PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 109–120 DOI: 10.2298/PIM141208013V

GEOMETRY OF PENTAGONAL QUASIGROUPS

Stipe Vidak

Abstract. Pentagonal quasigroups are IM-quasigroups in which the additional identity of pentagonality holds. Motivated by the example $C(q)$, where q is a solution of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$, some basic geometric concepts are introduced and studied in a general pentagonal quasigroup. Such concepts are parallelogram, midpoint of a segment, regular pentagon and regular decagon. Some theorems of Euclidean plane which use these concepts are stated and proved in pentagonal quasigroups.

1. Introduction

A quasigroup $(Q, \cdot)$ is a groupoid in which each of the equations $a \cdot x = b$ and $y \cdot a = b$ has a unique solution for given $a, b \in Q$. A quasigroup $(Q, \cdot)$ is called *IM-quasigroup* if it satisfies the identities of *idempotency* and *mediality*:

$$aa = a \tag{1.1}$$

$$ab \cdot cd = ac \cdot bd. \tag{1.2}$$

The immediate consequences of these identities are the identities known as *elasticity*, *left distributivity* and *right distributivity*:

$$ab \cdot a = a \cdot ba, \tag{1.3}$$

$$a \cdot bc = ab \cdot ac, \tag{1.4}$$

$$ab \cdot c = ac \cdot bc. \tag{1.5}$$

Definition 1.1. *Pentagonal quasigroup* is an IM-quasigroup $(Q, \cdot)$ in which the identity of *pentagonality* holds:

$$(ab \cdot a)b \cdot a = b. \tag{1.6}$$

In pentagonal quasigroups, along with pentagonality and the identities which are valid in any IM-quasigroup, some other identities hold. They are stated in the following theorem (see [**2**]).

2010 *Mathematics Subject Classification:* 20N05.
Key words and phrases: IM-quasigroup, parallelogram, midpoint, regular pentagon, regular decagon.
Communicated by Žarko Mijajlović.

Theorem 1.1. *In every IM-quasigroup* $(Q, \cdot)$ *identity* (1.6) *and the identities*

$$(ab \cdot a)c \cdot a = bc \cdot b, \tag{1.7}$$

$$(ab \cdot a)a \cdot a = ba \cdot b, \tag{1.8}$$

$$ab \cdot (ba \cdot a)a = b \tag{1.9}$$

are mutually equivalent and they imply the identity

$$a(b \cdot (ba \cdot a)a) \cdot a = b \tag{1.10}$$

for every a, b, $c \in Q$.

Example 1.1. The basic example of the pentagonal quasigroup is $C(q) = (\mathbb{C}, *)$, where $*$ is defined by $a * b = (1-q)a + qb$ and q is a solution of the equation

$$q^4 - 3q^3 + 4q^2 - 2q + 1 = 0. \tag{1.11}$$

The Equation (1.11) has four complex solutions:

$$q_{1,2} = \frac{1}{4}\Big(3 + \sqrt{5} \pm i\sqrt{2\big(5 + \sqrt{5}\,\big)}\,\Big), \quad q_{3,4} = \frac{1}{4}\Big(3 - \sqrt{5} \pm i\sqrt{2\big(5 - \sqrt{5}\,\big)}\,\Big).$$

The most general example of the pentagonal quasigroup is given by the representation theorem proved in [**2**].

Theorem 1.2. *Pentagonal quasigroup* $(Q, \cdot)$ *exists if and only if there exists an Abelian group* $(Q, +)$ *with an automorphism* φ *that satisfies*

$$\varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + \mathbf{1} = 0 \tag{1.12}$$

and $a \cdot b = a + \varphi(b - a)$ *for all* $a, b \in Q$.

The example $C(q)$ motivates the introduction of many geometric concepts in pentagonal quasigroups. We can regard elements of the set $\mathbb{C}$ as points of the Euclidean plane. For any two different points a, $b \in \mathbb{C}$ the equality $a * b = (1-q)a + qb$ can be written in the form

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0}.$$

That means that the points a, b and $a * b$ are vertices of a triangle directly similar to the triangle with vertices 0, 1 and q. Each q_i, $i = 1, 2, 3, 4$ gives a certain type of triangle and we get so called *characteristic triangles* for pentagonal quasigroups. In $C(q_1)$ the point $a * b$ is the third vertex of the regular pentagon determined by its adjacent vertices a and b. Any identity in the pentagonal quasigroup $C(q) = (\mathbb{C}, *)$ can be interpreted as a theorem of the Euclidean geometry. Some figures which show that can be found in [**2**].

Here we define and study some basic geometric concepts in pentagonal quasigroups. We give the definition of the parallelogram and the midpoint of a segment and give an example of a quasigroup in which segments can have multiple midpoints. We give definitions of the regular pentagon and the regular decagon and their centres. In the last section we use defined concepts to state and prove two well known theorems of Euclidean plane in pentagonal quasigroups.

