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VERTEX DECOMPOSABLE GRAPH

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Abstract. Let G be a simple graph on the vertex set $V(G)$ and $S = \{x_{11}, \dots, x_{n1}\}$ a subset of $V(G)$. Let $m_1, \dots, m_n \geqslant 2$ be integers and $G_1, \dots, G_n$ connected simple graphs on the vertex sets $V(G_i) = \{x_{i1}, \dots, x_{im_i}\}$ for $i = 1, \dots, n$. The graph $G(G_1, \dots, G_n)$ is obtained from G by attaching G_i to G at the vertex x_{i1} for $i = 1, \dots, n$. We give a characterization of $G(G_1, \dots, G_n)$ for being vertex decomposable. This generalizes a result due to Mousivand, Seyed Fakhari, and Yassemi.

1. Introduction

Let G be a finite simple graph (no loops or multiple edges) on the vertex set $V(G) = \{x_1, \dots, x_m\}$ and edge set $E(G)$. Adding a *whisker* to G at x_i means adding a new vertex y and edge $\{x_i, y\}$ to G. Villarreal showed that the graph obtained from G by adding a whisker to every vertex of G is Cohen–Macaulay [**7**, Proposition 2.2]. Moreover, it was shown that this graph is also unmixed and vertex decomposable [**4**, Theorem 4.4]. Adding a whisker to a vertex of G is the same as saying that attaching the complete graph K_2 to a vertex of G. Let $S = \{x_{11}, \dots, x_{n1}\}$ be a subset of $V(G)$. The graph $G(G_1, \dots, G_n)$ is obtained from G by *attaching* G_i to G at the vertex x_{i1} for $i = 1, \dots, n$. Recently, Mousivand, Fakhari and Yassemi characterized when the graph obtained by attaching arbitrary graphs to every vertex of a given graph is vertex decomposable [**6**, Proposition 2.3]. We generalize this result (see Theorem 3.1).

2. Preliminaries

We recall some definitions and properties that will be used later. Let K be a field. To any finite simple graph G (no loops or multiple edges) with vertex set $V(G) = \{x_1, \dots, x_m\}$ and edge set $E(G)$ one associates *edge ideal* $I(G) \subset K[x_1, \dots, x_m]$ whose generators are all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$.

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The graph G is called *Cohen–Macaulay* over K if $K[x_1, \dots, x_m]/I(G)$ has this property.

Let G be a graph with the vertex set $V(G) = \{x_1, \dots, x_m\}$ and the edge set $E(G)$. The *induced subgraph* $G|_W$ for $W \subseteq V(G)$ is defined by

$$G|_W := (W, \{e \in E(G); e \subset W\}).$$

For $W \subseteq V(G)$ we denote $G|_{V(G) \setminus W}$ by $G \setminus W$. If $W = \{x\}$, we write $G \setminus x$ instead of $G \setminus \{x\}$. For any $x \in V(G)$ we denote the *open neighbor* set of x in G by $N_G(x)$, i.e., $N_G(x) := \{y \in V(G) \mid \{x, y\} \in E(G)\}$, $N_G[x] := N_G(x) \cup \{x\}$. For $W \subseteq V(G)$, we set $N_G[W] = \bigcup_{x \in W} N_G[x]$.

A graph is *complete* if every two of its vertices are adjacent. The complete graph with n vertices is denoted by K_n.

A *simplicial complex* Δ on a finite set V is a collection of subsets of V which is closed under inclusion. Members of Δ are called *faces*. The maximum faces of Δ with respect to the inclusion are called the *facets*. The *dimension* of a face F is $|F| - 1$ and the dimension of a complex Δ is the maximum of the dimensions of its facets. If all the facets of Δ have the same dimension we say that Δ is *pure* and a complex with a unique facet is called a *simplex*. Let Δ be a simplicial complex on the vertex set V. The *induced subcomplex* of $\Delta|_S$ for $S \subseteq V$ is defined by $\Delta|_S := \{F \in \Delta \mid F \subseteq S\}$.

