PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113) (2016), 249–255 DOI: 10.2298/PIM141215014T

ON GENERALIZATIONS OF INJECTIVE MODULES

Burcu Nişancı Türkmen

Abstract. As a proper generalization of injective modules in term of supplements, we say that a module M has *the property* (SE) (respectively, *the property* (SSE)) if, whenever $M \subseteq N$, M has a supplement that is a direct summand of N (respectively, a strong supplement in N). We show that a ring R is a left and right artinian serial ring with $\mathrm{Rad}(R)^2 = 0$ if and only if every left R-module has the property (SSE). We prove that a commutative ring R is an artinian serial ring if and only if every left R-module has the property (SE).

1. Introduction

In this paper all rings are associative with identity and all modules are unital left modules. Let R be such a ring and let M be an R-module. The notation $K \subseteq M$ ($K \subset M$) means that K is a (proper) submodule of M. A nonzero submodule $K \subseteq M$ is called *essential* in M, written as $K \trianglelefteq M$, if $K \cap L \neq 0$ for every nonzero submodule of M. Dually, a proper submodule $S \subset M$ is called *small* (in M), denoted by $S \ll M$, if $M \neq S + K$ for every proper submodule K of M [**11**]. Following [**11**], a module M is called *supplemented* if every submodule of M has a supplement in M. A submodule $K \subseteq M$ is a supplement of a submodule L in M if and only if $M = L + K$ and $L \cap K \ll K$.

In [**7**], Mohamed and Müller call a module M $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M. A module M is called *strongly supplemented* or *lifting* if every submodule L of M has a strong supplement K in M, i.e., $M = L + K$, $L \cap K \ll K$ and $(L \cap K) \oplus L' = L$ for some submodule L' of L. Clearly, every strongly supplemented module is $\oplus$-supplemented.

Let M be a module. A module N is said to be *extension* of M provided $M \subseteq N$. As a generalization of injective modules, since every direct summand is a supplement, Zöschinger defined in [**12**] a module M with *the property* (E) if it has a supplement in every extension. He studied the various properties of a module M

2010 *Mathematics Subject Classification:* Primary 16D10; Secondary 16D50.
Key words and phrases: supplement, module with the properties (SE) and (SSE), artinian serial ring.
Communicated by Zoran Petrović.

with the property (E) in the same paper. We consider the following conditions for a module M:

(SE) In any extension N of M, M has a supplement that is a direct summand of N.

(SSE) M has a strong supplement in every extension.

Now we have these implications on modules:
injective $\Rightarrow$ module with (SSE) $\Rightarrow$ module with (SE) $\Rightarrow$ module with (E).
Some examples are given to show that these inclusions are proper.

In this study, we obtain some elementary facts about the properties (SE) and (SSE). Especially, we give a relation for the module where every submodule has the property (SE). We obtain that a semisimple R-module M has the property (E) if and only if M has the property (SSE). We prove that a module M over a von Neumann regular ring has the property (SE) if and only if it is injective. We illustrate a module with the property (SE) where factor module doesn't have the property (SE). We give a characterization of commutative artinian serial rings via the property (SE). We also show that a ring R is a left and right artinian serial ring with $\mathrm{Rad}(R)^2 = 0$ if and only if every left R-module has the property (SSE).

2. Modules with the properties (SE) and (SSE)

Recall that a submodule K of a module M is the *weak supplement* of a submodule L in M if $M = K + L$ and $K \cap L \ll M$. By the radical of a module M, denoted by $\mathrm{Rad}(M)$, we will indicate the sum of all small submodules of a module M, or equivalently, intersection of all maximal submodules of M. If $M = \mathrm{Rad}(M)$, that is, M has no maximal submodules, M is called *radical*.

PROPOSITION 2.1. *Let M be a semisimple R-module. Then, the following statements are equivalent.*

(1) *M has the property* (E).
(2) *M has the property* (SE).
(3) *M has a weak supplement in every extension N.*
(4) *For every module N with $M \subseteq N$, there exists a submodule K of N such that $N = M + K$ and $M \cap K \subseteq \mathrm{Rad}(N)$.*
(5) *M has the property* (SSE).