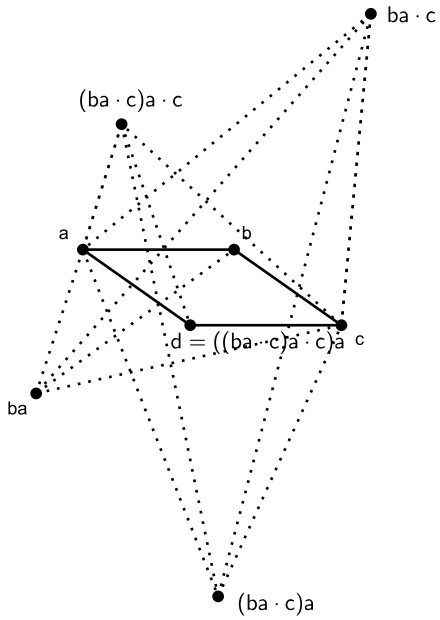

FIGURE 1. Definition of parallelogram

2. Parallelogram and midpoint of a segment

DEFINITION 2.1. Let $(Q,\cdot)$ be a quasigroup. Elements of Q are called *points*. A pair of points a and b is called a *segment* and is denoted by $\{a,b\}$. Cyclic n-tuple of points $a_1,a_2,\ldots,a_n$ is called *n-gon* and is denoted by $(a_1,a_2,\ldots,a_n)$.

Given four elements a,b,c,d of a medial quasigroup $(Q,\cdot)$ the concept of parallelogram was defined in [**5**]. However, it is necessary to define many nontrivial geometric concepts in a medial quasigroup in order to define it properly. If we observe medial quasigroups with the additional identity of idempotency, IM-quasigroups, the definition of parallelogram becomes much more elegant (see [**6**]). In many subclasses of IM-quasigroups [**7, 8**] this definition can become even simpler due to an additional identity, which determines that subclass. In pentagonal quasigroups the definition is motivated by Figure 1.

DEFINITION 2.2. Let $(Q,\cdot)$ be a pentagonal quasigroup. We say that $a,b,c,d \in Q$ form a *parallelogram* and we denote it by $\mathrm{Par}(a,b,c,d)$ if

$$d = ((ba\cdot c)a\cdot c)a. \tag{2.1}$$

DEFINITION 2.3. Let Q be a set and let $P\subset Q^4$ be a relation. Structure (Q,P) is called *parallelogram space* if:

i) For every $a,b,c\in Q$ there is only one $d\in Q$ such that $P(a,b,c,d)$ holds.
ii) If (e,f,g,h) is any cyclic permutation of (a,b,c,d) or (d,c,b,a), then $P(a,b,c,d)$ implies $P(e,f,g,h)$, for every $a,b,c,d\in Q$.

iii) For every $a, b, c, d, e, f \in Q$ $P(a, b, c, d)$ and $P(c, d, e, f)$ imply $P(a, b, f, e)$.

Our goal is to show that the structure (Q, Par), where Q is a pentagonal quasigroup and $\mathrm{Par} \subset Q^4$ a quaternary relation defined by (2.1) is a parallelogram space.

PROPOSITION 2.1. *If* $(Q, \cdot)$ *is a pentagonal quasigroup and* $a, b, c, d \in Q$*, then* $\mathrm{Par}(a, b, c, d) \Rightarrow Par(b, c, d, a)$.

PROOF. $\mathrm{Par}(a, b, c, d)$ is equivalent to $d = ((ba \cdot c)a \cdot c)a$ and we have to prove the identity

$$a = ((cb \cdot d)b \cdot d)b.$$

Using (1.6) we get $a = (ba \cdot b)a \cdot b$. Right cancellation by b gives

$$(ba \cdot b) \cdot a = (cb \cdot d)b \cdot d.$$

If we apply (1.5) on the left hand side and plug in $d = ((ba \cdot c)a \cdot c)a$ on the right-hand side, we get

$$(ba \cdot a) \cdot ba = (cb \cdot d)b \cdot ((ba \cdot c)a \cdot c)a.$$

Using (1.2) and cancelling from the right by ba, we need to prove

$$ba \cdot a = (cb \cdot d)((ba \cdot c)a \cdot c).$$

Now we have:

$$\begin{aligned}(cb \cdot d)((ba \cdot c)a \cdot c) &\overset{(1.2)}{=} (cb \cdot (ba \cdot c)a) \cdot dc \overset{(1.2)}{=} (c(ba \cdot c) \cdot ba) \cdot dc \\ &\overset{(1.4)}{=} (c(ba \cdot c) \cdot ba)d \cdot (c(ba \cdot c) \cdot ba)c \\ &\overset{(1.3),(1.6)}{=} (c(ba \cdot c) \cdot ba)d \cdot ba.\end{aligned}$$

