of MDBCSP. Let decision variables x and z be defined as in (3.1) and (3.3), respectively. Let us choose arbitrary vertex k from V', and define it as origin from which the external flow streams into the network. Therefore, $t_k = 1$ and $u_k = |V'|$, while $t_i = 0$ and $u_i = 0$ for all $i \neq k$. In order to obtain values of v_e the search of graph G' should be performed starting from the vertex k (for example we can use depth first search or breadth first search). Let $s(v)$ be an enumeration of vertices from V' generated by search process. For example, $s(k) = 1$, for the last searched vertex s is equal to $|V'|$, while $s(v) = 0$ for all vertices from $V \setminus V'$. For each edge $e \in E'$, the value of variable v_e is defined as follows. If a graph is searched from vertex i_e to j_e through edge e ($s(j_e) = s(i_e) + 1$), then v_e is equal to the number of vertices in the search subtree rooted by j_e. Otherwise v_e is defined as a number of vertices in the search subtree rooted by i_e multiplied by -1. For all other edges from E' and all edges from $E \setminus E'$ it holds that $v_e = 0$.

Since the subgraph G' has maximal degree of d, then it is obvious that (4.2) holds. Constraints (4.4) and (4.5) are directly satisfied from the definitions of variables t and u.

If $e \in E \setminus E'$, then from the definition of variables x and v, it holds $x_e = 0$ and $v_e = 0$ and constraints (4.6) and (4.7) are satisfied. In the other case, when $e \in E'$, implying $x_e = 1$. Since v_e is defined as number of vertices (or multiplied by -1), then it must be in $[-n, n]$, and constraints (4.6) and (4.7) are satisfied.

If $i \in V \setminus V'$, then all edges e such that $i_e = i$ or $j_e = i$ are in $E \setminus E'$. Therefore, $u_i = 0, v_e = 0$ and $z_i = 0$ and constraint (4.8) is satisfied. If $i = k$ then $u_i = |V'|$ and $z_i = 1$. The overall number of vertices in the subtrees of its successors is equal to the $|V'| - 1$ (all vertices from V' are enveloped except vertex k). Therefore, by the definition of variable v, it holds that

$$\sum_{e:i_e=i} v_e - \sum_{e:j_e=i} v_e = \sum_{\substack{e:i_e=k \\ s(j_e)<s(k)}} v_e + \sum_{\substack{e:i_e=k \\ s(j_e)>s(k)}} v_e - \sum_{\substack{e:j_e=k \\ s(i_e)<s(k)}} v_e - \sum_{\substack{e:j_e=k \\ s(i_e)>s(k)}} v_e$$

Since $s(k) = 1$, $s(i_e) > 1$ and $s(j_e) > 1$ then the first and third sums are equal to zero. Since all elements of the second sum are positive, and all elements in the fourth sum are negative, overall sum is equal to overall number of vertices in the subtrees of successors of k i.e. to $|V'|-1$.

$$\sum_{e:i_e=i} v_e - \sum_{e:j_e=i} v_e = \sum_{\substack{e:i_e=i\\ s(j_e)<s(i)}} v_e + \sum_{\substack{e:i_e=i\\ s(j_e)>s(i)}} v_e - \sum_{\substack{e:j_e=i\\ s(i_e)<s(i)}} v_e - \sum_{\substack{e:j_e=i\\ s(i_e)>s(i)}} v_e$$

In the first and third sum exists only one element whose value is different from zero. The difference of the first and third sums is equal to the number of vertices in subtree rooted by i. Since that difference of the second and fourth sums is equal to the number of vertices in subtrees rooted by successors of i, implying that the left-hand side of constraint (4.8) is equal to one. For $i \in V'$ $z_i = 1$, which is the right-hand side of constraint (4.8).

For $i \in V \setminus V'$, $e \in E \setminus E'$ implying $z_i = 0$ and $\sum_{e:e\ni i} 0 = 0$ so constraint (4.9) is satisfied by default. In the second case when $i \in V'$, at least one edge $e \in E'$ is incident to the vertex i, i.e. $x_e = 1$, then right-hand side is grater or equal to 1, which is equal to left-hand side.

