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EXISTENCE RESULTS FOR NONLINEAR IMPULSIVE q_k-INTEGRAL BOUNDARY VALUE PROBLEMS

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Abstract. We investigate a nonlinear impulsive q_k-integral boundary value problem by means of Leray–Schauder degree theory and contraction mapping principle. The conditions ensuring the existence and uniqueness of solutions for the problem are presented. An illustrative example is discussed.

1. Introduction

We investigate the existence and uniqueness of solutions for a nonlinear impulsive q_k-integral boundary value problem

$$\begin{aligned} D_{q_k}u(t) &= f(t,u(t)), \quad 0<q_k<1, \quad t\in J', \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k=1,2,\dots,m, \\ u(T) &= \textstyle\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} g(s,u(s))\,d_{q_i}s, \end{aligned} \tag{1.1}$$

where D_{q_k} are q_k-derivatives $(k=0,1,2,\dots,m)$, $f,g\in C(J\times\mathbb{R},\mathbb{R})$, $I_k\in C(\mathbb{R},\mathbb{R})$, $J=[0,T](T>0)$, $0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=T$, $J'=J\backslash\{t_1,t_2,\dots,t_m\}$, and $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t=t_k$ $(k=1,2,\dots,m)$ respectively.

The study of q-difference equations, initiated with the pioneer work of Jackson [**1**], has been developed over the years. The concept of q-calculus corresponds to the classical calculus without the idea of limit. This subject is also known as quantum calculus and finds its applications in a variety of disciplines such as special functions, super-symmetry, control theory, operator theory, combinatorics, initial and boundary value problems of q-difference equations, etc. For the systematic development of q-calculus, we refer the reader to the books [**2**–**4**] and papers [**5**–**10**].

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The importance of q-difference equations lies in the fact that these equations are always completely controllable and appear in the q-optimal control problems [**11**]. The variational q-calculus is regarded as a generalization of the continuous variational calculus due to the presence of an extra-parameter q that may be physical or economical in its nature. The variational calculus on the q-uniform lattice includes the study of the q-Euler equations and its applications to the isoperimetric and Lagrange problems and commutation equations. In other words, it suffices to solve the q-Euler-Lagrange equation for finding the extremum of the functional involved instead of solving the Euler–Lagrange equation [**12**]. Further details can be found in [**13**–**16**].

The initial and boundary value problems of impulsive fractional differential equations have been extensively investigated by many researchers, for instance, see [**17**–**25**]) and references therein. In a recent paper [**26**], the authors discussed the existence and uniqueness of solutions for impulsive q_k-difference equations.

Motivated by [**26**], the present work is devoted to the study of impulsive q_k-difference equations with integral boundary condition. The paper is organized as follows. In Section 2, we present some basic concepts of the topic and an auxiliary lemma. Section 3 contains the main results, while an illustrative example is discussed in Section 4.

2. Preliminaries

Let us set $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, T]$ and introduce the space: $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k),\ k = 0, 1, \cdots, m$, and $u(t_k^+)$ exist for $k = 1, 2, \cdots, m\}$ with the norm $\|u\| = \sup_{t \in J} |u(t)|$. Obviously $PC(J, \mathbb{R})$ is a *Banach* space.

Next we recall some basic concepts of q_k-calculus [**26**]. For $0 < q_k < 1$ and $t \in J_k$, we define the q_k-derivatives of a real valued continuous function f as

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t). \tag{2.1}$$

Higher order q_k-derivatives are given by

$$D^0_{q_k} f(t) = f(t), \quad D^n_{q_k} f(t) = D_{q_k} D^{n-1}_{q_k} f(t), \quad n \in \mathbb{N}, \quad t \in J_k.$$