PROOF. (1) $\Rightarrow$ (2) Let N be any extension of M. By (1), we have $N = M + K$ and $M \cap K \ll K$ for some submodule $K \subseteq M$. Since M is semisimple module, then there exists a submodule X of M such that $M = (M \cap K) \oplus X$. So $(M \cap K) \cap X = K \cap X = 0$. Therefore $N = M + K = [(M \cap K) \oplus X] + K = K \oplus X$. This means that M has the property (SE).

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5) Let $M \subseteq N$. Then there exists a submodule K of N such that $N = M + K$ and $M \cap K \subseteq \mathrm{Rad}(N)$. By [**1**, 2.8 (9)], we obtain that $Soc(\mathrm{Rad}(N)) \ll N$. Since M is semisimple, we can write the decomposition $M = (M \cap K) \oplus X$ for some submodule $X \subseteq M$. It follows that $M \cap K = Soc(M \cap K) \subseteq Soc(\mathrm{Rad}(N)) \ll N$. Applying [**1**, 2.2(3)], $M \cap K$ is a small submodule of N. Since $N = M + K = X \oplus K$,

we obtain that $M \cap K \ll K$ by [**1**, 2.2(6)]. Hence, K is a strong supplement of M in N.

(5) $\Rightarrow$ (1) is trivial. $\square$

A module M is called *semilocal* if $\frac{M}{\mathrm{Rad}(M)}$ is semisimple, and a ring R is called *semilocal* if ${}_RR$ (or R_R) is semilocal. Lomp proved in [**6**, Theorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal. Using this fact we obtain the following:

Corollary 2.1. *Let M be a semisimple module over a semilocal ring R. Then, M has the property* (SSE).

Proof. Let N be an R-module with $M \subseteq N$. Since R is semilocal, we obtain that N is semilocal by [**6**, Theorem 3.5]. Therefore, there exists a submodule K of N such that $N = M + K$ and $M \cap K \subseteq \mathrm{Rad}(N)$. Applying Proposition 2.1, we derive that M has the property (SSE). $\square$

Let R be a ring and M be a left R-module. Take two sets I and J, and for every $i \in I$ and $j \in J$, an element r_{i_j} of R such that, for every $i \in I$, only finitely many r_{i_j} are nonzero. Furthermore, take an element m_i of M for every $i \in I$. These data describe a system of linear equations in M:

$$\sum_{j \in J} r_{i_j} x_j = m_i \text{ for every } i \in I.$$

The goal is to decide whether this system has a solution, i.e., whether there exist elements x_j of M for every $j \in J$ such that all the equations of the system are simultaneously satisfied (Note that we do not require that only finitely many of the x_j are nonzero here). Now consider such a system of linear equations, and assume that any subsystem consisting of only finitely many equations is solvable (The solutions to the various subsystems may be different). If every such "finitely-solvable" system is itself solvable, then the module M is called *algebraically compact*. For example, every injective module is algebraically compact.

Corollary 2.2. *Let R be a commutative noetherian ring. Then, the following three statements are equivalent for a semisimple left R-module M.*

(1) *M has the property (SE).*
(2) *M is algebraically compact.*
(3) *Almost all isotopic components of M are zero.*

Proof. It follows from Proposition 2.1 and [**12**, Proposition 1.6]. $\square$

It is clear that every injective module has the property (SSE), but the following example shows that a module with the property (SSE) need not be injective. Firstly, we need the following crucial lemma.

Lemma 2.1. *Every simple module has the property* (SSE).

Proof. Let M be a simple module and N be any extension of M. Since M is simple, then $M \ll N$ or $M \oplus K = N$ for a submodule K of N. In the first case, N

is a strong supplement of M in N. In the second case, K is a strong supplement of M in N. So, in each case M has a strong supplement in N, that is, M has the property (SSE). $\square$

Recall from [**2**] that a ring R is called *von Neumann regular* if every element $a \in R$ can be written in the form axa, for some $x \in R$. More formally, a ring R is regular in the sense of von Neumann if and only if the following equivalent conditions hold:

(1) $\frac{R}{I}$ is a projective R-module for every finitely generated ideal I.
(2) Every finitely generated left ideal is generated by an idempotent.
(3) Every finitely generated left ideal is a direct summand of R.