Constraints (4.10) are satisfied by the definition of all variables.

For $e \in E \setminus E'$ constraints (4.3) are trivially satisfied since $x_e = 0$. In the other case when $e \in E'$, then $i_e, j_e \in E'$ implying $x_e = z_{i_e} = z_{j_e} = 1$, which means that constraints (4.3) are satisfied.

From the constraint (3.4), $\sum_{e\in E} w(e) \cdot x_e = \sum_{e\in E'} w(e)$, therefore $\max \sum_{e\in E} w(e) \cdot x_e \geqslant \sum_{e\in E'} w(e)$.

In the last case when $i \in V'$ and $i \neq k$, it holds that $u_i = 0$ and $z_i = 0$. □

THEOREM 4.2. *If the values of variables x, z, t, v, u are the optimal solution of the new MILP model, then the subgraph $G' = (V', E')$ where $V' = \{v \in V | z_v = 1\}$ and $E' = \{e \in E | x_e = 1\}$, is the optimal solution of the MDBCS problem.*

PROOF. Let values (x, z, t, v, u) be a solution of the MILP model specified by (4.1) and constraints (4.2)–(4.10). Let us define graph $G' = (V', E')$, where $V' = \{v \in V | z_v = 1\}$ and $E' = \{e \in E | x_e = 1\}$. In continuation it will be proven that G' is well defined, connected and d-bounded subgraph of G.

- If $e \in E'$ then $x_e = 1$. From constraints (4.3) it follows that $z_{i_e} = z_{j_e} = 1$ meaning that both $i_e, j_e \in V'$. Therefore, the graph G' is well defined.
- The subgraph G' has no isolated vertices because of (4.9), for each $v \in V$, $(\forall e \in E')(v \notin e) \Rightarrow \sum_{e:e\ni v} x_e = \sum_{e:e\ni v, e\in E\setminus E'} x_e = 0 \Rightarrow z_v = 0 \Rightarrow v \in V \setminus V'$. Directly this means that if $v \in V'$, there must be at least one $e \in E'$ incident to v.
- From constraints (4.2) it follows that G' has no vertex of degree greater than d;
- Now we can prove that V' is connected. It is sufficient to prove that for every partition of V' into the two disjoint subsets V_1', V_2' with $|V_1'|, |V_2'| \geqslant 2$ there exists at least one edge $e \in E$, which is incident to vertices from both subsets. Suppose, without loss of generality, that $k \notin V_1'$.

$$\sum_{i\in V_1'} \left(u_i + \sum_{e:j_e=i} v_e - \sum_{e:i_e=i} v_e \right) = \sum_{i\in V_1'} z_i$$

The right-hand side R is equal to the $|V_1'|$. The left-hand side L is equal to

$$\begin{aligned}
&\sum_{i\in V_1'} u_i + \sum_{i\in V_1'}\sum_{e:j_e=i} v_e - \sum_{i\in V_1'}\sum_{e:i_e=i} v_e\\
&= 0 + \sum_{e:i_e,j_e\in V_1'} v_e + \sum_{e:i_e\in V_2',j_e\in V_1'} v_e + \sum_{e:j_e\in V_1',i_e\in V\setminus V'} v_e\\
&\quad - \sum_{e:i_e,j_e\in V_1'} v_e - \sum_{e:i_e\in V_1',j_e\in V_2'} v_e - \sum_{e:i_e\in V_1',j_e\in V\setminus V'} v_e\\
&= \sum_{e:i_e\in V_2',j_e\in V_1'} v_e - \sum_{e:i_e\in V_1',j_e\in V_2'} v_e
\end{aligned}$$

Since $L = R$

$$\sum_{e:i_e\in V_2',j_e\in V_1'} v_e - \sum_{e:i_e\in V_1',j_e\in V_2'} v_e = |V_1'|,$$

therefore at least one sum is different from zero and, consequently

$$(\exists e)(((i_e\in V_2'\wedge j_e\in V_1')\vee(i_e\in V_1'\wedge j_e\in V_2'))v_e\neq 0.$$