The q_k-integral of a function f is defined by

$$_{t_k}\mathcal{I}_{q_k} f(t) := \int_{t_k}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k), \quad t \in J_k, \tag{2.2}$$

provided the series converges. If $a \in (t_k, t)$ and f is defined on the interval (t_k, t), then

$$\int_a^t f(s)\, d_{q_k} s = \int_{t_k}^t f(s)\, d_{q_k} s - \int_{t_k}^a f(s)\, d_{q_k} s.$$

Observe that

$$D_{q_k}(_{t_k}\mathcal{I}_{q_k} f(t)) = D_{q_k} \int_{t_k}^t f(s)\, d_{q_k} s = f(t),$$

$$
\begin{aligned}
{}_{t_k}\mathcal{I}_{q_k}(D_{q_k}f(t)) &= \int_{t_k}^{t} D_{q_k}f(s)\,d_{q_k}s = f(t),\\
{}_{a}\mathcal{I}_{q_k}(D_{q_k}f(t)) &= \int_{a}^{t} D_{q_k}f(s)\,d_{q_k}s = f(t)-f(a), \quad a\in(t_k,t).
\end{aligned}
$$

In the case $t_k = 0$ and $q_k = q$ in (2.1) and (2.2), then $D_{q_k}f = D_q f$, ${}_{t_k}\mathcal{I}_{q_k}f = {}_0\mathcal{I}_q f$, where D_q and ${}_0\mathcal{I}_q$ are the well-known q-derivative and q-integral of the function $f(t)$ and are defined by

$$
D_q f(t) = \frac{f(t)-f(qt)}{(1-q)t}, \quad {}_0\mathcal{I}_q f(t) = \int_0^t f(s)\,d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n).
$$

Lemma 2.1. *For a given $\sigma(t)\in C(J,\mathbb{R})$, a function $u\in PC(J,\mathbb{R})$ is a solution of the following impulsive q_k-integral boundary value problem*

$$
\begin{cases}
D_{q_k}u(t) = \sigma(t), \quad 0<q_k<1, \quad t\in J',\\
\Delta u(t_k) = I_k(u(t_k)), \quad k=1,2,\dots,m,\\
u(T) = \sum\limits_{i=0}^{m}\int_{t_i}^{t_{i+1}} g(s,u(s))\,d_{q_i}s,
\end{cases}
\tag{2.3}
$$

if and only if u satisfies the q_k-integral equation

$$
u(t) = \begin{cases}
\int_0^t \sigma(s)\,d_{q_0}s + \sum\limits_{i=0}^{m}\int_{t_i}^{t_{i+1}}[g(s,u(s))-\sigma(s)]d_{q_i}s - \sum\limits_{i=1}^{m} I_i(u(t_i)), \quad t\in J_0;\\
\int_{t_k}^{t}\sigma(s)\,d_{q_k}s + \sum\limits_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum\limits_{i=0}^{m}\int_{t_i}^{t_{i+1}}[g(s,u(s))-\sigma(s)]d_{q_i}s\\
-\sum\limits_{i=k+1}^{m} I_i(u(t_i)), \quad t\in J_k.
\end{cases}
\tag{2.4}
$$

Proof. Let u be a solution of q_k-integral boundary value problem (2.3). For $t\in J_0$, applying the operator ${}_0\mathcal{I}_{q_0}$ on both sides of $D_{q_0}u(t)=\sigma(t)$, we get

$$
u(t) = u(0) + {}_0\mathcal{I}_{q_0}\sigma(t) = u(0) + \int_0^t \sigma(s)\,d_{q_0}s.
$$

Thus, $u(t_1^-) = u(0) + \int_0^{t_1}\sigma(s)\,d_{q_0}s$. For $t\in J_1$, applying the operator ${}_{t_1^+}\mathcal{I}_{q_1}$ on both sides of $D_{q_1}u(t)=\sigma(t)$ yields

$$
u(t) = u(t_1^+) + \int_{t_1}^{t}\sigma(s)\,d_{q_1}s.
$$

Taking into account the condition: $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, we obtain

$$
u(t) = u(0) + \int_{t_1}^{t}\sigma(s)\,d_{q_1}s + \int_0^{t_1}\sigma(s)\,d_{q_0}s + I_1(u(t_1)), \quad \forall t\in J_1.
$$