Example 2.1. (1) Consider the simple $\mathbb{Z}$-module $\frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is prime. By Lemma 2.1, M has the property (SSE). On the other hand, it is not injective.

(2) (See [**3**, 6.1]) Let V be a countably infinite-dimensional left vector space over a division ring S. Let $R = End({}_SV)$ be the ring of left linear operators on V. Then R is a von Neumann regular ring. Claim that the simple left R-module V is not injective. Assume the contrary that ${}_RV$ is injective. Consider a basis $\{v_i | i \in \mathbb{N}\}$ of V. For each $i \in \mathbb{N}$, let us define $f_i \in R$ by $f_i(v_i) = v_i$ and $f_i(v_i) = 0$ for $i \neq j$. Set $A = \sum_i Rf_i$. Then A is a left ideal of R. Consider a left R-homomorphism $\varphi : A \to_R V$ defined by $\varphi(\sum_i r_i f_i) = \sum_i r_i v_i$, where $r_i \in R$ is zero for all but finitely many i. Since ${}_RV$ is injective, there exists $v \in V$ such that $\varphi(f_i) = f_i v$ for every $i \in \mathbb{N}$. This gives $v_i = f_i v$ for every $i \in \mathbb{N}$. Now if $v = d_1v_1 + d_2v_2 + \cdots + d_nv_n$, then any $i \in \mathbb{N} \setminus \{1, 2, \ldots, n\}$, we have $f_i v = 0$, a contradiction. This shows ${}_RV$ is not injective. Thus R is not a left V-ring as the simple left R-module V is not injective. By Lemma 2.1, the left R-module V has the property (SSE).

Lemma 2.2. *Let M be a module with the property* (SE). *Suppose that N is an extension of M such that* $\mathrm{Rad}(N) = 0$. *Then, M is a direct summand of N.*

Proof. Let N be any extension of M. Since M has the property (SE), there exist submodules K and K' of N such that $N = M + K$, $M \cap K \ll K$ and $N = K \oplus K'$. By the hypothesis, $M \cap K \subseteq \mathrm{Rad}(N) = 0$. It follows that $N = M \oplus K$. $\square$

A ring R is said to be *left V-ring* if every simple left R-module is injective. It is well known that R is a left V-ring if and only if $\mathrm{Rad}(M) = 0$ for every left R-module M (3).

Proposition 2.2. *For a module M over a left V-ring R, the following statements are equivalent.*

(1) *M is injective.*
(2) *M has the property* (SSE).
(3) *M has the property* (SE).

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.
(3) $\Rightarrow$ (1) follows from Lemma 2.2. $\square$

Corollary 2.3. *Let R be a commutative von Neumann regular ring. Then, an R-module M has the property* (SE) *if and only if it is injective.*

Proof. Since R is a commutative von Neumann regular ring, it is a left V-ring. Hence, the proof follows from Proposition 2.2. □

Now, we give simple facts which are used for the proof of the following theorem. For a monomorphism $f : M \to M$, we have:

(1) If A, B are submodules of M such that $A \subseteq B \subseteq \mathrm{Im}(f)$, then $A \ll B$ implies $f^{-1}(A) \ll f^{-1}(B)$.
(2) If $M = A + B$ for some submodules A, B of M with one of contained in $\mathrm{Im}(f)$, then $M = f^{-1}(A) + f^{-1}(B)$.

Theorem 2.1. *For an R-module M, the following statements are equivalent.*

(1) *For an extension N of M and a submodule K such that $N = M + K$, K contains a supplement of M in N that is a direct summand of K.*
(2) *Every submodule of M has the property* (SE).

