PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 121–124 DOI: 10.2298/PIM150430027L

A NOTE ON GAUTSCHI'S INEQUALITY AND APPLICATION TO WALLIS' AND STIRLING'S FORMULA

Martin Lukarevski

Abstract. We present novel elementary proofs of Stirling's approximation formula and Wallis' product formula, both based on Gautschi's inequality for the Gamma function.

1. Introduction

The Gamma function, defined for $x > 0$ by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

also called the second Eulerian integral is, according to Philip Davis, "undoubtedly the most fundamental of the so-called 'higher mathematical functions'. It is simple enough for juniors in college to meet, but deep enough to have called forth contributions from the finest mathematicians" (see the excellent article [**3**] for its intricate and intriguing history). For two sequences (a_n) and (b_n), $a_n \sim b_n$ means that their ratio tends to 1 as n tends to infinity. $f(x) \sim g(x)$ denotes the same for functions. The Scottish mathematician James Stirling (1692–1770) gave in 1730 in *Methodus Differentialis* the famous approximation

$$n! \sim \sqrt{2\pi n}\,(n/e)^n,$$

which has many applications in probability theory and statistics. Integration by parts yields that Gamma function satisfies the fundamental recurrence relation $\Gamma(x+1) = x\Gamma(x)$. It follows that $\Gamma(n+1) = n!$ for every positive integer n. Being the most natural extension of the factorial, Gamma function itself has the analogous approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x}\,(x/e)^x.$$

2010 *Mathematics Subject Classification:* 41A60.

Key words and phrases: Gautschi's inequality; Gamma function; Stirling's approximation; Wallis' product.

Communicated by Gradimir Milovanović.

We will refer to them as little and big Stirling approximation respectively. In 1656 the Englishman John Wallis (1616–1703) showed in his work *Arithmetica Infinitorum* the remarkable infinite product

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots=\frac{\pi}{2}.$$

We show that the results follow very easily from an elementary inequality for the Gamma function, known as Gautschi's inequality (see [**4**]). For the proofs we need only three facts about Gamma. It is logarithmically convex function, which means that $\log\Gamma(x)$ is convex

$$\Gamma(sx+(1-s)y)\leqslant\Gamma(x)^s\Gamma(y)^{1-s},\qquad \text{for } x,y>0,\quad 0<s<1.$$

Gamma has the value

$$\Gamma\Big(n+\frac{1}{2}\Big)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}, \tag{1.1}$$

and it satisfies the Legendre duplication formula

$$\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma\Big(x+\frac{1}{2}\Big)/\sqrt{\pi}. \tag{1.2}$$

The value is obtained from $\Gamma(1/2)=\sqrt{\pi}$ and repeated use of $\Gamma(x+1)=x\Gamma(x)$. For the first and the third facts, and much more about Γ, the reader is referred to Artin [**2**].

2. Gautschi's inequality

The following result is Gautschi's inequality (see [**4**]).

Theorem 1. *For all $x>0$ and $0<s<1$ it holds*

$$(x+s)^{s-1}\leqslant\frac{\Gamma(x+s)}{\Gamma(x+1)}\leqslant x^{s-1}.$$

In the following theorem the two approximations are consequence of the asymptotic expansion of the binomial coefficient $\binom{2n}{n}$.

Theorem 2. *It holds*

$$(n+1)(n+2)\cdots(n+n)\sim\sqrt{2}\,2^{2n}(n/e)^n, \tag{2.1}$$

$$\binom{2n}{n}\frac{1}{2^{2n}}\sim\frac{1}{\sqrt{\pi n}}. \tag{2.2}$$

Remark 1. One can conjecture the asymptotics $\binom{2n}{n}\frac{1}{2^{2n}}\sim C/\sqrt{n}$, for some constant C, from some weak estimates. For example,

$$\frac{\sqrt{5}}{4}\frac{1}{\sqrt{n+1/4}}\leqslant\binom{2n}{n}\frac{1}{2^{2n}}\leqslant\frac{3}{2\sqrt{7}}\frac{1}{\sqrt{n+2/7}}$$

is given in [**5**, p. 12, Aufgabe 3]. Even the crude estimate $\binom{2n}{n}\frac{1}{2^{2n}}\geqslant\frac{1}{2n}$ is fruitful in number theory, see [**1**, Chapter 2].

