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A CHARACTERIZATION OF PGL$(2, p^n)$ BY SOME IRREDUCIBLE COMPLEX CHARACTER DEGREES

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Abstract. For a finite group G, let $\mathrm{cd}(G)$ be the set of irreducible complex character degrees of G forgetting multiplicities and $X_1(G)$ be the set of all irreducible complex character degrees of G counting multiplicities. Suppose that p is a prime number. We prove that if G is a finite group such that $|G| = |\mathrm{PGL}(2,p)|$, $p \in \mathrm{cd}(G)$ and $\max(\mathrm{cd}(G)) = p+1$, then $G \cong \mathrm{PGL}(2,p)$, $SL(2,p)$ or $\mathrm{PSL}(2,p) \times A$, where A is a cyclic group of order $(2, p-1)$. Also, we show that if G is a finite group with $X_1(G) = X_1(\mathrm{PGL}(2,p^n))$, then $G \cong \mathrm{PGL}(2,p^n)$. In particular, this implies that $\mathrm{PGL}(2,p^n)$ is uniquely determined by the structure of its complex group algebra.

1. Introduction and preliminaries

Throughout this paper, let G be a finite group, p a prime number, n a natural number and let all characters of the groups be complex characters (that is, characters afforded by irreducible complex representations). The set of irreducible characters of G is denoted by $\mathrm{Irr}(G)$ and we write $\mathrm{cd}(G)$ for the set of irreducible character degrees of G forgetting multiplicities. Denote by $X_1(G)$ the first column of the ordinary character table of G. Thus $X_1(G)$ can be considered as the set of all irreducible character degrees of G counting multiplicities.

It is known that non-abelian simple groups are uniquely determined by their character tables. It was shown in [**9**] that the symmetric groups are also uniquely determined by their character tables. Hupert [**5**] conjectured that if G is a finite group and S is a finite non-abelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(S)$, then $G \cong S \times A$, where A is an abelian group. He verified the conjecture for the Suzuki groups, the family of simple groups $\mathrm{PSL}_2(q)$, for even q, and many of the sporadic simple groups. The authors proved in [**12, 8, 3**] that each Mathieu-groups, $\mathrm{PSL}(2,p)$, can be uniquely determined by their orders and their largest and second largest irreducible character degrees, respectively.

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Here we prove the following.

THEOREM 1.1. *If* $|G| = |\operatorname{PGL}(2,p)|$ *and*

(1) $p \in \operatorname{cd}(G)$, (2) $\max(\operatorname{cd}(G)) = p+1$,

then $G \cong \operatorname{PGL}(2,p), \operatorname{SL}(2,p)$ *or* $\operatorname{PSL}(2,p) \times A$, *where* A *is a cyclic group of order* $(2, p-1)$.

Tong-Viet [**10**] shows that the simple classical groups of Lie type are uniquely determined by the first column of their character tables. Here we prove

THEOREM 1.2. *For the natural number* n, $\operatorname{PGL}(2,p^n)$ *is uniquely determined by the first column of its character table.*

Let $\mathbb{C}$ be the complex number field. Denote by $\mathbb{C}G$ the group algebra of G. The Brauer's Problem asks which groups can be determined by the structure of their complex group algebras. As a consequence of our results, we show that $\operatorname{PGL}(2,p^n)$ is uniquely determined by the structure of its complex group algebra.

Throughout the paper, we use the following notations: For a natural number n, $\pi(n)$ is the set of prime divisors of n and $\pi(G)$ is $\pi(|G|)$. For a prime r, the set of r-Sylow subgroups of G is denoted by $\operatorname{Syl}_r(G)$ and $n_r(G) = |\operatorname{Syl}_r(G)|$. Let s be a prime and let m be a natural number. We use $s^e \parallel m$ when $s^e \mid m$ but $s^{e+1} \nmid m$. The s-part of m is denoted by $|m|_s$, i.e., $|m|_s = s^e$ if $s^e \parallel m$. If $\gcd(\mathrm{s},\mathrm{m}) = 1$ and s is odd, then we denote by $e(s,m)$ multiplicative order of m modulo s, i.e., the smallest natural number n satisfying the condition $m^n \equiv 1 \pmod{s}$. Also, we write H ch G if H is a characteristic subgroup of G. Set $H_G = \cap_{g\in G} H^g$. If $\chi = \sum_{i=1}^{N} n_i\chi_i$, where for every $1 \leqslant i \leqslant N$, $\chi_i \in \operatorname{Irr}(G)$, then those χ_i with $n_i > 0$ are called irreducible constituents of χ.