Let us now prove $(c(ba \cdot c) \cdot ba)d = ((ba \cdot (ba \cdot a)) \cdot ba)(ba \cdot a)$. We have:

$$\begin{aligned}(c(ba \cdot c) \cdot ba)d &= (c(ba \cdot c) \cdot ba) \cdot ((ba \cdot c)a \cdot c)a \\ &\overset{(1.2)}{=} (c(ba \cdot c) \cdot ((ba \cdot c)a \cdot c))(ba \cdot a) \\ &\overset{(1.3)}{=} ((c \cdot ba)c \cdot ((ba \cdot c)a \cdot c))(ba \cdot a) \\ &\overset{(1.5)}{=} ((c \cdot ba) \cdot (ba \cdot c)a)c \cdot (ba \cdot a) \\ &\overset{(1.2),(1.3)}{=} ((c \cdot ba)c \cdot (ba \cdot a))c \cdot (ba \cdot a) \\ &\overset{(1.7)}{=} ((ba \cdot (ba \cdot a)) \cdot ba)(ba \cdot a).\end{aligned}$$

In the end using (1.6) we have $((ba \cdot (ba \cdot a)) \cdot ba)(ba \cdot a) \cdot ba = ba \cdot a$. Hence $a = ((cb \cdot d)b \cdot d)b$, i.e. $\mathrm{Par}(b, c, d, a)$. $\square$

PROPOSITION 2.2. *If* $(Q, \cdot)$ *is a pentagonal quasigroup and* $a, b, c, d \in Q$*, then* $\mathrm{Par}(a, b, c, d) \Rightarrow \mathrm{Par}(c, b, a, d)$.

Proof. Since $\mathrm{Par}(a,b,c,d) \Leftrightarrow d = ((ba\cdot c)a\cdot c)a$, we have:

$$\begin{aligned} d &= ((ba\cdot c)a\cdot c)a \overset{(1.6)}{=} ((ba\cdot c)a\cdot c)((ca\cdot c)a\cdot c) \\ &\overset{(1.5)}{=} ((ba\cdot c)a\cdot (ca\cdot c)a)c \overset{(1.5)}{=} ((ba\cdot c)(ca\cdot c)\cdot a)c \\ &\overset{(1.5)}{=} ((ba\cdot ca)c\cdot a)c \overset{(1.5)}{=} ((bc\cdot a)c\cdot a)c, \end{aligned}$$

which is equivalent to $\mathrm{Par}(c,b,a,d)$. □

Lemma 2.1. *If $(Q,\cdot)$ is a pentagonal quasigroup and $a,b,c,d\in Q$, then*

$$\mathrm{Par}(a,b,c,d) \;\Leftrightarrow\; \exists\, p,\, q\in Q \text{ such that } pa = qb \text{ and } pd = qc.$$

Proof. Let us put $p = (ab\cdot a)b$ and $q = b$. Then by (1.1) and (1.6) $pa = qb$ holds. If we show that

$$(ab\cdot a)b\cdot ((ba\cdot c)a\cdot c)a = bc,$$

we will show that $\mathrm{Par}(a,b,c,d)$ implies $pd = qc$ with the above defined p and q. In fact, proving that we will prove the other implication as well, because with such defined p and q we have $pa = qb$ and $pd = qc$, and we must show $d = ((ba\cdot c)a\cdot c)a$. When we show

$$(ab\cdot a)b\cdot ((ba\cdot c)a\cdot c)a = bc,$$

we see that $((ba\cdot c)a\cdot c)a$ satisfies the equation $pd = qc$ with given p and qc, and since in a quasigroup such solution must be unique, d must be equal to $((ba\cdot c)a\cdot c)a$.

Let us now show that $(ab\cdot a)b\cdot ((ba\cdot c)a\cdot c)a = bc$. We have:

$$\begin{aligned} (ab\cdot a)b\cdot ((ba\cdot c)a\cdot c)a &\overset{(1.4)}{=} ((ab\cdot a)b\cdot ((ba\cdot c)a\cdot c))\cdot ((ab\cdot a)b\cdot a) \\ &\overset{(1.6)}{=} ((ab\cdot a)b\cdot ((ba\cdot c)a\cdot c))b \\ &\overset{(1.2)}{=} (((ab\cdot a)\cdot (ba\cdot c)a)\cdot bc)b \\ &\overset{(1.5)}{=} ((ab\cdot (ba\cdot c))a\cdot bc)b \overset{(1.2)}{=} (((a\cdot ba)\cdot bc)a\cdot bc)b \\ &\overset{(1.3)}{=} (((ab\cdot a)\cdot bc)a\cdot bc)b \overset{(1.7)}{=} ((b\cdot bc)b\cdot bc)b \overset{(1.6)}{=} bc. \end{aligned}$$ □