3. Proof of the little Stirling and Wallis' formula

We derive the Stirling's formula by putting together the two approximations (2.1) and (2.2).

$$n! = \frac{(n+1)(n+2)\cdot \ldots \cdot (n+n)}{\binom{2n}{n}} \sim \sqrt{2}\, 2^{2n}(n/e)^n \frac{\sqrt{\pi n}}{2^{2n}} = \sqrt{2\pi n}\,(n/e)^n.$$

Gautschi's inequality can be rewritten in an equivalent form

$$\Big(\frac{x}{x+s}\Big)^{1-s} \leqslant \frac{\Gamma(x+s)}{x^s\Gamma(x)} \leqslant 1,$$

and as a corollary one gets the classical asymptotic relation, see [**7**]

$$\lim_{x\to\infty} \frac{\Gamma(x+s)}{x^s\Gamma(x)} = 1. \tag{3.1}$$

Wallis' formula now appears as a special case of this very interesting limit. It follows that $\lim_{n\to\infty} a_n = 1$ where

$$a_n := \frac{\Gamma^2(n+1)}{\Gamma\big(n+\frac{1}{2}\big)\Gamma\big(n+\frac{3}{2}\big)}.$$

Using (1.1), we see that

$$a_n = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi}.$$

Hence Wallis' product formula.

Remark 2. The two results are closely related. A standard way of deriving Stirling's formula is from Wallis' product and this in turn is derived from calculating the integrals $\int_0^{\pi/2}\sin^n x\,dx$. Conversely, one can use Stirling's formula to show Wallis' product. It is not surprising that the Gamma function is intrinsic to both of them.

4. Proof of the big Stirling

Theorem 3. *For* $f(x) := \frac{\Gamma(x+1)}{\sqrt{x}}(e/x)^x$ *we have* $\lim_{x\to\infty} f(x) = \sqrt{2\pi}$.

Proof. $f(x) = \int_0^\infty e^{x-t}\frac{t^{x-1}}{x^{x-1/2}}\,dt$. With two consecutive changes of variables, $t := u^2$ the first, and $u := \sqrt{x}+y$ the second, the function is transformed to

$$f(x) = \int_{-\sqrt{x}}^{\infty} e^{-2\sqrt{x}y}\Big(1+\frac{y}{\sqrt{x}}\Big)^{2x-1} e^{-y^2}\,dy,$$

and as in [**6**] is easily seen to converge to some finite constant $C := \lim_{x\to\infty} f(x)$. It remains to determine this constant. One can proceed as in [**6**] and use the delicate Lebesgue dominated convergence theorem to pass under the integral sign. But we can avoid that in the following way $C = \lim_{x\to\infty}\frac{f(x)^2}{f(2x)}$. In the calculation of the last fraction we use Legendre duplication formula (1.2) for $\Gamma(2x+1)$

$$\Gamma(2x+1) = 2^{2x}\Gamma\Big(x+\frac{1}{2}\Big)\Gamma(x+1)/\sqrt{\pi}.$$

Then

$$\frac{f(x)^2}{f(2x)} = \frac{\Gamma^2(x+1)}{x\Gamma(2x+1)}(e/x)^{2x}\sqrt{2x}\,(2x/e)^{2x} = \frac{\Gamma(x)\sqrt{2x}}{\Gamma\left(x+\frac{1}{2}\right)}\sqrt{\pi}.$$

Letting x tend to infinity and using limit (3.1), we get that $C = \sqrt{2\pi}$, completing the proof of the big Stirling approximation formula. $\square$

References

1. M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, Springer-Verlag, Berlin, 1998
2. E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, San Francisco, 1964
3. P. J. Davis, *Leonhard Euler's integral: historical profile of the Gamma function*, Am. Math. Monthly **66** (1959), 849–869
4. W. Gautschi, *Some elementary inequalities relating to the Gamma and incomplete Gamma function*, J. Math. Phys. **38** (1959), 77–81
5. F. Neiss, *Determinanten und Matrizen*, 7. Aufl. Berlin, 1967
6. J. M. Patin, *A very short proof of Stirling's formula*, Am. Math. Monthly **96** (1989), 41–42
7. J. G. Wendel, *Note on the Gamma function*, Am. Math. Monthly **55** (1948), 563–564

Department of Mathematics and Statistics
University "Goce Delčev"
Štip
Macedonia
Martin.Lukarevski@ugd.edu.mk

(Received 24 11 2014)
(Revised 30 04 2015)