In the following lemmas, for $\chi \in \operatorname{Irr}(G)$ and the normal subgroup N of G, χ_N is the restriction of χ to N and for $\theta \in \operatorname{Irr}(N)$, θ^G is the induced character on G. For Theorem 1.1, we need some facts about the relation between $\operatorname{Irr}(G)$ and $\operatorname{Irr}(G/N)$, when for some $\chi \in \operatorname{Irr}(G)$, $\chi_N = \theta \in \operatorname{Irr}(N)$.

LEMMA 1.1. (Gallagher's Theorem) [**6**, Corollary 6.17] *Let* $N \trianglelefteq G$ *and let* $\chi \in \operatorname{Irr}(G)$ *be such that* $\chi_N = \theta \in \operatorname{Irr}(N)$. *Then the characters* $\beta\chi$ *for* $\beta \in \operatorname{Irr}(G/N)$ *are irreducible distinct for distinct* β *and are all of the irreducible constituents of* θ^G.

In order to find the normal abelian subgroups of the given groups in Theorems 1.1 and 1.2, we need the following well-known lemma.

LEMMA 1.2 (Ito's Theorem). [**6**, Theorem 6.15] *Let* $A \trianglelefteq G$ *be abelian. Then* $\chi(1) \mid [G\colon A]$, *for all* $\chi \in \operatorname{Irr}(G)$.

The interest of Lemma 1.3 is that it allows one to obtain some information about cd of the normal subgroup N of G by considering some elements of $\operatorname{cd}(G)$ and $[G\colon N]$, which will be needed in the proofs of Theorems 1.1 and 1.2.

LEMMA 1.3. [**6**, Theorem 6.2 and Corollary 11.29] *Let* $N \trianglelefteq G$ *and* $\chi \in \operatorname{Irr}(G)$. *Let* θ *be an irreducible constituent of* χ_N *and suppose that* $\theta_1 = \theta, \ldots, \theta_t$ *are distinct conjugates of* θ *in* G. *Then* $\chi_N = e\sum_{i=1}^{t}\theta_i$, *where* $e = [\chi_N, \theta]$. *Also,* $\chi(1)/\theta(1) \mid [G\colon N]$.

Applying Lemma 1.4 to the proof of Theorem 1.1 (Case b) leads us to obtain some prime divisors of the elements of cd of some normal subgroups of G, by considering their normal abelian Sylow subgroups.

LEMMA 1.4 (Ito–Michler's Theorem). [**4**, Theorem 19.10 and Remark 19.11] *Let $\rho(G)$ be the set of all prime divisors of the elements of* $\mathrm{cd}(G)$. *Then $p \notin \rho(G)$ if and only if G has a normal abelian p-Sylow subgroup.*

In this paper, we need $\mathrm{cd}(\mathrm{SL}(2, q))$, $\mathrm{cd}(\mathrm{PSL}(2, q))$, $\mathrm{cd}(\mathrm{PGL}(2, q))$ and $\mathrm{cd}(G)$, where G is an extension of $\mathrm{PSL}(2, q)$, frequently. So we bring them in Lemma 1.5 for making it easy to use.

LEMMA 1.5. [**11**, Theorem A and Corollary C] *If q is a power of an odd prime number, then*

(i) $\mathrm{cd}(\mathrm{SL}(2, q)) = \{1, q-1, (q-1)/2, q, q+1, (q+1)/2\}$;
(ii) $\mathrm{cd}(\mathrm{PSL}(2, q)) = \{1, q-1, q, q+1, (q+\varepsilon)/2\}$, *where* $\varepsilon = (-1)^{(q-1)/2}$;
(iii) $\mathrm{cd}(\mathrm{PGL}(2, q)) = \{1, q-1, q, q+1\}$;
(iv) *if* $q > 3$ *and* $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{Aut}(\mathrm{PSL}(2, q))$ *such that* $[G : \mathrm{PSL}(2, q)] = 2$ *and* $G \neq \mathrm{PGL}(2, q)$, *then* $2(q-1) \in \mathrm{cd}(G)$.