Proposition 2.3. *If $(Q,\cdot)$ is a pentagonal quasigroup and $a,b,c,d,e,f\in Q$, then*

$$\mathrm{Par}(a,b,c,d),\, \mathrm{Par}(c,d,e,f) \;\Rightarrow\; \mathrm{Par}(a,b,f,e).$$

Proof. By Lemma 2.1 we have

$$\begin{aligned} \mathrm{Par}(a,b,c,d) &\;\Leftrightarrow\; \exists\, p_1,\, q_1\in Q \text{ such that } p_1a = q_1b \text{ and } p_1d = q_1c, \\ \mathrm{Par}(c,d,e,f) &\;\Leftrightarrow\; \exists\, p_2,\, q_2\in Q \text{ such that } p_2c = q_2d \text{ and } p_2f = q_2e, \end{aligned}$$

wherefrom we conclude that $p_1 = q_2$ and $q_1 = p_2$. By Lemma 2.1 we have

$$\mathrm{Par}(a,b,f,e) \;\Leftrightarrow\; \exists\, p_3,\, q_3\in Q \text{ such that } p_3a = q_3b \text{ and } p_3e = q_3f.$$

If we put $p_3 = p_1 = q_2$ and $q_3 = q_1 = p_2$, then we immediately see that $p_3a = q_3b$ and $p_3e = q_3f$, which implies $\mathrm{Par}(a,b,f,e)$. □

Theorem 2.1. *If* $(Q,\cdot)$ *is a pentagonal quasigroup, and* $\mathrm{Par} \subset Q^4$ *relation on* Q *defined by* (2.1)*, then* (Q, Par) *is a parallelogram space.*

Proof. Property i) is obvious.

Property ii) follows from Propositions 2.1 and 2.2, because all cyclic permutations of (a,b,c,d) and (d,c,b,a) can be achieved from (a,b,c,d) by successive applications of permutations from those propositions.

Property iii) follows from Proposition 2.3. □

The previous theorem proves that the parallelogram, defined in a pentagonal quasigroup, possesses all the properties of the parallelogram in medial and IM-quasigroups stated and proved in [**5**] and [**6**]. The same holds for the concept of the midpoint of a segment whose definition uses the definition of the parallelogram.

Let $(Q,\cdot)$ be a medial quasigroup. We say that $m \in Q$ is the midpoint of the segment $\{a,b\}$, $a,b \in Q$ if $\mathrm{Par}(a,m,b,m)$ holds. Using (2.1) the midpoint of a segment can be defined in a pentagonal quasigroup in the following way.

Definition 2.4. Let $\{a,b\}$ be a segment in a pentagonal quasigroup $(Q,\cdot)$ for $a,b \in Q$. We say that $m \in Q$ is the *midpoint* of the segment $\{a,b\}$ and denote it by $M(a,m,b)$ if

$$m = ((ma \cdot b)a \cdot b)a \tag{2.2}$$

Applying Theorem 1.2 several times we get

$$\begin{aligned}((ma \cdot b)a \cdot b)a =& (\mathbf{1}-\varphi)^5(m) + \varphi(\mathbf{1}-\varphi)^4(a) + \varphi(\mathbf{1}-\varphi)^3(b) \\ &+ \varphi(\mathbf{1}-\varphi)^2(a) + \varphi(\mathbf{1}-\varphi)(b) + \varphi(a).\end{aligned}$$

Plugging that into identity (2.2) and using (1.12) several times we get $2m = a+b$, where $+$ is the addition of the Abelian group $(Q,+)$ associated to the pentagonal quasigroup $(Q,\cdot)$.

Obviously, in the quasigroup $C(q_i) = (\mathbb{C}, *)$, $i = 1,2,3,4$, the midpoint m of any segment $\{a,b\}$ is unique and can be expressed as $m = \frac{a+b}{2}$. However, that is not the case for all pentagonal quasigroups.

Example 2.1. Pentagonal quasigroup $(Q_{16},\cdot)$ is constructed from the group $\mathbb{Z}_2^4$ and the automorphism $\varphi(x,y,z,w) = (w,x,y,z+w)$ satisfying (1.12).

Let $a,b \in Q_{16}$ be such that $a \neq b$. We notice that there exists no $x \in Q_{16}$ such that $M(a,x,b)$, while for every $x \in Q_{16}$ $M(a,x,a)$ holds. Thus the segment $\{a,a\}$ has sixteen distinct midpoints in Q_{16}. The reason behind that lies in the Abelian group associated to the quasigroup $(Q_{16},\cdot)$ by Theorem 1.2, which is $\mathbb{Z}_2^4$. Element x is midpoint of the segment $\{a,b\}$ if and only if $2x = a+b$ holds. Since two different elements of $\mathbb{Z}_2^4$ that add up to the identity element do not exist, there exists no $x \in Q_{16}$ such that $M(a,x,b)$. Since every element of the group $\mathbb{Z}_2^4$ is of order 2, for every $x \in Q_{16}$, we have $M(a,x,a)$.