From that

$$(\exists e)(((i_e\in V_2'\wedge j_e\in V_1')\vee(i_e\in V_1'\wedge j_e\in V_2'))x_e = 1,$$

because if $v_e > 0$, then from (4.6) it is $x_e = 1$. Otherwise, if $v_e < 0$ then from (4.7) it follows that $x_e = 1$. Therefore, there exists at least one edge $e \in E$, which is incident to vertices from both subsets V_1' and V_2'.

- For previously defined E' it holds that

$$\sum_{e\in E'} w(e) = \sum_{e\in E} w(e)\cdot x_e = \mathrm{obj}_{\mathrm{MILP}}(x,z,t,u,v)$$

  Since the optimal solution value of the MDBCSP is maximum of $w(G')$ over all d-bounded connected subgraphs G', and $G' = (V', E')$ satisfies all MDBCSP conditions and have objective equal to $\mathrm{obj}_{\mathrm{MILP}}(x,z,t,u,v)$, then the optimal solution value of the MDBCSP on given graph G is greater or equal to $\mathrm{obj}_{\mathrm{MILP}}(x,z,t,u,v)$. □

The nature of variables from presented MILP model can be also illustrated on Example 1.1.

Let us consider graph from the left side in Figure 1. As was previously mentioned, for $d = 2$, the optimal solution value is 99. One optimal solution is represented as follows.

Value of $t_2 = 1$ and corresponding $u_2 = 15$ while all others $t_i = 0$ and $u_i = 0$ for $i \neq 2$.

Values of z-variables are $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_8 = z_{10} = z_{11} = z_{15} = z_{16} = z_{19} = z_{21} = z_{22} = z_{23} = 1$, while $z_7 = z_9 = z_{12} = z_{13} = z_{14} = z_{17} = z_{18} = z_{20} = z_{24} = z_{25} = z_{26} = 0$.

Following x-variables have zero values: $x_{(2,4)} = x_{(5,10)} = x_{(6,7)} = x_{(6,9)} = x_{(10,12)} = x_{(10,13)} = x_{(10,14)} = x_{(10,16)} = x_{(14,17)} = x_{(15,18)} = x_{(16,20)} = x_{(17,25)} = x_{(23,24)} = x_{(23,25)} = x_{(23,26)}$. The corresponding v-variables have also zero values.

All other x-variables have value equal to 1, and the corresponding v-variables have values: $v_{(1,2)} = -6$, $v_{(1,3)} = 5$, $v_{(3,4)} = 4$, $v_{(4,5)} = 3$, $v_{(5,6)} = 2$, $v_{(6,8)} = 1$, $v_{(10,11)} = -7$, $v_{(10,15)} = 6$, $v_{(15,19)} = 5$, $v_{(16,21)} = 1$, $v_{(16,23)} = -2$, $v_{(19,22)} = 4$, $v_{(22,23)} = 3$, and $v_{(2,11)} = 8$.

The proposed MILP formulation, using CPLEX 10.1 solver, produced result in 1.119 seconds with 192 constrains, while the CPLEX with existing ILP formulation failed to produce result with the 3600 seconds time limit with 643568 constraints, running on the PC with Intel i3 2.2 GHz processor and 4 GB RAM.

5. Conclusions

This paper is devoted to the maximum degree-bounded connected subgraph problem. We introduced a mixed integer linear programming formulation with polynomial number of constraints, with proof of its correctness. Therefore the problem can be solved exactly, by standard ILP solvers, on instances of higher dimensions than those proposed in the literature.

The future work can be directed in several ways. It would be desirable to investigate the application of some exact method using the proposed MILP formulation. Further research also should be directed to solving similar problems.

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Department of Natural Sciences and Mathematics
Military Academy
Belgrade
Serbia
max@tmf.bg.ac.rs

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