Since for every odd prime divisor r of $|\,\mathrm{PGL}(2, p^n)|$, $\mathrm{PGL}(2, p^n)$ has exactly one irreducible character degree divisible by r, we may apply the following lemma to the proof of Step 3 of Theorem 1.2.

LEMMA 1.6. [**7**, Theorem C and Corollary 7.5] *Let G be a finite group with exactly one irreducible character degree divisible by p. Assume that G is not p-solvable, and let $U = O_p(G)$ and $K/U = O_{p'}(G/U)$. Then K is the unique largest normal p-solvable subgroup of G. Also, G/K has a simple socle S/K, and $[G:S]$ is not divisible by p. In particular, $S/K \cong M_{11}$, J_1 or* $\mathrm{PSL}(2, q)$, *where q is a power of the prime r.*

Lemmas 1.7 and 1.8 will be needed in Step 3 of the proof of Theorem 1.2 and the proof of Theorem 1.1, respectively.

LEMMA 1.7. [**6**, Theorem 12.15] *If* $|\,\mathrm{cd}(G)| \leqslant 3$, *then G is solvable.*

LEMMA 1.8. [**12**] *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and* $|G/K| \mid |\,\mathrm{Out}(K/H)|$.

The following lemma follows immediately by checking the order of finite simple groups of Lie type over a finite field of order q for showing that the non-abelian chief factor of G is isomorphic to $\mathrm{PSL}(2, p)$.

LEMMA 1.9. *Let H be a finite simple group of Lie type over a finite field of order q, where $q = r^t$ for a prime r. If $p \in \pi(H)$ and $e(p, q) = i$, then $(q^i - 1) < |H|_r$ except in the following cases:*

(i) $i = 2$ *and* $H = \mathrm{PSL}(2, q)$;
(ii) $i = 6$ *and* $H = PSU(3, q)$;
(iii) $i = 4$ *and* $H = {}^2B_2(q)$, *where* $q = 2^{2m+1}$, $m \geqslant 1$;
(vi) $i = 6$ *and* $H = {}^2G_2(q)$, *where* $q = 3^{2m+1}$, $m \geqslant 1$.

2. Proof of Theorem 1.1

Throughout this section, let G be a group satisfying the conditions of the main theorem. Since $p, p+1 \in \mathrm{cd}(G)$, fix $\chi, \phi \in \mathrm{Irr}(G)$ such that $\chi(1) = p$ and $\phi(1) = p+1$.

I. Let $p = 3$ and $P \in \mathrm{Syl}_3(G)$. If $n_3(G) = 1$, then, since P is a cyclic group of order 3, Ito's theorem forces $3 = \chi(1) \mid [G : P] = 8$, which is impossible. Thus $n_3(G) = 4$ and hence, $P_G = 1$, so $G = G/P_G \hookrightarrow S_4$. But $|G| = |S_4|$, so $G \cong S_4 \cong \mathrm{PGL}(2, 3)$, as claimed. The same reasoning completes the proof in the case when $p = 2$.

II. Let $p > 3$. We claim that G is not solvable. On the contrary, suppose that G is solvable. We are going to get a contradiction in the following cases:

Case a. Let $(p-1)/2$ be even. Let H be a Hall subgroup of G of order $2p(p-1)$. Thus $[G:H] = (p+1)/2$. Hence $G/H_G \hookrightarrow S_{(p+1)/2}$. Since $p > (p+1)/2$, $p \in \pi(H_G)$. Let $P \in \mathrm{Syl}_p(H_G)$. We can see that $n_p(H_G) = 1$, so P ch $H_G \trianglelefteq G$ and hence, $P \trianglelefteq G$. But since $|P| = p$, P is abelian and hence, by Ito's theorem, $p = \chi(1) \mid [G : P]$, which is a contradiction.