Repeating the above process, it is found that

$$
u(t) = u(0) + \int_{t_k}^{t}\sigma(s)\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)), \quad t\in J_k.
\tag{2.5}
$$

Substituting $t = T$ in (2.5), we have

$$u(T) = u(0) + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \sigma(s)\, d_{q_i} s + \sum_{i=1}^{m} I_i(u(t_i)). \tag{2.6}$$

Using the boundary condition given by (2.3) in (2.6), we obtain

$$u(t) = \int_{t_k}^{t} \sigma(s)\, d_{q_k} s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sigma(s)\, d_{q_i} s + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - \sigma(s)] d_{q_i} s \\ - \sum_{i=k+1}^{m} I_i(u(t_i)), \quad t \in J_k.$$

Conversely, assume that u satisfies q_k-integral equation (2.4). Then, by applying the operator D_{q_k} on both sides of (2.4) and using $t = T$, we obtain (2.3). □

3. Main results

By Lemma 2.1, the nonlinear impulsive q_k-integral boundary value problem (1.1) can be transformed into an equivalent fixed point problem: $u = \mathcal{G}u$, where the operator $\mathcal{G} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is defined by

$$(\mathcal{G}u)(t) = \int_{t_k}^{t} f(s, u(s))\, d_{q_k} s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(s, u(s))\, d_{q_i} s \\ + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - f(s, u(s))] d_{q_i} s - \sum_{i=k+1}^{m} I_i(u(t_i)).$$

One can notice that the existence of a fixed point of the operator $\mathcal{G}$ implies the existence of a solution of problem (1.1).

To show the existence of solutions for problem (1.1), we rely on Leray–Schauder degree theory and Banach fixed point theorem.

THEOREM 3.1. *Assume that* (H_1) *there exist nonnegative constants* a, b, c, d *and* e *such that* $\frac{(2a+c)T+me}{1-(2b+d)T} > 0$ *and*

$$|f(t, u)| \leqslant a + b|u|, \quad |g(t, u)| \leqslant c + d|u|, \quad |I_k(u)| \leqslant e, \quad k = 1, 2, \dots, m,$$

for all $t \in J,\ u \in \mathbb{R}$. *Then impulsive* q_k*-integral boundary value problem* (1.1) *has at least one solution.*

PROOF. In the first step, it will be shown that the operator $\mathcal{G} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. Let $\mathcal{H} \subset PC(J, \mathbb{R})$ be bounded. Then, for $\forall t \in J,\ u \in \mathcal{H}$, we have $|f(t, u)| \leqslant \mathcal{L}_1$, $|g(t, u)| \leqslant \mathcal{L}_2$, $|I_k(u)| \leqslant \mathcal{L}_3$, where $\mathcal{L}_i$ $(i = 1, 2, 3)$ are constants and $k = 1, 2, \dots, m$. Hence, for $(t, u) \in J \times \mathcal{H}$, the following inequality holds

$$|(\mathcal{G}u)(t)| \leqslant \int_{t_k}^{t} |f(s, u(s))| d_{q_k} s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s, u(s))| d_{q_i} s$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}[|g(s,u(s))|+|f(s,u(s))|]d_{q_i}s+\sum_{i=k+1}^{m}|I_i(u(t_i))| \\
&\leqslant \mathcal{L}_1(t-t_k)+\mathcal{L}_2\sum_{i=0}^{k-1}(t_{i+1}-t_i)+(\mathcal{L}_1+\mathcal{L}_2)\sum_{i=0}^{m}(t_{i+1}-t_i)+(m-k)\mathcal{L}_3 \\
&\leqslant T\mathcal{L}_1+T(\mathcal{L}_1+\mathcal{L}_2)+(m-k)\mathcal{L}_3 \\
&\leqslant T(2\mathcal{L}_1+\mathcal{L}_2)+m\mathcal{L}_3=:\mathcal{L}\ \ \text{(constant)}.
\end{aligned}
$$