The following theorem will be used repeatedly in the next section.

Theorem 2.2. *In the pentagonal quasigroup* $(Q,\cdot)$ *the identity*

$$((((ba \cdot a)a \cdot a)a \cdot a)a \cdot a)a = (ab \cdot a)b \tag{2.3}$$

holds for every $a, b \in Q$.

PROOF. Obviously $\mathrm{Par}(a, b, a, ((ba \cdot a)a \cdot a)a)$ holds. Using property ii) of the parallelogram space, $\mathrm{Par}(a, ((ba \cdot a)a \cdot a)a, a, b)$ holds as well. That gives the identity $((((ba \cdot a)a \cdot a)a \cdot a)a \cdot a)a \cdot a = b$. Applying (1.6) to the right-hand side and cancelling from the right by a gives the desired identity. □

3. Regular pentagon and regular decagon

Motivated by Figure 2 the following concept can be defined in a general pentagonal quasigroup.

DEFINITION 3.1. Let (a, b, c, d, e) be a pentagon in a pentagonal quasigroup $(Q, \cdot)$. We say that (a, b, c, d, e) is a *regular pentagon* and denote it by $\mathrm{RP}(a, b, c, d, e)$ if $ab = c$, $bc = d$ and $cd = e$ hold.

We immediately note that any regular pentagon in a pentagonal quasigroup is uniquely determined by its pair of adjacent vertices. The remaining vertices can be expressed in terms of these two vertices using operation in the quasigroup.

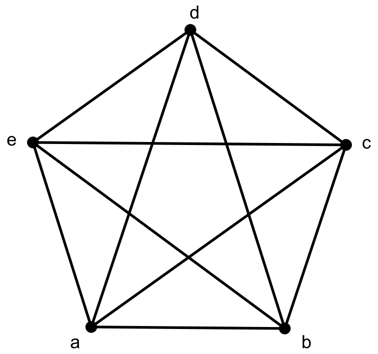

FIGURE 2. Regular pentagon (a, b, c, d, e)

THEOREM 3.1. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d, e \in Q$ *such that* $\mathrm{RP}(a, b, c, d, e)$ *holds. Then* $d = ba \cdot a$ *and* $e = (ab \cdot a)b$ *hold.*

PROOF. We directly compute

$$d = bc = b \cdot ab \overset{(1.3)}{=} ba \cdot b, \quad e = cd = ab \cdot (ba \cdot b) \overset{(1.5)}{=} (a \cdot ba)b \overset{(1.3)}{=} (ab \cdot a)b. \quad \square$$

THEOREM 3.2. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d, e \in Q$ *such that* $\mathrm{RP}(a, b, c, d, e)$ *holds. Then* $de = a$ *and* $ea = b$ *hold.*

PROOF. Using the previous theorem we compute

$$de = (ba \cdot b) \cdot (ab \cdot a)b \overset{(1.5)}{=} (ba \cdot (ab \cdot a))b \overset{(1.5)}{=} (b \cdot ab)a \cdot b \overset{(1.3)}{=} (ba \cdot b)a \cdot b \overset{(1.6)}{=} a,$$

$$ea = (ab \cdot a)b \cdot a \overset{(1.6)}{=} b. \quad \square$$

Next two corollaries are direct consequences of the previous theorem.

COROLLARY 3.1. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d, e \in Q$. *Then*

$$\mathrm{RP}(a, b, c, d, e) \Leftrightarrow ab = c,\ bc = d,\ cd = e,\ de = a,\ ea = b.$$

COROLLARY 3.2. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d, e \in Q$. *Then* $\mathrm{RP}(a, b, c, d, e)$ *implies* $\mathrm{RP}(p, q, r, s, t)$, *where* (p, q, r, s, t) *is any cyclic permutation of* (a, b, c, d, e).

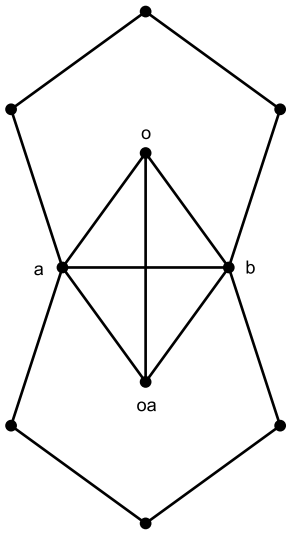

FIGURE 3. Centre of regular pentagon

Figure 3 motivates introduction of the following geometric concept in pentagonal quasigroups.