Case b. Let $(p+1)/2$ be even. If there exists an odd prime r and a natural number α such that $r^\alpha \parallel (p+1)/4$, then let H be a Hall subgroup of G of order $p(p-1)(p+1)/r^\alpha$. Thus $[G:H] = r^\alpha$. Hence, $G/H_G \hookrightarrow S_{r^\alpha}$. But $r^\alpha \leqslant (p+1)/4$. Thus $p \in \pi(H_G)$. Let $P \in \mathrm{Syl}_p(H_G)$. Since $|P| = p$, P is abelian. Also, $n_p(H_G) = kp+1 \mid (p-1)(p+1)/r^\alpha$. First suppose that $k \geqslant 1$. Then there exists a natural number t such that $r^\alpha t(kp+1) = (p-1)(p+1)$. Therefore, $p \mid tr^\alpha + 1$ and hence there exists a natural number s such that $ps = r^\alpha t + 1$. Hence, $(ps-1)(kp+1) = p^2 - 1$, which implies $k = s = 1$. Therefore, $r^\alpha \mid p-1$. On the other hand, $r \mid p+1$, so $r \mid \gcd(p-1, p+1) = 2$, which is a contradiction. It follows that $n_p(H_G) = 1$. Thus $P \trianglelefteq G$ and hence, Ito's theorem implies that $p = \chi(1) \mid [G:P]$, which is impossible.

Let $p+1 = 2^\alpha$, for some natural number α. Let H be a Hall subgroup of G of order $2p(p+1) = 2^{\alpha+1}p$. Then $[G : H] = (p-1)/2$ and $G/H_G \hookrightarrow S_{(p-1)/2}$. Thus $p \in \pi(H_G)$. Let $P \in \mathrm{Syl}_p(H_G)$. Since $|P| = p$, P is abelian. If $n_p(H_G) = 1$, then $P \trianglelefteq G$, so applying Ito's theorem to P and χ leads us to get a contradiction. Therefore, $n_p(H_G) \neq 1$, so we can see at once that $n_p(H_G) = p+1 = 2^\alpha$ and hence, $[H : H_G] \mid 2$. Let $\theta \in \mathrm{Irr}(H_G)$ such that $[\phi_{H_G}, \theta] \neq 0$. Then Lemma 1.3 shows that $p+1 = \phi(1) \mid \theta(1)[G:H_G] = \theta(1)[G:H][H : H_G]$, so either $|H| = |H_G|$ and $p+1 \mid \theta(1)$ or $|H_G| = |H|/2$ and $(p+1)/2 \mid \theta(1)$. Also, $p \in \pi(H_G)$ and $n_p(H_G) \neq 1$, so Ito–Michler's Theorem guarantees that there exists $\eta \in \mathrm{cd}(H_G)$ such that $p \mid \eta(1)$. It is known that $\Sigma_{\alpha \in \mathrm{Irr}(H_G)} \alpha(1)^2 = |H_G|$. Thus either $|H_G| = |H|$ and $p^2 + (p+1)^2 \leqslant |H_G|$ or $|H_G| = |H|/2$ and $p^2 + ((p+1)/2)^2 \leqslant |H_G|$. This forces either $p^2 + (p+1)^2 \leqslant 2p(p+1)$ or $p^2 + (p+1)^2/4 \leqslant p(p+1)$, which is impossible.

Therefore, G is not solvable. Now, Lemma 1.8 shows that G has a normal series as $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of m copies of a non-abelian simple group S. Since $p \| |G|$, we deduce that exactly one of the following holds:

$$p \mid |G/K|, \quad p \mid |H|, \quad p \mid |K/H|.$$

Thus the proof falls into the following cases:

