PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 211–216 DOI: 10.2298/PIM1613211Z

HÖLDER'S REVERSE INEQUALITY AND ITS APPLICATIONS

Chang-Jian Zhao and Wing Sum Cheung

Abstract. We establish a new reverse Hölder integral inequality and its discrete version. As applications, we prove Radon's, Jensen's reverse and weighted power mean inequalities and their discrete versions.

1. Introduction

The well-known classical Hölder inequality can be stated as follows.

Theorem 1.1. *Let a_i and b_i $(i = 1, \dots, n)$ be positive real sequences. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\Big(\sum a_i^p\Big)^{1/p}\Big(\sum b_i^q\Big)^{1/q} \geqslant \sum a_i b_i. \tag{1.1}$$

Here and in what follows $\sum$ means $\sum_{i=1}^n$.

The integral version of (1.1) is the following.

Theorem 1.2. *Let $f(x)$ and $g(x)$ be positive continuous functions on $[a, b]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\Big(\int_a^b f^p(x)\,dx\Big)^{1/p}\Big(\int_a^b g^q(x)\,dx\Big)^{1/q} \geqslant \int_a^b f(x)\,g(x)\,dx. \tag{1.2}$$

Hölder's inequality plays an important role in different branches of modern mathematics such as classical real and complex analysis, numerical analysis, probability and statistics, differential equations and et al. In recent years some authors [**1**,**2**,**6**,**7**,**9**,**10**,**16**–**19**] have given considerable attention to Hölder's inequality

2010 *Mathematics Subject Classification:* 26D15.
Key words and phrases: Hölder's inequality, Radon's inequality, Jensen's inequality, power mean inequality.
The first author's research is supported by National Natural Science Foundation of China (11371334). The second author's research is supported by National Natural Science Foundation of China (11371334) and a HKU Seed Grant for Basic Research.
Communicated by Gradimir Milovanović.

together by its integral version and various generalizations. Some reverse versions were recent established [**4**, **13**, **15**, **18**].

We establish a new reverse Hölder integral inequality and with its discrete form. As applications, we prove Radon's, Jensen's reverse and weighted power mean inequalities.

2. Hölder's reverse inequalities

LEMMA 2.1. *If a, b are positive real numbers and $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, then* (*see* [**14**])

$$S\Big(\frac{a}{b}\Big)a^{1/p}b^{1/q} \geqslant \frac{a}{p} + \frac{b}{q}, \tag{2.1}$$

where

$$S(h) = \frac{h^{1/(h-1)}}{e \log h^{1/(h-1)}}, \quad h \neq 1.$$

THEOREM 2.1 (Hölder's reverse inequality). *Let $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $f(x)$ and $g(x)$ are non-negative continuous functions and $f^{1/p}(x)\, g^{1/q}(x)$ is integrable on $[a, b]$, then*

$$\Big(\int_a^b f(x)^p dx\Big)^{1/p}\Big(\int_a^b g(x)^q dx\Big)^{1/q} \leqslant \int_a^b S\Big(\frac{Y f^p(x)}{X g^q(x)}\Big) \cdot f(x)g(x)\, dx, \tag{2.2}$$

where

$$X = \int_a^b f^p(x)\, dx, \quad Y = \int_a^b q^q(x)\, dx,$$

and $S(h)$ is as in Lemma 2.1.

PROOF. Let

$$a = \frac{f^p(x)}{X}, \quad b = \frac{g^q(x)}{Y}.$$

By using Lemma 2.1, we have

$$S\Big(\frac{Y f^p(x)}{X g^q(x)}\Big) \cdot \frac{f(x)g(x)}{X^{1/p}Y^{1/q}} \geqslant \frac{1}{p}\frac{f^p(x)}{X} + \frac{1}{q}\frac{g^p(x)}{Y}.$$

Therefore

$$\frac{\int_a^b S\big(\frac{Y f^p(x)}{X g^q(x)}\big) f(x)g(x)dx}{X^{1/p}Y^{1/q}} \geqslant \frac{1}{p}\frac{\|f(x)\|_p^p}{X} + \frac{1}{q}\frac{\|g(x)\|_p^p}{Y} = 1.$$

This proof is completed. □

REMARK 2.1. Obviously, inequality (2.2) is just an inverse of inequality (1.2). Moreover, let $f(x)$ and $g(x)$ reduce to positive real sequences a_i and b_i ($i = 1, \ldots, n$), respectively and with appropriate changes in the proof of (2.1), we have

$$\Big(\sum a_i^p\Big)^{1/p}\Big(\sum b_i^q\Big)^{1/q} \leqslant \sum S\Big(\frac{Y' a_i^p}{X' b_i^q}\Big) a_i b_i,$$

where $X' = \sum a_i^p$, $Y' = \sum b_i^q$.