DEFINITION 3.2. Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, c, d, e \in Q$ such that $\mathrm{RP}(a, b, c, d, e)$ holds. We say that $o \in Q$ is the *centre* of the regular pentagon (a, b, c, d, e) if $oa \cdot b = o$ holds.

THEOREM 3.3. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* $a, b, c, d, e, o \in Q$ *such that* $\mathrm{RP}(a, b, c, d, e)$ *holds and* o *is the centre of* (a, b, c, d, e). *Then* $ob \cdot c = o$, $oc \cdot d = o$, $od \cdot e = o$ *and* $oe \cdot a = o$ *hold.*

PROOF. We compute $ob \cdot c = ob \cdot ab \overset{(1.5)}{=} oa \cdot b = o$. The other identities are proved analogously. $\square$

The centre of a regular pentagon in pentagonal quasigroups does not have to exist as explained in the next example.

TABLE 1. Pentagonal quasigroup Q_5

$\cdot$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

EXAMPLE 3.1. The quasigroup Q_5 (Table 1) is constructed from the Abelian group $\mathbb{Z}_5$ using the automorphism $\varphi(x) = 2x$ which satisfies (1.12).

In $(Q_5, \cdot)$ the statement $\mathrm{RP}(0, 1, 2, 3, 4)$ holds. The pentagon $(0, 1, 2, 3, 4)$ is determined by its vertices 0 and 1. Now we want to try all possible candidates for the centre o of $(0, 1, 2, 3, 4)$:

$$00 \cdot 1 = 2 \neq 0,\ 10 \cdot 1 = 3 \neq 1,\ 20 \cdot 1 = 4 \neq 2, 30 \cdot 1 = 0 \neq 3,\ 40 \cdot 1 = 1 \neq 4.$$

Hence, we conclude that the regular pentagon $(0, 1, 2, 3, 4)$ in Q_5 does not have a centre. Algebraic justification for that is the use of Theorem 1.2. If we apply $a \cdot b = a + \varphi(b - a)$ to the identity $oa \cdot b = o$, we get successively:

$$\begin{aligned} (\mathbf{1} - \varphi)(oa) + \varphi(b) &= o \\ (\mathbf{1} - \varphi)^2(o) + (\mathbf{1} - \varphi)\varphi(a) + \varphi(b) &= o \\ o - 2\varphi(o) + \varphi^2(o) - o &= -\varphi(\mathbf{1} - \varphi)(a) - \varphi(b) \\ \varphi(2 \cdot \mathbf{1} - \varphi)(o) &= \varphi(\mathbf{1} - \varphi)(a) + \varphi(b). \end{aligned}$$

Since φ is an isomorphism, it is equivalent to

$$(2 \cdot \mathbf{1} - \varphi)(o) = (\mathbf{1} - \varphi)(a) + b.$$

The function $2 \cdot \mathbf{1} - \varphi = 0$ is not invertible, so there exists no $o \in Q_5$ which satisfies the previous equation.

Figure 4 gives motivation for the next definition.

DEFINITION 3.3. We say that points $a, b, c, d, e, f, g, h, i, j$ of the pentagonal quasigroup $(Q, \cdot)$ form a *regular decagon* with the *centre* $o \in Q$ if

$$ba \cdot b = cb \cdot c = dc \cdot d = ed \cdot e = fe \cdot f = gf \cdot g = hg \cdot h = ih \cdot i = ji \cdot j = o.$$

We denote it by $\mathrm{RD}_o(a, b, c, d, e, f, g, h, i, j)$ or, if we want to omit the centre, by $\mathrm{RD}(a, b, c, d, e, f, g, h, i, j)$.

We note that any regular decagon in a pentagonal quasigroup is uniquely determined by its pair of adjacent vertices.

LEMMA 3.1. *Let $(Q, \cdot)$ be a pentagonal quasigroup and $a, b, o \in Q$. Then $ba \cdot b = o$ is equivalent to $b = (ao \cdot o)o$.*

PROOF. Multiplying the identity $ba \cdot b = o$ from the right by a and b respectively we get $(ba \cdot b)a \cdot b = oa \cdot b$. Applying (1.6) we get $a = oa \cdot b$ and (1.9) finally gives $b = (ao \cdot o)o$. $\square$

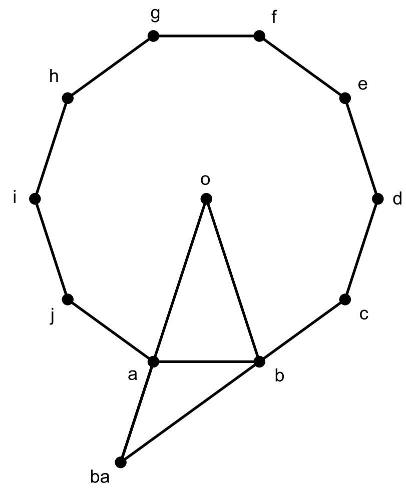

FIGURE 4. Regular decagon $(a, b, c, d, e, f, g, h, i, j)$ with the centre o

THEOREM 3.4. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $o \in Q$. *Then* $\mathrm{RD}_o(a, b, c, d, e, f, g, h, i, j)$ *implies* $j = co$ *and* $j = (((ao{\cdot}o)o{\cdot}o)o{\cdot}o)o$.