i. Let $p \mid |G/K|$. We know that K/H is isomorphic to m copies of a non-abelian simple group S. Thus $\mathrm{Out}(K/H) \cong \mathrm{Out}(S) \wr S_m$. Also Lemma 1.8 shows that $|G/K| \mid |\mathrm{Out}(K/H)|$. Therefore $p \mid |\mathrm{Out}(S)|$ or $p \mid |S_m|$. If $p \mid |S_m|$, then $m \geqslant p$. But the order of the smallest simple group is 60 and hence, $60^p \leqslant |K/H|$. It follows that $60^p \leqslant p(p^2-1)$, a contradiction. Hence, $p \mid |\mathrm{Out}(S)|$ and $p \nmid |S|$. Now, considering the order of the outer automorphism groups of alternating groups and simple sporadic groups leads us to see that S is a simple group of Lie type over a finite field of order q, where $q = p_0^f$ for some prime number p_0 and some natural number f such that $p \mid f$ (see [**2**]). Since $p \nmid |S|$ and $|S| \mid |G|$, $|S| \mid (p^2-1)$ and since $p \geqslant 5$, $q \mid |S|$ and $p_0 \geqslant 2$, we deduce that $2^p \leqslant p_0^p \leqslant p_0^f = q \leqslant p^2-1$, which is a contradiction.

ii. Let $p \mid |H|$. Let $\theta \in \mathrm{Irr}(H)$ be a constituent of χ_H. Then Lemma 1.3 implies that $\chi(1)/\theta(1) \mid [G\colon H]$, and since $p \nmid [G\colon H]$, $\theta(1) = p$. So $\chi_H = \theta$ and now, Gallagher's theorem shows that for every $\beta \in \mathrm{Irr}(G/H)$, $\beta\chi \in \mathrm{Irr}(G)$. So for every $\beta \in \mathrm{Irr}(G/H)$, $p\beta(1) \in \mathrm{cd}(G)$. But by our assumption, $\max(\mathrm{cd}(G)) = p+1$, so for every $\beta \in \mathrm{Irr}(G/H)$, $\beta(1) = 1$ and hence, G/H is abelian, which is a contradiction.

iii. Let $p \mid |K/H|$. Since K/H is isomorphic to the direct product of m copies of S, we must have $p^m \mid |K/H|$. But we know that $p \parallel |G|$. This implies that K/H is a simple group such that p is the maximal prime divisor of its order. Also $|K/H| \mid p(p^2-1)$. Now, these conditions on K/H rule out the case that K/H is a sporadic simple group.

If K/H is an alternating group, then $S \cong A_n$, for some $n \geqslant 5$, so $p \leqslant n$ and $n!/2 = |A_n| = |S| \leqslant p(p^2-1) \leqslant n(n^2-1)$. This implies that $p = n = 5$ and hence, $K/H \cong A_5 \cong \mathrm{PSL}(2,5)$.

Let K/H be a finite simple group of Lie type over a finite field of order q, where $q = r^u$ for a prime r. If $p \neq r$, then suppose $e(p,q) = i$. Since $|K/H| \mid |G|$, we deduce that one of the following holds:

1. Let $|K/H|_r = |p^2-1|_2$. Then $r = 2$ and since $\gcd(p-1, p+1) = 2$, we can see that $|p^2-1|_2 = 2|p-1|_2$ or $2|p+1|_2$. If $4 \mid q$, then either $i = 1$ or $p \nmid q-1$ and hence, $p \mid (q^i-1)/(q-1)$. If $i = 1$, then since $|K/H|_r = q^t$, we can see that $q-1 \mid |K/H|_r - 1$, so $p \mid |K/H|_r - 1 = |p^2-1|_2 - 1$, which is impossible. So $i \neq 1$ and hence, $p \mid (q^i-1)/(q-1)$. Thus $3p \leqslant (q-1)p \leqslant q^i - 1$ and $|K/H|_r \leqslant 2(p+1)$. Therefore, $(q^i-1) > |K/H|_r$ and so, K/H is isomorphic to one of the groups obtained in Lemma 1.9(1–3). If $i = 6$ and $K/H \cong PSU(3,q)$, then $p \mid (q^3+1)/(q+1)$. Thus $5p \leqslant (q^3+1) < 2q^3 \leqslant 2|K/H|_r \leqslant 4(p+1)$, which is impossible. If $i = 4$ and $K/H \cong {}^2B_2(q)$, where $q = 2^{2m+1}$, then since $2m+1$ is odd, q^2+1 is not prime, so we can see that $p \neq q^2+1$ and hence, $3p \leqslant q^2+1 \leqslant 2(p+1)+1$, which is impossible. Thus $K/H \cong \mathrm{PSL}(2,q)$. Now let $q = 2$. If $p \neq q^i - 1$, then we can see that $3p \leqslant q^i - 1$. Now applying the previous argument leads us to get a contradiction. If $p = 2^i - 1$, then i is prime and $2^i = p+1$. Since $|K/H|_2 = |p^2-1|_2$, we deduce that $|K/H|_2 = 2^{i+1}$. But $p \geqslant 5$, so $i \geqslant 3$. Thus checking the order of finite simple groups of Lie type leads us to get a contradiction.