Let F be a face of Δ. The *link* and the *deletion* of F from Δ are the simplicial complex defined by

$$\operatorname{link}_\Delta F := \{G \in \Delta \mid G \cap F = \emptyset, G \cup F \in \Delta\}, \quad \operatorname{del}_\Delta F := \{G \in \Delta \mid F \nsubseteq G\}.$$

If $F = \{x\}$, we write $\operatorname{link}_\Delta x$ (resp. $\operatorname{del}_\Delta x$) instead of $\operatorname{link}_\Delta \{x\}$ (resp. $\operatorname{del}_\Delta \{x\}$). See [**5**] for detailed information.

Let $G = (V(G), E(G))$ be a graph. A subset F of $V(G)$ is called an *independent set* if no two vertices of F are adjacent. The independence complex of G, denoted by $\operatorname{Ind}(G)$, is the simplicial complex on the vertex set $V(G)$, defined by

$$\operatorname{Ind}(G) := \{F \subseteq G \mid F \text{ is an independent set of G}\}.$$

A simplicial complex Δ is called *vertex decomposable* if it is a simplex or else has a vertex x such that

(1) both $\operatorname{del}_\Delta x$ and $\operatorname{link}_\Delta x$ are vertex decomposable, and
(2) there is no face of $\operatorname{link}_\Delta x$ which is also a facet of $\operatorname{del}_\Delta x$.

A *shedding vertex* is the vertex x which satisfies condition (2).

Remark 2.1. Our definition of shedding vertex is slightly different with the definition in [**6**], where a shedding vertex is the one which satisfies conditions (1) and (2).

Vertex decomposability was introduced in the pure case by Billera and Provan [**2**] and extended to nonpure complexes by Björner and Wachs [**3**]. A graph G is called *vertex decomposable* if $\operatorname{Ind}(G)$ has this property. In [**8**] Woodroofe translated the notion of vertex decomposability for graphs as follows.

A graph G is *vertex decomposable* if it is totally disconnected (with no edges) or else has some vertex x such that

(1) both $G \smallsetminus N_G[x]$ and $G \smallsetminus x$ are vertex decomposable, and
(2) for every independent set S in $G \smallsetminus N_G[x]$, there exists some $y \in N_G(x)$ such that $S \cup \{y\}$ is independent in $G \smallsetminus x$.

A vertex x which satisfies condition (2) is called a *shedding vertex* for G.

A graph G is called *chordal* if every induced cycle of G of length $\geqslant 4$ has a chord. In [**8**] Woodroofe proved that every chordal graph is vertex decomposable.

3. Vertex decomposable graph

Here we prove our main result. We write $B \dot\cup C$ for disjoint union.

Theorem 3.1. *Let G be a simple graph on the vertex set $V(G)$ and $S = \{x_{11}, \dots, x_{n1}\}$ a subset of $V(G)$. Let $m_1, \dots, m_n \geqslant 2$ be integers and $G_1, \dots, G_n$ connected graphs on the vertex set $V(G_i) = \{x_{i1}, \dots, x_{im_i}\}$ for $i = 1, \dots, n$ and suppose that x_{i1} is a shedding vertex of G_i for $i = 1, \dots, n$. Then $G(G_1, \dots, G_n)$ is vertex decomposable if and only if*

(i) *$G_i \smallsetminus x_{i1}$ and $G_i \smallsetminus N_{G_i}[x_{i1}]$ are vertex decomposable, for $i = 1, \dots, n$, and*
(ii) *for any face μ of $\Delta = \operatorname{Ind}(G)$ with $\mu \subseteq S$, the complex $\operatorname{link}_\Delta \mu|_{\bar S}$ is vertex decomposable, where $\bar S = V(G) \smallsetminus S$.*

Proof. ($\Leftarrow$) The graph $G(G_1, \dots, G_n)$ is obtained from G by attaching G_i to G at the vertex x_{i1} for $i = 1, \dots, n$. We prove the assertion by induction on $n = |S|$. Set $G' = G(G_1, \dots, G_n)$. If $S = \emptyset$, then $\mu = \emptyset$, $G' = G$ and $\operatorname{link}_\Delta \mu|_{\bar S} = \operatorname{Ind}(G)$ is vertex decomposable by hypothesis.