PROOF. Multiple use of Lemma 3.1 gives:

$$\begin{aligned}
&j = (io \cdot o)o = ((ho \cdot o)o \cdot o)o \cdot o = ((((go \cdot o)o \cdot o)o \cdot o)o \cdot o)o \\
&\overset{(2.3)}{=} (og \cdot o)g = (og \cdot o) \cdot (fo \cdot o)o \overset{(1.5)}{=} (og \cdot (fo \cdot o))o \overset{(1.2),(1.3)}{=} ((of \cdot o) \cdot go)o \\
&\overset{(1.5)}{=} (of \cdot g)o \cdot o = (of \cdot (fo \cdot o)o)o \cdot o \overset{(1.9)}{=} fo \cdot o = ((eo \cdot o)o \cdot o)o \\
&= (((do \cdot o)o \cdot o)o \cdot o)o \cdot o = (((((co \cdot o)o \cdot o)o \cdot o)o \cdot o)o \cdot o)o \\
&\overset{(1.4)}{=} ((oc \cdot o)c \cdot o)o \overset{(1.6)}{=} co = (bo \cdot o)o \cdot o = (((ao \cdot o)o \cdot o)o \cdot o)o.
\end{aligned}$$
□

COROLLARY 3.3. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $o \in Q$. *Then* $\mathrm{RD}_o(a, b, c, d, e, f, g, h, i, j)$ *implies the following identities*

$$a = do, b = eo, c = fo, d = go, e = ho, f = io, g = jo, h = ao, i = bo, j = co.$$

THEOREM 3.5. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $o \in Q$. *Then* $\mathrm{RD}_o(a, b, c, d, e, f, g, h, i, j)$ *implies* $aj \cdot a = o$.

PROOF. Using Theorem 3.4 we get

$$\begin{aligned}
aj \cdot a &= (a \cdot (((ao \cdot o)o \cdot o)o \cdot o)o)a \overset{(1.6)}{=} (((oa \cdot o)a \cdot o) \cdot (((ao \cdot o)o \cdot o)o \cdot o)o)a \\
&\overset{(1.5)}{=} ((oa \cdot o)a \cdot (((ao \cdot o)o \cdot o)o \cdot o))o \cdot a \overset{(1.2)}{=} (((oa \cdot o) \cdot ((ao \cdot o)o \cdot o)o) \cdot ao)o \cdot a \\
&\overset{(1.5)}{=} ((oa \cdot ((ao \cdot o)o \cdot o))o \cdot ao)o \cdot a \overset{(1.5)}{=} ((oa \cdot ((ao \cdot o)o \cdot o))a \cdot o)o \cdot a \\
&\overset{(1.4),(1.9)}{=} ((a(oa \cdot o) \cdot a)o \cdot o)a \overset{(1.5),(1.6)}{=} (ao \cdot a)o \cdot a \overset{(1.6)}{=} o.
\end{aligned}$$
□

Corollary 3.4. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $o \in Q$. *Any nine of ten identities*

$$ba \cdot b = o, \quad cb \cdot c = o, \quad dc \cdot d = o, \quad ed \cdot e = o, \quad fe \cdot f = o,$$
$$gf \cdot g = o, \quad hg \cdot h = o, \quad ih \cdot i = o, \quad ji \cdot j = o, \quad aj \cdot a = o$$

imply the remaining one.

Corollary 3.5. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $o \in Q$. *Any nine of ten identities*

$$ba \cdot b = o, \quad cb \cdot c = o, \quad dc \cdot d = o, \quad ed \cdot e = o, \quad fe \cdot f = o,$$
$$gf \cdot g = o, \quad hg \cdot h = o, \quad ih \cdot i = o, \quad ji \cdot j = o, \quad aj \cdot a = o$$

imply $\mathrm{RD}_o(a, b, c, d, e, f, g, h, i, j)$.

4. Two theorems of Euclidean plane

The following theorem was proved in [**5**].

Theorem 4.1. *In a medial quasigroup any two of three statements* $\mathrm{Par}(a, b, c, d)$, $\mathrm{Par}(e, f, g, h)$, $\mathrm{Par}(ae, bf, cg, dh)$ *imply the remaining one.*

As an illustration of the concepts introduced before we give versions of two theorems of Euclidean plane in pentagonal quasigroups.