This is just an inverse of the well-known Hölder's inequality (1.1).

3. Applications of Hölder's reverse inequality

THEOREM 3.1 (Radon's reverse integral inequality). *Let $f(x)$ and $g(x)$ be positive and continuous functions. If $m > 0$, then*

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \leqslant \frac{\Big(\int_a^b S\Big(\frac{\tilde{Y} f^{m+1}(x)}{\tilde{X} g^{m+1}(x)}\Big) f(x)\,dx\Big)^{m+1}}{\big(\int_a^b g(x)\,dx\big)^m}, \tag{3.1}$$

where

$$\tilde{X} = \int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx, \quad \tilde{Y} = \int_a^b g(x)\,dx.$$

PROOF. Let $p = m+1, q = (m+1)/m$ and replacing $f(x)$ and $g(x)$ by $u(x)$ and $v(x)$ in (2.2), respectively, we have

$$\begin{aligned}\Big(\int_a^b u(x)^{m+1}\,dx\Big)^{1/(m+1)} &\Big(\int_a^b v(x)^{(m+1)/m}dx\Big)^{m/(m+1)} \\ &\leqslant \int_a^b S\Big(\frac{\hat{Y} u^{m+1}(x)}{\hat{X} v^{(m+1)/m}(x)}\Big) u(x)v(x)\,dx,\end{aligned} \tag{3.2}$$

where $\hat{X} = \int_a^b u^{m+1}(x)\,dx$, $\hat{Y} = \int_a^b v^{(m+1)/m}(x)\,dx$. Taking $u(x) = \big(\frac{f(x)}{g(x)}\big)^{1/(m+1)}$, $v(x) = f^{m/(m+1)}(x)\,g^{1/(m+1)}(x)$ in (3.2), we obtain

$$\begin{aligned}&\int_a^b S\Big(\frac{\bar{Y}}{\bar{X} g^{(m+1)/m}(x)}\Big) f(x)\,dx \\ &\quad \geqslant \Big(\int_a^b \frac{f(x)}{g(x)}\,dx\Big)^{1/(m+1)} \Big(\int_a^b f(x)\,g^{1/m}(x)\,dx\Big)^{m/(m+1)},\end{aligned}$$

where

$$\bar{X} = \int_a^b \frac{f(x)}{g(x)}\,dx, \quad \bar{Y} = \int_a^b f(x)\,g^{1/m}(x)\,dx.$$

Hence

$$\int_a^b \frac{f(x)}{g(x)}\,dx \leqslant \frac{\Big(\int_a^b S\Big(\frac{\bar{Y}}{\bar{X} g^{(m+1)/m}(x)}\Big) f(x)dx\Big)^{m+1}}{\big(\int_a^b f(x) g^{1/m}(x)\,dx\big)^m}.$$

Replacing $f(x)$ and $g(x)$ by $u(x)$ and $v(x)$, respectively, and leting $u(x) = f(x)$ and $v(x) = \big(\frac{g(x)}{f(x)}\big)^m$, we get

$$\int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx \leqslant \frac{\Big(\int_a^b S\Big(\frac{\tilde{Y} f^{m+1}(x)}{\tilde{X} g^{m+1}(x)}\Big) f(x)\,dx\Big)^{m+1}}{\big(\int_a^b g(x)\,dx\big)^m},$$

where

$$\tilde{X} = \int_a^b \frac{f^{m+1}(x)}{g^m(x)}\,dx, \quad \tilde{Y} = \int_a^b g(x)\,dx. \qquad \square$$

REMARK 3.1. Let $f(x)$ and $g(x)$ reduce to positive real sequences a_i and b_i $(i = 1, \dots, n)$, respectively and with appropriate changes in the proof of (3.1), we have

$$\sum \frac{a_i^{m+1}}{b_i^m} \leqslant \frac{\Big(\sum S\Big(\frac{\tilde{Y}' a_i^{m+1}}{\tilde{X}' b_i^{m+1}}\Big) a_i\Big)^{m+1}}{\big(\sum b_i\big)^m},$$

where $\tilde{X}' = \sum a_i^{m+1}/b_i^m$, and $\tilde{Y}' = \sum b_i$.