Assume that $n \geqslant 1$ and the assertion holds for any graph G and every $S \subseteq V(G)$ such that $|S| \leqslant n-1$. Suppose $S = \{x_{11}, \dots, x_{n1}\}$. We prove that x_{11} is a shedding vertex of G'. Assume that A is an independent set in $G' \smallsetminus N_{G'}[x_{11}]$. We have $A = B \dot\cup C$ where $B = A \cap [V(G_2) \cup \dots \cup V(G_n) \cup (V(G) \smallsetminus \{x_{21}, \dots, x_{n1}\})]$ and $C = A \cap V(G_1)$. Since $N_{G_1}[x_{11}] \subseteq N_{G'}[x_{11}]$, it follows that C is an independent set in $G_1 \smallsetminus N_{G_1}[x_{11}]$. There exists an $x_{1k} \in N_{G_1}(x_{11})$ such that $C \cup \{x_{1k}\}$ is independent set in $G_1 \smallsetminus x_{11}$, because x_{11} is a shedding vertex of G_1. For any $y \in V(G_2) \cup \dots \cup V(G_n) \cup (V(G) \smallsetminus \{x_{21}, \dots, x_{n1}\})$, we have $\{x_{1k}, y\} \notin E(G')$. Thus $A \cup \{x_{1k}\}$ is independent in $G' \smallsetminus x_{11}$ and so x_{11} is a shedding vertex of G'.

Set $S_1 = \{x_{21}, \dots, x_{n1}\}$ and $\Delta' = \operatorname{Ind}(G \smallsetminus x_{11})$. We have $G' \smallsetminus x_{11} = G'_1 \dot\cup G'_2$ where $G'_1 = G_1 \smallsetminus x_{11}$ and $G'_2 = (G \smallsetminus x_{11})(G_2, \dots, G_n)$. Let $\mu \in \Delta'$ with $\mu \subseteq S_1$. Thus $\mu \in \Delta$ and $\mu \subseteq S$. We have

$$\begin{aligned}\operatorname{link}_{\Delta'} \mu|_{\bar S_1} &= \{F \in \Delta' \mid F \subseteq \bar S_1,\ F \cup \mu \in \Delta',\ F \cap \mu = \emptyset\} \\ &= \{F \in \Delta \mid F \subseteq \bar S,\ F \cup \mu \in \Delta,\ F \cap \mu = \emptyset\} = \operatorname{link}_\Delta \mu|_{\bar S}.\end{aligned}$$

Thus we have that $\operatorname{link}_{\Delta'} \mu|_{\bar S_1}$ is vertex decomposable by hypothesis. Therefore G'_2 is vertex decomposable by induction and G'_1 is vertex decomposable by hypothesis. It now follows from [**8**, Lemma 20] that $G' \smallsetminus x_{11}$ is vertex decomposable.

Now consider $G' \smallsetminus N_{G'}[x_{11}]$. If $S \subseteq N_{G'}[x_{11}]$, then

$$G' \smallsetminus N_{G'}[x_{11}] = (G \smallsetminus N_G[x_{11}]) \dot\cup (G_1 \smallsetminus N_{G_1}[x_{11}] \dot\cup G_2 \smallsetminus x_{21} \dot\cup \cdots \dot\cup G_n \smallsetminus x_{n1}).$$

Since $S \subseteq V(G)$, it follows that $S \subseteq N_G[x_{11}]$ and $V(G \smallsetminus N_G[x_{11}]) \subseteq \bar S$.