Theorem 4.2. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, a_1, b_1, c_1, d_1, e_1, $\in Q$ *such that* $\mathrm{RP}(a, b, c, d, e)$, $M(a, a_1, b)$, $M(b, b_1, c)$ *hold. Then*

$$\mathrm{RP}(a_1, b_1, c_1, d_1, e_1) \Rightarrow M(c, c_1, d),\ M(d, d_1, e),\ M(e, e_1, a).$$

Proof. Statements $M(a, a_1, b)$ and $M(b, b_1, c)$ are respectively equivalent to $\mathrm{Par}(a, a_1, b, a_1)$ and $\mathrm{Par}(b, b_1, c, b_1)$. By Theorem 4.1 we have $\mathrm{Par}(ab, a_1b_1, bc, a_1b_1)$. Since we have $\mathrm{RP}(a, b, c, d, e)$, we get $\mathrm{Par}(c, a_1b_1, d, a_1b_1)$ or $M(c, a_1b_1, d)$. The statement $\mathrm{RP}(a_1, b_1, c_1, d_1, e_1)$ gives $a_1b_1 = c_1$. Hence we have $M(c, c_1, d)$. The statements $M(d, d_1, e)$ and $M(e, e_1, a)$ are proved analogously. □

Theorem 4.3. *Let* $(Q, \cdot)$ *be a pentagonal quasigroup and* a, b, c, d, e, f, g, h, i, j, $\in Q$. *We have*

$$\mathrm{RD}(a, b, c, d, e, f, g, h, i, j) \Rightarrow \mathrm{RP}(a, c, e, g, i),\ \mathrm{RP}(b, d, f, h, j).$$

Proof. Let us denote by o the centre of the regular decagon $(a, b, c, d, e, f, g, h, i, j)$. Since $cb \cdot c = o$, by Lemma 3.1 we have $c = (bo \cdot o)o$. Corollary 3.3 gives $b = eo$, wherefrom using (1.6) we get $e = (ob \cdot o)b$. Also, we have $ba \cdot b = o$, wherefrom using (1.6) we get $ba = (bo \cdot b)o$. Using (1.3) and (1.10) on that we get $a = ((bo \cdot b)o \cdot (((bo \cdot b)o \cdot b)b \cdot b))b$. Furthermore, we have:

$$a = ((bo \cdot b)o \cdot (((bo \cdot b)o \cdot b)b \cdot b))b \overset{(1.6)}{=} ((bo \cdot b)o \cdot (ob \cdot b))b$$
$$\overset{(1.5)}{=} ((bo \cdot b)o \cdot b) \cdot (ob \cdot b)b \overset{(1.6)}{=} o \cdot (ob \cdot b)b.$$

Now we compute:

$$ac = (o \cdot (ob \cdot b)b) \cdot (bo \cdot o)o \overset{(1.4)}{=} (o(ob \cdot b) \cdot ob) \cdot (bo \cdot o)o$$
$$\overset{(1.5),(1.9)}{=} (o(ob \cdot b) \cdot (bo \cdot o)o)b \overset{(1.4)}{=} (((o \cdot ob) \cdot ob) \cdot (bo \cdot o)o)b$$
$$\overset{(1.5),(1.9)}{=} ((o \cdot ob) \cdot (bo \cdot o)o)b \cdot b \overset{(1.4),(1.8),(1.9)}{=} ((bo \cdot b)b \cdot b)b \overset{(1.8)}{=} (ob \cdot o)b.$$

That shows $ac = e$. Identities $ce = g$ and $eg = i$ are proved analogously. Those three identities imply $\mathrm{RP}(a, c, e, g, i)$. The statement $\mathrm{RP}(b, d, f, h, j)$ is proved analogously. □

References

1. K. Toyoda, *On axioms of linear functions*, Proc. Imp. Acad. Tokyo **17** (1941), 221–227.
2. S. Vidak, *Pentagonal quasigroups*, Quasigroups Relat. Syst. **22** (2014), 147–158.
3. V. Volenec, *Admissible identities in complex IM-quasigroups I*, Rad JAZU, Mat. Znan. **4** (1985), 61–85.
4. ______, *Admissible identities in complex IM-quasigroups II*, Rad JAZU, Mat. Znan. **5** (1986), 45–78.
5. ______, *Geometry of medial quasigroups*, Rad JAZU, Mat. Znan. **5** (1986), 79–91.
6. ______, *Geometry of IM-quasigroups*, Rad HAZU, Mat. Znan. **10** (1991), 139–146.
7. ______, *GS-quasigroups*, Čas. Pěst. Mat. **115** (1990), 307–318.
8. ______, *Hexagonal quasigroups*, Arch. Math., Brno **27a** (1991), 113–122.

Department of Mathematics
University of Zagreb
Zagreb
Croatia
svidak@math.pmf.hr

(Received 17 12 2014)