This is just an inverse of following well-known the Radon inequality [**5**]

$$\sum \frac{a_i^{m+1}}{b_i^m} \geqslant \frac{\big(\sum a_i\big)^{m+1}}{\big(\sum b_i\big)^m}.$$

THEOREM 3.2 (Jensen's reverse integral inequality). *Let $f(x)$ and $p(x)$ be positive continuous functions and $\int_a^b p(x)\,dx = 1$. If $0 < s < t$, then*

$$\Big(\int_a^b S\Big(\frac{f^t(x)}{\check{X}}\Big) f^s(x) p(x)\,dx\Big)^{1/s} \geqslant \Big(\int_a^b p(x) f^t(x)\,dx\Big)^{1/t}, \tag{3.3}$$

where

$$\check{X} = \int_a^b p(x) f^t(x) t\,dx.$$

PROOF. From the hypotheses, we have

$$\begin{aligned}\int_a^b S\Big(\frac{f^t(x)}{\check{X}}\Big) f^s(x) p(x)\,dx \\ = \int_a^b S\Big(\frac{\check{Y}[p^{s/t}(x) f^s(x)]^{t/s}}{\check{X}[p^{1-s/t}(x)]^{t/(t-s)}}\Big) p^{s/t}(x) f^s(x) \cdot p^{1-s/t}(x)\,dx,\end{aligned} \tag{3.4}$$

where $\check{X} = \int_a^b p(x) f^t(x)\,dx$ and $\check{Y} = \int_a^b p(x)\,dx$. In view if $p = \frac{t}{s}$ and then $q = \frac{t}{t-s}$, and by using (2.2) on the right-hand side of (3.4), we have

$$\begin{aligned}\int_a^b S\Big(\frac{\check{Y}[p^{s/t}(x) f^s(x)]^{t/s}}{\check{X}[p^{1-s/t}(x)]^{t/(t-s)}}\Big) p^{s/t}(x) f^s(x) \cdot p^{1-s/t}(x)\,dx \\ \geqslant \Big(\int_a^b (p^{s/t}(x) f^s(x))^{t/s} dx\Big)^{s/t} \Big(\int_a^b [(p(x))^{1-s/t}]^{t/(t-s)} dx\Big)^{(t-s)/t}.\end{aligned} \tag{3.5}$$

From (3.4), (3.5) and in view of $\int_a^b p(x)\,dx = 1$, we obtain

$$\int_a^b S\Big(\frac{f^t(x)}{\check{X}}\Big) f^s(x)\, p(x)\,dx \geqslant \Big(\int_a^b p(x) f^t(x)\,dx\Big)^{s/t}.$$

Hence

$$\Big(\int_a^b S\Big(\frac{f^t(x)}{\check{X}}\Big) f^s(x)\, p(x)\,dx\Big)^{1/s} \geqslant \Big(\int_a^b p(x) f^t(x)\,dx\Big)^{1/t}. \qquad \square$$

REMARK 3.2. If $f(x)$ and $p(x)$ reduce to positive real sequences a_i and λ_i $(i = 1, \dots, n)$, respectively and with appropriate changes in (3.3), we have

$$\Big(\sum S\Big(\frac{a_i^t}{\breve{X}'}\Big) a_i^s \lambda_i\Big)^{1/s} \geqslant \Big(\sum \lambda_i a_i^t\Big)^{1/t},$$

where $\breve{X}' = \sum \lambda_i a_i^t$.

This is just an inverse of the following well-known Jensen's inequality [3]

$$\Big(\sum a_i^s \lambda_i\Big)^{1/s} \leqslant \Big(\sum \lambda_i a_i^t\Big)^{1/t},$$

THEOREM 3.3 (Reverse weighted power mean integral inequality). *Let $f(x)$ and $p(x)$ be positive and continuous functions. If $r > 0$ and $\int_a^b p(x)\,dx = 1$, then*

$$\Big(\int_a^b S\Big(\frac{\int_a^b p(x) f^{2r}(x)\,dx}{f^{2r}(x)}\Big) p(x) f^r(x)\,dx\Big)^{1/r} \geqslant \Big(\int_a^b p(x) f^{2r}(x)\,dx\Big)^{1/2r}. \tag{3.6}$$

PROOF. From the hypotheses, we have

$$\begin{aligned}\int_a^b S\Big(\frac{\int_a^b p(x) f^{2r}(x)\,dx}{f^{2r}(x)}\Big) p(x) f^r(x)\,dx \\ = \int_a^b S\Big(\frac{p(x)\int_a^b p(x) f^{2r}(x)\,dx}{p(x) f^{2r}(x)\int_a^b p(x)\,dx}\Big) p^{1/2}(x) \cdot p^{1/2}(x) f^r(x)\,dx.\end{aligned} \tag{3.7}$$

From (2.2), we obtain

$$\Big(\int_a^b f(x)^2 dx\Big)^{1/2}\Big(\int_a^b g(x)^2 dx\Big)^{1/2} \leqslant \int_a^b S\Big(\frac{\breve{Y} f^2(x)}{\breve{X} g^2(x)}\Big) f(x)\, g(x)\,dx, \tag{3.8}$$

where $\breve{X} = \int_a^b f^2(x)\,dx$, $\breve{Y} = \int_a^b g^2(x)\,dx$. By using (3.8) on the right-hand side of (3.7), we have

$$\begin{aligned}\Big(\int_a^b S\Big(\frac{\int_a^b p(x) f^{2r}(x)\,dx}{f^{2r}(x)}\Big) p(x) f^r(x)\,dx\Big)^2 \\ \geqslant \int_a^b p(x)\,dx \cdot \int_a^b p(x) f^{2r}(x)\,dx = \int_a^b p(x) f^{2r}(x)\,dx.\end{aligned}$$