Set $\mu = \{x_{11}\}$. We have $\mu \in \Delta$, $\mu \subseteq S$ and $\mathrm{link}_\Delta\, \mu|_{\bar{S}} = \mathrm{Ind}(G \setminus N_G[x_{11}])$ is vertex decomposable by hypothesis. Also, for $2 \leqslant i \leqslant n$, $G_i \setminus x_{i1}$ and $G_1 \setminus N_{G_1}[x_{11}]$ are vertex decomposable and hence $(G_1 \setminus N_{G_1}[x_{11}] \,\dot\cup\, G_2 \setminus x_{21} \,\dot\cup \cdots \dot\cup\, G_n \setminus x_{n1})$ is vertex decomposable by [**8**, Lemma 20]. Thus $G' \setminus N_{G'}[x_{11}]$ is vertex decomposable.

Assume $S \not\subseteq N_{G'}[x_{11}]$. After relabeling the vertices of G, we assume that $S \cap N_G[x_{11}] = \{x_{11}, \dots, x_{l1}\}$ and

$$G' \setminus N_{G'}[x_{11}] = G_0(G_{l+1}, \dots, G_n) \,\dot\cup\, (G_1 \setminus N_{G_1}[x_{11}] \,\dot\cup\, G_2 \setminus x_{21} \,\dot\cup \cdots \dot\cup\, G_l \setminus x_{l1})$$

where $G_0 = (G \setminus N_G[x_{11}])$. Set $S_1 = S \setminus N_G[x_{11}] = \{x_{l+11}, \dots, x_{n1}\}$. Let $\mu \in \mathrm{link}_\Delta\, x_{11}$ and $\mu \subseteq S_1$. We have

$$\begin{aligned}
\mathrm{link}_{\mathrm{link}_\Delta x_{11}}\, \mu|_{\bar{S}_1} &= \{F \in \mathrm{link}_\Delta\, x_{11} \mid F \subseteq \bar{S}_1,\ F \cup \mu \in \mathrm{link}_\Delta\, x_{11},\ F \cap \mu = \emptyset\} \\
&= \{F \in \Delta \mid F \subseteq \bar{S},\ F \cup \mu \cup \{x_{11}\} \in \Delta,\ F \cap (\mu \cup \{x_{11}\}) = \emptyset\} \\
&= \mathrm{link}_\Delta\, (\mu \cup \{x_{11}\})|_{\bar{S}}.
\end{aligned}$$

From hypothesis, we have that $\mathrm{link}_\Delta\, (\mu \cup \{x_{11}\})|_{\bar{S}}$ is vertex decomposable and so $\mathrm{link}_{\mathrm{link}_\Delta x_{11}}\, \mu|_{\bar{S}_1}$ is vertex decomposable for any face μ of $\mathrm{link}_\Delta\, x_{11}$ with $\mu \subseteq S_1$. Note that $\mathrm{Ind}(G_0) = \mathrm{link}_\Delta\, x_{11}$ and hence $G_0(G_{l+1}, \dots, G_n)$ is vertex decomposable by induction. Also, $G_1 \setminus N_{G_1}[x_{11}]$, $G_2 \setminus x_{21}, \dots, G_l \setminus x_{l1}$ are vertex decomposable by hypothesis. Thus $G' \setminus N_{G'}[x_{11}]$ is vertex decomposable by [**8**, Lemma 20].