2. Let $|K/H|_r \mid (p+1)/2$ or $|K/H|_r \mid (p-1)/2$. Then $p \leqslant q^i - 1$ and $|K/H|_r \leqslant (p+1)/2$. Thus $(q^i-1) > |K/H|_r$ and so, K/H is isomorphic to one of the groups obtained in Lemma 1.9. If $i=6$ and $K/H \cong PSU(3,q)$, then $p \leqslant (q^3+1) \leqslant |K/H|_r + 1 \leqslant (p+1)/2+1$, which is impossible. If $i=4$ and $K/H \cong {}^2B_2(q)$, where $q = 2^{2m+1}$, then since $p \mid q^2+1$, $p \leqslant q^2+1 = |K/H|_r + 1 \leqslant (p+1)/2+1$ and hence, $p \leqslant 3$, which is impossible. The same reasoning rules out the case that $i=6$ and $K/H \cong {}^2G_2(q)$, where $q = 3^{2m+1}$. Thus $K/H \cong \mathrm{PSL}(2,q)$.

3. Let $|K/H|_r = |p-1|_2$ or $|K/H|_r = |p+1|_2$. If $|K/H|_r \neq p-1$ and $|K/H|_r \neq p+1$, then we can see that $|K/H|_r \leqslant (p+1)/3$ and hence, applying the same argument as that used in 2 leads us to $K/H \cong \mathrm{PSL}(2,q)$. Thus for some natural number t, $|K/H|_r = 2^t$ and either $|K/H|_r = p-1$ or $|K/H|_r = p+1$. Since $p \mid |K/H|$, checking the order of finite simple groups of Lie type shows that either $p=7$ and $K/H \cong \mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ or $K/H \cong \mathrm{PSL}(2,q)$.

But if $K/H \cong \mathrm{PSL}(2,q)$, where $p \nmid q$, then since $p \in \pi(K/H)$, either $p \mid q-1$ or $p \mid q+1$, so we have the following possibilities:

- If $p = q-1$, then $q = 2^\alpha$, for some natural number α and
$$|G| = p(p-1)(p+1) = q(q-1)(q-2) < q(q^2-1) = |K/H|,$$
which is impossible.
- If $p \mid q-1$ and $p \leqslant (q-1)/2$, then
$$|G| = p(p^2-1) \leqslant ((q-1)/2)((q+1)/2)((q-3)/2) < q(q^2-1)/2 = |K/H|,$$
which is a contradiction.
- If $p \mid q+1$, then applying the same argument as above leads to $q=4$ and $p=5$. Thus $K/H \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$.

These show that $p \mid q$. Since $p \parallel |G|$, we deduce that $p = q$. Thus considering the order of finite simple groups of Lie type over a finite field of order p forces $K/H = \mathrm{PSL}(2,p)$, and so $|H| = 2$ or $|G/K| = 2$. Let for some natural number d with $d \mid 2n$, $dp(q-1)$ or $dp(q+1)$ belongs to $\mathrm{cd}(G/K) = \mathrm{cd}(G)$, $|G/K| = 1$ and hence, $G = K$ and $G \cong 2 : \mathrm{PSL}(2,p)$. Thus either $G \cong Z(G) \times \mathrm{PSL}(2,p)$ or $G \cong \mathrm{SL}(2,p)$. If $|G/K| = 2$, then $|H| = 1$ and hence, $K = \mathrm{PSL}(2,p)$ and $G = \mathrm{PSL}(2,p) : 2 = \mathrm{PGL}(2,p)$. Thus the theorem is proved.

Corollary 2.1. *Let $|G| = |\,\mathrm{PGL}(2,p)|$. If $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}(2,p))$, then G is isomorphic to* $\mathrm{PGL}(2,p)$.