Hence

$$\Big(\int_a^b S\Big(\frac{\int_a^b p(x) f^{2r}(x)\,dx}{f^{2r}(x)}\Big) p(x) f^r(x)\,dx\Big)^{1/r} \geqslant \Big(\int_a^b p(x) f^{2r}(x)\,dx\Big)^{1/2r}. \qquad \square$$

REMARK 3.3. If $f(x)$ and $p(x)$ reduce to positive real sequences a_i and λ_i $(i = 1, \dots, n)$, respectively and with appropriate changes in (3.6), we have

$$\Big(\sum S\Big(\frac{\sum \lambda_i a_i^{2r}}{a_i^{2r}}\Big) \lambda_i a_i^r\Big)^{1/r} \geqslant \Big(\sum \lambda_i a_i^{2r}\Big)^{1/2r}.$$

This is just an inverse of the following well-known weighted power mean inequality [**5**]

$$\Big(\sum \lambda_i a_i^r\Big)^{1/r} \leqslant \Big(\sum \lambda_i a_i^{2r}\Big)^{1/2r}.$$

It is worth noting that literature [**11**] is relevant in spirit and methodology to the present paper, but for the Hilbert-type integral inequality and its connection with analytic number theory.

References

1. S. Abramovich, B. Mond, J. E. Pečarić, *Sharpening Hölder's inequality*, J. Math. Anal. Appl. **196**(1) (1995), 1131–1134.
2. J. M. Aldaz, *A stability version of Hölder's inequality*, J. Math. Anal. Appl. **343**(2) (2008), 842–852.
3. E. F. Beckenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1961.
4. C. BorellInverse, *Hölder inequalities in one and several dimensions*, J. Math. Anal. Appl. **41**(2)(1973), 300–312.
5. G. H. Hardy, J. E. Littlewood, Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.
6. Y. Kim, X. Yang, *Generalizations and refinements of Hölder's inequality*, Appl. Math. Lett. **25**(7) (2012), 1094–1097.
7. E. G. Kwon, E. K. Bae, *On a continuous form of Hölder inequality*, J. Math. Anal. Appl. **343**(1) (2008), 585–592.
8. B. Mond, J. E. Pečaric, *Remark on a recent converse of Hölder's inequality,* J. Math. Anal. Appl. **181**(1) (1994), 280–281.
9. G. S. Mudholkar, M. Freimer, P. Subbaiah, *An extension of Hölder's inequality*, J. Math. Anal. Appl. **102**(2) (1984), 435–441.
10. H. Qiang, Z. Hu, *Generalizations of Hölder's and some related inequalities*, Comput. Math. Appl. **61**(2) (2011), 392–396.
11. M. T. Rassias, B. C. Yang, *A multidimensional Hilbert-type integral inequality related to the Riemann zeta function*, in: *Applications of Mathematics and Informatics in Science and Engineering*, Springer, New York, 2014, 417–433.
12. R. P. Singh, R. Kumar, R. K. Tuteja, *Application of Hölder's inequality in information theory*, Inf. Sci. **152** (2003), 145–154.
13. J. Tian, *Reversed version of a generalized sharp Hölder's inequality and its applications*, Inf. Sci. **201**(2012), 61–69.
14. M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Jpn. **55** (2002), 538–588.
15. Y. V. Venkatesh, *Converse Hölder inequality and the L_p-instability of nonlinear time-varying feedback systems*, Nonlinear Anal., Theory Methods Appl. **12**(3) (1988), 247–258.
16. L. Wu, J. Sun, X. Ye, L. Zhu, *Hölder type inequality for Sugeno integral*, Fuzzy Sets Syst. **161**(17) (2010), 2337-2347.
17. S. H. Wu, *Generalization of a sharp Hölder's inequality and its application*, J. Math. Anal. Appl. **332**(1) (2007), 741–750.
18. W. Yang, *A functional generalization of Diamonda integral Hölder's nequality on time scales*, Appl. Math. Lett. **23** (2010), 1208–1212.
19. X. Yang, *Hölder's inequality*, Appl. Math. Lett. **16**(6) (2003), 897–903.

Department of Mathematics, China Jiliang University
Hangzhou, P. R. China
chjzhao@163.com, chjzhao@cjlu.edu.cn

(Received 24 04 2013)
(Revised 11 01 2016)

Department of Mathematics, The University of Hong Kong, Hong Kong
wscheung@hku.hk