($\Rightarrow$) Conversely, assume that $G' = G(G_1, \dots, G_n)$ is vertex decomposable. Set $\Delta' = \mathrm{Ind}(G')$. After relabeling the vertices of G, we assume that $\mu = \{x_{11}, \dots, x_{l1}\}$ and $\mu \in \Delta$. Hence $\{x_{11}, \dots, x_{l1}\}$ is independent set in G. We show that $\mathrm{link}_\Delta(\mu)|_{\bar{S}}$ is vertex decomposable. For $1 \leqslant j \leqslant l$, let A_j be a facet of $\mathrm{Ind}(G_j)$ with $x_{j1} \in A_j$. Note that for $1 \leqslant i \leqslant n$, G_i is connected graph and hence has no isolated vertex. Thus $\mathrm{Ind}(G_j)$ has no cone point and let for $l+1 \leqslant j \leqslant n$, A_j be a facet of $\mathrm{Ind}(G_j)$ with $x_{j1} \notin A_j$. It follows that $A = \bigcup_{j=1}^n A_j$ is a face of Δ'. One can check that $\mathrm{link}_\Delta(\mu)|_{\bar{S}} = \mathrm{link}_{\Delta'}(A)$.

Suppose $F \in \mathrm{link}_\Delta(\mu)|_{\bar{S}}$. It is clear that for $1 \leqslant i \leqslant n$, $x_i \notin F$ and $F \cup \mu \in \Delta$ and so $F \cup A$ is independent set in G' and hence $F \cup A \in \Delta'$. Also, $F \cap A = \emptyset$ because $F \cap \mu = \emptyset$. Since $\Delta \subseteq \Delta'$ and $F \in \Delta$, we have $F \in \Delta'$ and so $F \in \mathrm{link}_{\Delta'}(A)$.

We now consider the reverse inclusion. Let $F \in \mathrm{link}_{\Delta'}(A)$. We have $F \cap A = \emptyset$ and hence $F \cap A_j = \emptyset$ for $1 \leqslant j \leqslant n$. We claim that for $1 \leqslant j \leqslant n$, $F \cap (V(G_j)) = \emptyset$. Let there exists an $i \in \{1, \dots, n\}$ such that $F \cap V(G_i) \neq \emptyset$, and let $x \in F \cap V(G_i)$. Since $F \cap A_i = \emptyset$, $x \notin A_i$. On the other hand, one has $F \cup A \in \Delta'$. Therefore $A_i \cup \{x\}$ is an independent set in G_i and $A_i \subset A_i \cup \{x\}$, that contradiction with facet A_i and so $F \cap (V(G_i)) = \emptyset$ and $F \subseteq \bar{S}$. Since $\bar{S} \subseteq V(G)$, we have $F \subseteq V(G)$. Moreover, $\mu \subseteq A$, $F \cup A \in \Delta'$ and Δ' is simplicial complex, thus $F \cup \mu \in \Delta'$. Also, $F \cup \mu \in \Delta$, because $F, \mu \in \Delta$ and $\Delta \subseteq \Delta'$. Since $F \cap A = \emptyset$ and $\mu \subseteq A$, we have $F \cap \mu = \emptyset$. Therefore $F \in \mathrm{link}_\Delta(\mu)|_{\bar{S}}$. Since $\Delta' = \mathrm{Ind}(G')$ is vertex decomposable, $\mathrm{link}_{\Delta'}(A)$ is vertex decomposable by [**1**, Theorem 2.5] and so $\mathrm{link}_\Delta(\mu)|_{\bar{S}}$ is vertex decomposable for any $\mu \subseteq S$ and $\mu \in \Delta$.

Now, we prove that $G_i \setminus x_{i1}$ and $G_i \setminus N_{G_i}[x_{i1}]$ are vertex decomposable for $1 \leqslant i \leqslant n$. Assume G is a connected graph (otherwise consider connected components of G). Then there exists a $y \in V(G)$ such that $x_{i1}y \in E(G)$. Also, $\mathrm{link}_{\Delta'}\, y$ and

$\operatorname{link}_{\Delta'} x_{i1}$ are vertex decomposable, because Δ' is vertex decomposable [**1**, Theorem 2.5]. Note that $G_i \setminus x_{i1}$ and $G_i \setminus N_{G_i}[x_{i1}]$ are respectively connected components of $\operatorname{link}_{\Delta'} y$ and $\operatorname{link}_{\Delta'} x_{i1}$ and so they are vertex decomposable [**8**, Lemma 20]. $\square$

Let $S = \{x_{11}, \dots, x_{n1}\}$ be a subset of $V(G)$. The graph $G \cup W(S)$ is obtained from G by adding a whisker to every vertex of S.