Proof. (1) ⇒ (2) Let U be any submodule of M and N be any extension of U. We have the following pushout

$$\begin{array}{ccc} U & \xrightarrow{\subset} & N \\ \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow{\alpha} & F \end{array}$$

It follows that $F = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$. Since α is a monomorphism, by assumption, $M \cong \mathrm{Im}(\alpha)$ has a supplement V in F with $V \leqslant \mathrm{Im}(\beta)$, i.e., $F = \mathrm{Im}(\alpha) + V$, $\mathrm{Im}(\alpha) \cap V \ll V$ and there exists a submodule K of $\mathrm{Im}(\beta)$ such that $\mathrm{Im}(\beta) = V \oplus K$. Then $N = \beta^{-1}(\mathrm{Im}(\alpha)) + \beta^{-1}(V) = U + \beta^{-1}(V)$ and $U \cap \beta^{-1}(V) \ll \beta^{-1}(V)$. Since β is a monomorphism, $N = \beta^{-1}(\mathrm{Im}(\beta)) = \beta^{-1}(V) \oplus \beta^{-1}(K)$, which means $\beta^{-1}(V)$ is a direct summand of N. Therefore U has the property (SE).

(2) ⇒ (1) Let N be any extension of M. Suppose that a submodule K of N satisfies $N = M + K$. By the hypothesis, $M \cap K$ has the property (SE), so $M \cap K$ has a supplement L in K such that L is a direct summand of K, that is, $K = (M \cap K) + L$, $(M \cap K) \cap L \ll L$ and there exists a submodule L' of K such that $K = L \oplus L'$. Then $N = M + K = M + L$, $M \cap L = (M \cap K) \cap L \ll L$. Also L is a direct summand of K. □

Recall that a ring R is *left hereditary* if every factor module of an injective left R-module is injective [**10**].

Example 2.2. [**12**] Let $R = \prod_{i \in I} F_i$ be a ring, where each F_i is field for an infinite index set I. Then R is a commutative von Neumann regular ring. Since R is not noetherian, it is not semisimple and so, by the Theorem of Osofsky [**8**], there is a cyclic R-module (which is clearly a factor module of R), which is not injective, and hence it does not have the property (SE) by Corollary 2.3.

Recall from [**11**] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M.

Proposition 2.3. *Let I be a two sided ideal in R and $\overline{R} = \frac{R}{I}$. If an R-module M has the property* (SE)*, $IM = 0$ and IK is a fully invariant submodule of any module K, then an $\overline{R}$-module ${}_{\overline{R}}M$ also has the property* (SE).

Proof. Let N be any extension of an $\overline{R}$-module ${}_{\overline{R}}M$. Since $IM = 0$, then ${}_RN$ is an extension of M. By the assumption, M has a supplement ${}_RV$ in ${}_RN$ that is a direct summand in ${}_RN$. Therefore ${}_{\overline{R}}V$ is a supplement of ${}_{\overline{R}}M$. Since IN is a fully invariant submodule of any module N, then ${}_{\overline{R}}V$ is a direct summand in ${}_{\overline{R}}N$. □

Proposition 2.4. *Let M be a module with the property* (SE)*. If M is a fully invariant submodule in every extension, then M has the property* (SSE).

Proof. Let N be any extension of M. Since M has the property (SE), there exist submodules K, K' of M such that $N = M + K$, $M \cap K \ll K$ and $N = K \oplus K'$. Since M is a fully invariant submodule of N, then $M = M \cap N = (M \cap K) \oplus (M \cap K')$. Therefore M has the property (SSE). □

Observe from Proposition 2.4 that since all radical modules are fully invariant, a radical module with the property (SE) has the property (SEE).

In [**9**, 1.4] a module M is called *uniserial* if its lattice of submodules is a chain. M is said to be *serial* if M is a direct sum of uniserial modules. A ring R is *left serial* if the module ${}_RR$ is serial.

Theorem 2.2. *The following statements are equivalent for a commutative ring R.*

(1) *R is artinian serial.*
(2) *Every left R-module has the property* (SE).

Proof. (1) ⇒ (2) Let R be an artinian serial ring. By [**5**, Corollary 3.13], every R-module is $\oplus$-supplemented. Let N be any extension of M. Since N is $\oplus$-supplemented, then there exist submodules K, K' of N such that $N = M + K$, $M \cap K \ll K$ and $N = K \oplus K'$. So M has the property (SE).