This implies that $\|\mathcal{G}u\| \leqslant \mathcal{L}$. Furthermore, for any $t', t'' \in J_k$ $(k=0,1,2,\ldots,m)$ such that $t' < t'' < T$, we have

$$
\begin{aligned}
|(\mathcal{G}u)(t'')-(\mathcal{G}u)(t')| &\leqslant \left|\int_{t_k}^{t''} f(s,u(s))\,d_{q_k}s-\int_{t_k}^{t'} f(s,u(s))\,d_{q_k}s\right| \\
&\leqslant \int_{t'}^{t''}|f(s,u(s))|d_{q_k}s \leqslant \mathcal{L}_1(t''-t').
\end{aligned} \tag{3.1}
$$

As $t' \to t''$, the right-hand side of (3.1) tends to zero. Thus, $\mathcal{G}(\mathcal{H})$ is a relatively compact set. Therefore, by the Arzelá–Ascoli theorem, the operator $\mathcal{G}$ is compact. Also, continuity of functions f, g and I_k imply that $\mathcal{G}$ is a continuous operator. In consequence, it follows that the operator $\mathcal{G}$ is completely continuous.

Now let us define $H(\lambda,u)=\lambda\mathcal{G}u$, $u\in PC(J,\mathbb{R})$, $\lambda\in[0,1]$ and note that $h_\lambda(u)=u-H(\lambda,u)=u-\lambda\mathcal{G}u$ is completely continuous.

Next, we fix $R=\frac{(2a+c)T+me}{1-(2b+d)T}+1$ and define a set $B_R=\{u\in PC(J,\mathbb{R}) \mid \|u\|<R\}$. To arrive at the desired conclusion, it is sufficient to show that $\mathcal{G}:\bar{B}_R\to PC(J,\mathbb{R})$ satisfies

$$
u \neq \lambda\mathcal{G}u, \quad \forall u\in\partial B_R \quad \forall\lambda\in[0,1]. \tag{3.2}
$$

Suppose that (3.2) is not true. Then, there exists some $\lambda\in[0,1]$ such that $u=\lambda\mathcal{G}u$ for any $u\in\partial B_R$ and $t\in J$. Thus, we have

$$
\begin{aligned}
|u(t)|=|\lambda(\mathcal{G}u)(t)| &\leqslant \int_{t_k}^{t}|f(s,u(s))|d_{q_k}s+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}|f(s,u(s))|d_{q_i}s \\
&\quad+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\Big[|g(s,u(s))|+|f(s,u(s))|\Big]d_{q_i}s+\sum_{i=k+1}^{m}|I_i(u(t_i))| \\
&\leqslant \int_{t_k}^{t}(a+b|u(s)|)\,d_{q_k}s+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(a+b|u(s)|)\,d_{q_i}s \\
&\quad+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}(c+d|u(s)|+a+b|u(s)|)\,d_{q_i}s+\sum_{i=k+1}^{m}e \\
&\leqslant (a+b\|u\|)\Big[(t-t_k)+\sum_{i=0}^{k-1}(t_{i+1}-t_i)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[a+c+(b+d)\|u\|\Big]\sum_{i=0}^{m}(t_{i+1}-t_i)+(m-k)e\\
&\leqslant (2b+d)T\|u\|+(2a+c)T+me,
\end{aligned}
$$

which leads to a contradiction: $\|u\| \leqslant \frac{(2a+c)T+me}{1-(2b+d)T} < R$. Hence our supposition is false and (3.2) is true. Applying the homotopy invariance of topological degree, it follows that

$$
\begin{aligned}
\deg(h_\lambda, B_R, 0) &= \deg(I-\lambda\mathcal{G}, B_R, 0) = \deg(h_1, B_R, 0)\\
&= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\end{aligned}
$$

where I is the unit operator. Since $\deg(I-\mathcal{G}, B_R, 0) = 1$, the operator $\mathcal{G}$ has at least one fixed point in B_R by the solvability of topological degree. Thus, the impulsive q_k-integral boundary value problem (1.1) has at least one solution in B_R. $\square$