Proof. It follows immediately from the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

First let $n=1$. Since $X_1(\mathrm{PGL}(2,p)) = X_1(G)$, $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}(2,p))$ and $|G| = |\,\mathrm{PGL}(2,p)|$, because $|G| = \sum_{\chi\in \mathrm{Irr}(G)} \chi(1)^2$. Thus Corollary 2.1 completes the proof. Now let $n > 1$. Since $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}(2,p^n))$, we deduce by Lemma 1.5(iii) that $\mathrm{cd}(G) = \{1, p^n-1, p^n+1, p^n\}$. Thus for every $r \in \pi(G) - \pi((2,p-1))$, G has exactly one irreducible character degree divisible by r. We are going to complete the proof in the following steps.

Step 1. If M is a nontrivial normal solvable subgroup of G, then p is odd, $M = Z(G)$ and $|M| = 2$.

Proof . Let N be a normal minimal subgroup of G such that $N \leqslant M$. Then there exists $r \in \pi(G)$ such that N is an r-elementary abelian subgroup. Thus Ito's theorem and our assumption force $p^n, p^n - 1, p^n + 1 \mid [G : N]$ and hence $p^n(p^{2n} - 1)/(2, p - 1) \mid [G : N] = |\,\mathrm{PGL}(2, p^n)|/|N|$. Therefore, $|N| = (2, p - 1)$. Since $M \neq 1$, we deduce that $|N| \neq 1$ and hence, $2 \mid p-1$ and $|N| = 2$. This forces $N \leqslant Z(G)$. Also, applying Ito's theorem and our assumption force $p^n, p^n-1, p^n+1 \mid [G\colon Z(G)]$ and hence, $|Z(G)| = |N| = 2$. We claim that $M = N$. If not, then we can assume that L/N is a normal minimal subgroup of G/N such that $L/N \leqslant M/N$. Thus there exists $s \in \pi(G)$ such that L/N is a s-elementary abelian subgroup. If $s \neq 2$, then since $N = Z(G)$, we deduce that G contains a normal subgroup K such that $K \cong L/N$, which is a contradiction with the above statements. Thus $s = 2$. If $|L| = 4$, then Ito's theorem and our assumption guarantee $p^n(p^{2n} - 1)/2 \mid [G\colon L]$, which is a contradiction. Thus $|L| > 4$. Now for $\varepsilon = \pm$, let $\chi_\varepsilon \in \mathrm{Irr}(G)$ such that $\chi_\varepsilon(1) = p^n + \varepsilon 1$ and $\theta_\varepsilon \in \mathrm{Irr}(L)$ such that $[\chi_\varepsilon, \theta_\varepsilon] \neq 0$. Thus Lemma 1.3 shows that $\chi_\varepsilon(1)/\theta_\varepsilon(1) \mid [G\colon L]$ and hence there exists $\theta \in \mathrm{Irr}(L)$ such that $|L|/2 \mid \theta(1)$. On the other hand, L/N is 2-elementary abelian and $|L/N| \geqslant 4$. Thus there exists $xN \in L/N$ such that $O(xN) = 2$ and $\langle xN \rangle \neq L/N$. Therefore, $\langle x \rangle N$ is a normal abelian subgroup of L of order 4 and hence Ito's theorem shows that $\theta(1) \mid |L|/4$, which is a contradiction. Therefore, $M = N$, as claimed.

Step 2. There exists $r \in \pi(G) - \{2\}$ such that G is not an r-solvable group.

Proof. Since by Step 1, G is not solvable, the result follows immediately from the definition of r-solvable groups.

Step 3. $G \cong \mathrm{PGL}(2, p^n)$.