(2) ⇒ (1) Let M be any R-module. By the hypothesis, every submodule of M has the property (SE). Then there exist submodules K and K' of M such that $M = U + K$, $U \cap K \ll K$ and $M = K \oplus K'$ for any submodule U of M. So M is $\oplus$-supplemented. It follows from [**5**, Corollary 3.13] that R is an artinian serial ring. □

In [**1**, 29.10] a ring R is left and right artinian serial with $\mathrm{Rad}(R)^2 = 0$ if and only if every R-module is strongly supplemented (or lifting) if and only if every R-module is extending. Here a module M is said to be *extending* if every submodule is essential in a direct summand of M. Every injective module is extending. The next result gives another characterization of an artinian serial ring in term of the property (SSE).

Theorem 2.3. *The following statements are equivalent for any ring R.*

(1) *R is a left and a right artinian serial ring with* $\mathrm{Rad}(R)^2 = 0$.
(2) *Every left R-module has the property* (SSE).

PROOF. (1) $\Rightarrow$ (2) Let R be a left and right artinian serial ring with the property $\mathrm{Rad}(R)^2 = 0$, M be any R-module and N be any extension of M. By [**1**, 29.10], every submodule of N has a strong supplement in N. So M has the property (SSE).

(2) $\Rightarrow$ (1) Let M be any R-module. By the hypothesis, every submodule of M has the property (SSE). So every submodule of M has a strong supplement in M. By [**1**, 29.10], a ring R is left and right artinian serial with $\mathrm{Rad}(R)^2 = 0$. $\square$

Zöschinger proved in [**12**] that over a local Dedekind domain, a module M has the property (SSE) if and only if the reduced part of M is semisimple. The following example shows that a module with the property (SE) need not be (SSE).

EXAMPLE 2.3. Let R be a local Dedekind domain (not field). Consider the factor ring $\frac{R}{\mathrm{Rad}(R)^n}$ of R, ($n \geqslant 3$). Then, $\frac{R}{\mathrm{Rad}(R)^n}$ is an artinian serial ring and so, by Theorem 2.2, the $\frac{R}{\mathrm{Rad}(R)^n}$-module $\frac{R}{\mathrm{Rad}(R)^n}$ has the property (SE). Therefore, the R-module $\frac{R}{\mathrm{Rad}(R)^n}$ has the property (SE). Note that $\frac{R}{\mathrm{Rad}(R)^n}$ is reduced. Since $\frac{R}{\mathrm{Rad}(R)^n}$ is not semisimple, it hasn't the property (SSE).

References

1. J. Clark, C. Lomp, N. Vajana, R. Wisbauer, *Frontiers in Mathematics, Lifting Modules. Supplements and Projectivity in Module Theory*, Birkhäuser, Basel, 2006.
2. N. Jacobson, *Basic Algebra II*, 2nd ed., W. H. Freeman and Company, New York, 1989.
3. S. K. Jain, A. K. Srivastava, A. A. Tuganbaev, *Oxford Mathematical Monograps, Cyclic Modules and The Structure of Rings*, Oxford Science Publications, New York, 2012.
4. F. Kasch, *Modules and Rings*, Academic Press Inc., New York, 1982.
5. D. Keskin, P. F. Smith, W. Xue, *Rings whose modules are ⊕-supplemented*, J. Algebra **218** (1999), 470–487.
6. C. Lomp, *On semilocal modules and rings*, Comm. Algebra **27**(4) (1999), 1921–1935.
7. S. H. Mohamed, B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lect. Notes 147, Cambridge University Press, Cambridge, 1990.
8. B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.
9. G. Puninski, *Serial Rings*, Kluwer, Dordrecht, Boston, London, 2001.
10. D. W. Sharpe, P. Vamos, *Injective Modules*, Cambridge University Press, 1972.
11. R. Wisbauer, *Foundations of Modules and Rings*, Gordon and Breach, 1991.
12. H. Zöschinger, *Moduln, die in jeder erweiterung ein komplement haben*, Math. Scand. **35** (1974), 267–287.

Faculty of Art and Science
Amasya University
Amasya
Turkey
burcunisancie@hotmail.com

(Received 29 07 2014)
(Revised 16 01 2015)