To prove the uniqueness of solutions, we list the following assumptions:
(H_2) there exist nonnegative continuous functions $M(t)$ and $N(t)$ such that

$$
\begin{aligned}
|f(t,u)-f(t,v)| &\leqslant M(t)|u-v|,\\
|g(t,u)-g(t,v)| &\leqslant N(t)|u-v|,
\end{aligned}
\quad \text{for all } t \in J,\ u, v \in \mathbb{R}.
$$

(H_3) there exists a positive constant K such that

$$
|I_k(u)-I_k(v)| \leqslant K|u-v|, \quad u, v \in \mathbb{R}, \quad k = 1, 2, \ldots, m.
$$

In the sequel, we set

$$
\begin{aligned}
&M^* = \max_{t\in J}|f(t,0)|, \quad N^* = \max_{t\in J}|g(t,0)|, \quad \gamma = \sum_{i=0}^{m} {}_{t_i}\mathcal{I}_{q_i}(2M+N)(t_{i+1}) + mK,\\
&\beta = (2M^*+N^*)T, \quad B_r = \big\{u \in PC(J,\mathbb{R}) \;\big|\; \|u\| \leqslant r\big\}, \quad r \geqslant \frac{\beta}{1-\gamma}.
\end{aligned}
$$

Theorem 3.2. *Let $\gamma < 1$ and the conditions $(H_2)-(H_3)$ hold. Then the impulsive q_k-integral boundary value problem* (1.1) *has a unique solution in B_r.*

Proof. Firstly, we show that the operator $\mathcal{G}$ maps B_r into itself. For $\forall t \in J_k, u \in B_r$, by (H_2) and (H_3), we find that

$$
\begin{aligned}
|(\mathcal{G}u)(t)| \leqslant& \int_{t_k}^{t}|f(s,u(s))|d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}|f(s,u(s))|d_{q_i}s\\
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\big[|g(s,u(s))|+|f(s,u(s))|\big]d_{q_i}s + \sum_{i=k+1}^{m}|I_i(u(t_i))|\\
\leqslant& \int_{t_k}^{t}\big[|f(s,u(s))-f(s,0)|+|f(s,0)|\big]d_{q_k}s\\
&+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\big[|f(s,u(s))-f(s,0)|+|f(s,0)|\big]d_{q_i}s
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\Big[|g(s,u(s))-g(s,0)|+|g(s,0)| \\
&+|f(s,u(s))-f(s,0)|+|f(s,0)|\Big]d_{q_i}s+\sum_{i=k+1}^{m}|I_i(u(t_i))| \\
\leqslant&\int_{t_k}^{t}(M(s)|u(s)|+M^*)\,d_{q_k}s+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(M(s)|u(s)|+M^*)\,d_{q_i}s \\
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\big[(M+N)(s)|u(s)|+(M^*+N^*)\big]d_{q_i}s+mK\|u\| \\
\leqslant&\left[{}_{t_k}\mathcal{I}_{q_k}M(t)+\sum_{i=0}^{k-1}{}_{t_i}\mathcal{I}_{q_i}M(t_{i+1})+\sum_{i=0}^{m}{}_{t_i}\mathcal{I}_{q_i}(M+N)(t_{i+1})+mK\right]\|u\| \\
&+M^*t+(M^*+N^*)T \\
\leqslant&\left[\sum_{i=0}^{m}{}_{t_i}\mathcal{I}_{q_i}(2M+N)(t_{i+1})+mK\right]\|u\|+(2M^*+N^*)T \\
\leqslant&\,\gamma\|u\|+\beta\leqslant r,
\end{aligned}
$$

which implies that $\mathcal{G}(B_r)\subset B_r$.