Corollary 3.1. [**1**, Theorem 4.4] *Let* $\operatorname{Ind}(G)$ *be the independence complex of a graph* G *on a vertex set* $V(G)$ *and let* $S \subseteq V(G)$. *Then* $\operatorname{Ind}(G \cup W(S))$ *is vertex decomposable if and only if* $\operatorname{Ind}((G \setminus N_G[\mu])|_{\bar{S}})$ *is vertex decomposable for all* $\mu \in \operatorname{Ind}(G)$ *with* $\mu \subseteq S$.

Proof. Adding a whisker to a vertex of G is the same as attaching the complete graph K_2 to a vertex of G, and we have $G \cup W(S) = G(\underbrace{K_2, \dots, K_2}_{n \text{ times}})$.

Note that $\operatorname{link}_{\operatorname{Ind}(G)}(\mu)|_{\bar{S}} = \operatorname{Ind}(G \setminus N_G[\mu])|_{\bar{S}} = \operatorname{Ind}((G \setminus N_G[\mu])|_{\bar{S}})$. Now, apply Theorem 3.1. $\square$

The cycle of length n is denoted by C_n. The induced cycle C_n of G is called *chordless cycle* of length n. In [**8**, Theorem 1] Woodroofe showed that if G is a graph with no chordless cycles of length other from 3 or 5, then G is vertex decomposable.

Corollary 3.2. *Let* G *be a simple graph on the vertex set* $V(G)$ *and* $S = \{x_{11}, \dots, x_{n1}\}$ *a subset of* $V(G)$. *Let* $m_1, \dots, m_n \geqslant 2$ *be integers and* $G_1, \dots, G_n$ *connected simple graphs on the vertex sets* $V(G_i) = \{x_{i1}, \dots, x_{im_i}\}$ *for* $i = 1, \dots, n$. *Assume that* x_{i1} *is a shedding vertex of* G_i *for* $i = 1, \dots, n$. *If* $G_i \setminus x_{i1}$ *and* $G_i \setminus N_{G_i}[x_{i1}]$ *for* $i = 1, \dots, n$ *are vertex decomposable and* $G \setminus S$ *is a graph with no chordless cycles of length other than* 3 *or* 5, *then* $G(G_1, \dots, G_n)$ *is vertex decomposable.*

Proof. Note that $\operatorname{link}_{\operatorname{Ind}(G)}(\mu)|_{\bar{S}} = \operatorname{Ind}(G \setminus N_G[\mu])|_{\bar{S}} = \operatorname{Ind}((G \setminus N_G[\mu])|_{\bar{S}})$. For every $\mu \subseteq S$, $(G \setminus N_G[\mu])|_{\bar{S}}$ is an induced subgraph of $G|_{\bar{S}}$, so it is a graph with no chordless cycles of length other from 3 or 5, and hence it is vertex decomposable by [**8**, Theorem 1]. Now, apply Theorem 3.1. $\square$

Therefore, we infer the following result.