Proof. By Step 2, there exists $r \in \pi(G) - \{2\}$ such that G is not r-solvable. Also, $\mathrm{cd}(G) = \{1, p^n, p^n + 1, p^n - 1\}$. Thus Lemma 1.6 shows that if $U = O_r(G)$ and $K/U = O_{r'}(G/U)$, then G/K has a simple socle S/K (which is isomorphic to M_{11}, J_1 or $\mathrm{PSL}(2, q)$), and $[G : S]$ is not divisible by p. Since $r \neq 2$, step 1 shows that $U = 1$. Also, $\mathrm{cd}(G/K) = \{\chi(1) : \chi \in \mathrm{Irr}(G),\ K \leqslant \ker(\chi)\}$ and Lemma 1.7 shows that $|\,\mathrm{cd}(G/K)| \geqslant 4$. Therefore, $\mathrm{cd}(G/K) = \mathrm{cd}(G)$. Thus $p^n, p^n + 1, p^n - 1 \mid |G/K|$. This shows that $|G|/2 \mid |G/K|$. Thus considering the order of $\mathrm{Aut}(M_{11})$ and $\mathrm{Aut}(J_1)$ guarantees that S/K is not isomorphic to M_{11} and J_1 and hence, $S/K \cong \mathrm{PSL}(2, q)$. If $p \mid [G : S]$, then Theorem A in [**11**] shows that for some natural number with $d \mid 2n$, $dp(q - 1)$ or $dp(q + 1)$ belongs to $\mathrm{cd}(G/K) = \mathrm{cd}(G)$, which is a contradiction. Thus $p \nmid [G : S]$. So $p^n \in \mathrm{cd}(S/K)$ and $p^n \mid |S/K|$. If $p \nmid q$, then considering the elements of $\mathrm{cd}(S/K)$ mentioned in Lemma 1.5(ii) shows that $p^n \mid q + 1$ or $p^n \mid q - 1$. If $p^n = q + 1$ or $q - 1$, then $|S/K| = p^n(p^n - 1)(p^n - 2)$ or $p^n(p^n + 1)(p^n + 2)$ which divides $|G|$ and hence $p^n - 2 \mid p^n + 1$ or $p^n + 2 \mid p^n - 1$, which is impossible. Thus $p^n \mid q$ and since $p^n \parallel |G|$, we deduce that $p^n = q$ and hence, $S/K \cong \mathrm{PSL}(2, p^n)$. If $p = 2$, then $|S/K| = |G|$ and hence, $S = G$, as claimed. Now let p be odd. If $K \neq 1$, then $G/S = 1$ and by step 1, $K = Z(G)$, which is a cyclic group of order 2 and hence $G \cong \mathrm{SL}(2, p^n)$ or $\mathrm{PSL}(2, p^n) \times Z(G)$. But $\mathrm{cd}(\mathrm{SL}(2, p^n)), \mathrm{cd}(\mathrm{PSL}(2, p^n)) \neq \mathrm{cd}(\mathrm{PGL}(2, p^n))$, by Lemma

1.5, which is a contradiction. Thus $K = 1$ and $|G/S| = 2$. Since $K = 1$ and S is a socle of G, we can see that $C_G(S) = 1$ and hence, $G/S \lesssim \mathrm{Out}(S)$. But we know that if p is an odd prime, then $\mathrm{Out}(S) = \mathrm{Out}(\mathrm{PSL}(2, p^n)) = (\langle\delta\rangle \times \langle\gamma\rangle)$, where δ is a diagonal automorphism of order 2 and γ is a field automorphism of order n. Also, $\mathrm{PSL}(2, p).\langle\delta\rangle = \mathrm{PGL}(2, p)$. If $G \ncong \mathrm{PGL}(2, p^n)$, then since $[G : S] = 2$, we deduce that G contains a field automorphism ϕ of order 2 and hence, $G = \mathrm{PSL}(2, p^n).\langle\phi\rangle$. Thus Lemma 1.5(iv) shows that $2(p^n - 1) \in \mathrm{cd}(G) = \mathrm{cd}(\mathrm{PGL}(2, p^n))$, which is a contradiction. This shows that $G \cong \mathrm{PGL}(2, p^n)$, as claimed.

REMARK 3.1. By Molien's theorem [**1**, Theorem 2.13] $X_1(\mathrm{PGL}(2, p^n)) = X_1(G)$ if and only if $\mathbb{C}\,\mathrm{PGL}(2, p^n) = \mathbb{C}G$. Thus Theorem 1.2 shows that if $\mathbb{C}G = \mathbb{C}\,\mathrm{PGL}(2, p^n)$, then $G \cong \mathrm{PGL}(2, p^n)$.

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