Next, we show that $\mathcal{G}$ is a contractive map. For each $u,v\in PC(J,\mathbb{R})$, it follows by (H_2) and (H_3) that

$$
\begin{aligned}
&|(\mathcal{G}u)(t)-(\mathcal{G}v)(t)| \\
\leqslant&\int_{t_k}^{t}|f(s,u(s))-f(s,v(s))|d_{q_k}s+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}|f(s,u(s))-f(s,v(s))|d_{q_i}s \\
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\Big[|g(s,u(s))-g(s,v(s))|+|f(s,u(s))-f(s,v(s))|\Big]d_{q_i}s \\
&+\sum_{i=k+1}^{m}|I_i(u(t_i))-I_i(v(t_i))| \\
\leqslant&\int_{t_k}^{t}M(s)\|u-v\|d_{q_k}s+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}M(s)\|u-v\|d_{q_i}s \\
&+\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}(M+N)(s)\|u-v\|d_{q_i}s+mK\|u-v\| \\
\leqslant&\left[{}_{t_k}\mathcal{I}_{q_k}M(t)+\sum_{i=0}^{k-1}{}_{t_i}\mathcal{I}_{q_i}M(t_{i+1})+\sum_{i=0}^{m}{}_{t_i}\mathcal{I}_{q_i}(M+N)(t_{i+1})+mK\right]\|u-v\| \\
\leqslant&\,\gamma\|u-v\|.
\end{aligned}
$$

This implies that $\|\mathcal{G}u - \mathcal{G}v\| \leqslant \gamma\|u - v\|$. Clearly $\mathcal{G}$ is a contraction in view of the assumption $\gamma < 1$. Hence, the conclusion of Theorem 3.2 follows by contraction mapping principle due to Banach. $\square$

4. Example

Consider the following nonlinear impulsive q_k-integral boundary value problem

$$
\begin{aligned}
&D_{\frac{2}{3+k}} u(t) = 5 + \frac{u(t)}{3 + u^2(t)}, \quad t \in [0,1], \quad t \neq \frac{k}{1+k},\\
&\Delta u\Big(\frac{k}{1+k}\Big) = 10 \sin u\Big(\frac{k}{1+k}\Big), \quad k = 1,2,\\
&u(1) = \int_0^{1/2} \big(3s + \tfrac{1}{5} u(s) e^{-u^2(s)}\big)\, d_{2/3}s\\
&\quad + \int_{1/2}^{2/3} \big(3s + \tfrac{1}{5} u(s) e^{-u^2(s)}\big)\, d_{1/2}s + \int_{2/3}^{1} \big(3s + \tfrac{1}{5} u(s) e^{-u^2(s)}\big)\, d_{2/5}s.
\end{aligned}
\tag{4.1}
$$

Here, $q_k = \frac{2}{3+k} (k = 0,1,2)$, $t_k = \frac{k}{1+k} (k = 1,2)$, $f(t,u) = 5 + \frac{u}{3+u^2}$, $I_k(u) = 10 \sin u$, $g(t,u) = 3t + \frac{1}{5} u e^{-u^2}$. Clearly $|f(t,u)| \leqslant 5 + \frac{1}{3}|u|$, $|g(t,u)| \leqslant 3 + \frac{1}{5}|u|$, $|I_k(u)| \leqslant 10$. Selecting $a = 5$, $b = \frac{1}{3}$, $c = 3$, $d = \frac{1}{5}$ and $e = 10$, all the conditions of Theorem 3.1 hold. Hence, by the conclusion of Theorem 3.1, there exists at least one solution for the problem (4.1).

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