Corollary 3.3. *Let* G *be a simple graph on the vertex set* $V(G)$ *and let* $S \subseteq V(G)$. *If* $G \setminus S$ *is a graph with no chordless cycles of length other than* 3 *or* 5, *then* $G \cup W(S)$ *is a vertex decomposable graph.*

Corollary 3.4. *Let* G *be a simple graph on the vertex set* $V(G)$ *and* $S = \{x_{11}, \dots, x_{n1}\}$ *a subset of* $V(G)$ *with* $|S| \geqslant |V(G)| - 3$. *Let* $m_1, \dots, m_n \geqslant 2$ *be integers and* $G_1, \dots, G_n$ *connected simple graphs on the vertex sets* $V(G_i) = \{x_{i1}, \dots, x_{im_i}\}$ *for* $i = 1, \dots, n$. *Assume* x_{i1} *is a shedding vertex of* G_i *for* $i = 1, \dots, n$. *Moreover, let* $G_i \setminus x_{i1}$ *and* $G_i \setminus N_{G_i}[x_{i1}]$ *for* $i = 1, \dots, n$ *be vertex decomposable. Then* $G(G_1, \dots, G_n)$ *is vertex decomposable.*

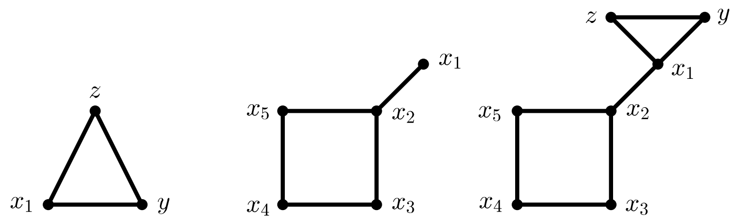
Proof. Since $|S| \geqslant |V(G)| - 3$, $G \setminus S$ is a graph on at most 3 vertices. So $G \setminus S$ is either a tree, edgeless graph, $K_2 \,\dot{\cup}\, K_1$ or C_3. Thus $G \setminus S$ is chordal and the assertion holds by Corollary 3.2. $\square$

Therefore the following result holds.

Corollary 3.5. *Let G be a simple graph, and let $S \subset V(G)$. Assume that $|S| \geqslant |V(G)| - 3$. Then $G \cup W(S)$ is a vertex decomposable graph.*

In the following example it is shown that the bound $|V(G)| - 3$ in Corollary 3.4 is sharp.

Example 3.1. Let G and G_1 be graphs shown in the figure below, and let $S = \{x_1\}$. Note that the only shedding vertices of $G(G_1)$ are x_1, y and z. Then $G(G_1)$ is not vertex decomposable, because $G(G_1) \smallsetminus N[y] = G(G_1) \smallsetminus N[z] = C_4$ and $G(G_1) \smallsetminus x_1 = K_2 \,\dot{\cup}\, C_4$ are not vertex decomposable.



G_1 $\quad G \quad$ $G(G_1)$

Corollary 3.6. *Let G be a simple graph on the vertex set $V(G)$ and $S = \{x_{11}, \dots, x_{n1}\}$ a subset of $V(G)$. Let $m_1, \dots, m_n \geqslant 2$ be integers and $G_1, \dots, G_n$ connected simple graphs on the vertex sets $V(G_i) = \{x_{i1}, \dots, x_{im_i}\}$ for $i = 1, \dots, n$. Assume that x_{i1} is a shedding vertex of G_i for $i = 1, \dots, n$. Moreover, let G_i, $G_i \smallsetminus x_{i1}$ and $G_i \smallsetminus N_{G_i}[x_{i1}]$ for $i = 1, \dots, n$ be vertex decomposable. Then*

(i) *If S is a vertex cover of G, then $G(G_1, \dots, G_n)$ is vertex decomposable.*
(ii) *If $G \smallsetminus S$ is a forest, then $G(G_1, \dots, G_n)$ is vertex decomposable.*
(iii) *If $G = C_n$ is a cycle and x a vertex of C_n, then $G(G_1)$ is vertex decomposable.*

Proof. If S is a vertex cover of G, then $G \smallsetminus S$ is an edgeless graph and so it is a chordal graph. Also, every forest is a chordal graph. Now, (i) and (ii) follow from Corollary 3.2. Finally, the resulting graph of removing a vertex from C_n is a tree and hence (iii) follows from (ii). □

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