PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 99(113)) (2016), 287–294 DOI: 10.2298/PIM1613287B

# ON THE LOCATION OF THE ZEROS OF CERTAIN POLYNOMIALS

**S. D. Bairagi, Vinay Kumar Jain, T. K. Mishra, and L. Saha**

Abstract. We extend Aziz and Mohammad's result that the zeros, of a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$, $ta_j \geqslant a_{j-1} > 0$, $j = 2, 3, \ldots, n$ for certain $t$ ( $> 0$), with moduli greater than $t(n-1)/n$ are simple, to polynomials with complex coefficients. Then we improve their result that the polynomial $P(z)$, of degree $n$, with complex coefficients, does not vanish in the disc

$$|z - ae^{i\alpha}| < a/(2n); a > 0, \max_{|z|=a} |P(z)| = |P(ae^{i\alpha})|,$$

for $r < a < 2, r$ being the greatest positive root of the equation

$$x^n - 2x^{n-1} + 1 = 0,$$

and finally obtained an upper bound, for moduli of all zeros of a polynomial, (better, in many cases, than those obtainable from many other known results).

## 1. Introduction and statement of results

While thinking about zeros of a polynomial, Aziz and Mohammad [**1**] proved the following result, thereby suggesting that certain zeros of a polynomial with non-negative increasing coefficients may be simple.

Theorem A. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ such that $a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 > 0$. Then all the zeros of $P(z)$, of modulus $\geqslant n/(n+1)$, are simple.*

In [**2**] Aziz and Mohammad obtained, a generalization as well as a refinement, of Theorem A.

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n > 1$, such that for some $t > 0$ one has $ta_j \geqslant a_{j-1} > 0$, $j = 2, 3, \ldots, n$, $a_0$ may be a real or a complex number. Then all the zeros of $P(z)$, of modulus greater than $t(n-1)/n$, are simple.*

2010 *Mathematics Subject Classification:* Primary 30C15; Secondary 30C10.
*Key words and phrases:* simple zeros, zero free region, refinement, upper bound for moduli of all zeros.
Communicated by Gradimir Milovanović.

In this paper, we extend Theorem B to polynomials with complex coefficients. More precisely we prove

THEOREM 1.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n(> 1)$ such that for certain $t > 0$*

$$t|a_j| \geqslant |a_{j-1}|, \quad j = 2, 3, \ldots, n \tag{1.1}$$

*and for certain real $\alpha$ and $\beta$: $|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2$, $j = 1, 2, 3, \ldots, n$, then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than*

$$\frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\right\}.$$

*In other words all the zeros of $p(z)$, with moduli greater than*

$$\frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\right\},$$

*are simple.*

THEOREM 1.2. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(> 1)$, with $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, \ldots, n$. If for certain $t > 0$*

$$t\alpha_j \geqslant \alpha_{j-1} \geqslant 0, \quad j = 2, 3, \ldots, n \quad \text{and} \quad \alpha_n > 0, \tag{1.2}$$

*then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than*

$$\frac{t(n-1)}{n}\left\{1 + \frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\Big(\frac{n}{t(n-1)}\Big)^{n-j}(j+1)|\beta_{j+1}|\right\}. \tag{1.3}$$

*In other words all the zeros of $p(z)$, with moduli greater than* (1.3) *are simple.*

Next we obtain the following result for polynomials with complex coefficients, similar to Theorem B, but not an extension of Theorem B to polynomials with complex coefficients.

THEOREM 1.3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(> 1)$ such that for certain $t > 0$: $t|a_j| \geqslant |a_{j-1}|$, $j = 2, 3, \ldots, n$. If $k$ is the greatest positive root of the trinomial equation $x^n - 2x^{n-1} + 1 = 0$, then $p(z)$ can not have a zero of order ($\geqslant 2$), with modulus greater than $kt(n-1)/n$. In other words all the zeros of $p(z)$, with moduli greater than $kt(n-1)/n$, are simple.*

Further in [**2**], Aziz and Mohammad also obtained a zero free region for polynomials with complex coefficients.

THEOREM C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients. Then for no real $a > 0$, $P(z)$ vanishes in the disk $|z - ae^{i\alpha}| < a/(2n)$, where*

$$\operatorname*{Max}_{|z|=a}|P(z)| = |P(ae^{i\alpha})|.$$

We obtain a refinement of Theorem C, for $r < a < 2$, $r$ being the greatest positive root of the equation $x^n - 2x^{n-1} + 1 = 0$.

THEOREM 1.4. *Let $p(z)$ be a polynomial of degree $n$ $(\geqslant 2)$ and $K$, the greatest positive root of the equation $x^{n+1} - 2x^n + 1 = 0$. If*

$$\max_{|z|=a} |p(z)| = |p(ae^{i\alpha_a})|, \quad a > 0, \tag{1.4}$$

$$R_a = \begin{cases} \frac{a^n(a-1)}{n(a^n-1)}, & 0 < a < K,\ a \neq 1, \\ 1/n^2, & a = 1, \\ 1/n, & a \geqslant K, \end{cases} \tag{1.5}$$

*then for no $a > 0$, $p(z)$ vanishes in the disc $|z - ae^{i\alpha_a}| < R_a$.*

Finally we obtain an upper bound for moduli of all zeros of a polynomial (better in many cases than those obtainable from many other known results).

THEOREM 1.5. *Let $p(z) = a_0 z^n + \sum_{k=m}^{n} a_k z^{n-k}$ be a polynomial, with*

$$r = \max_{m \leqslant k \leqslant n-1} \left| \frac{a_{k+1}}{a_k} \right| \tag{1.6}$$

*and let $\xi$ be unique positive root of the equation*

$$x^m - r x^{m-1} - \frac{|a_m|}{|a_0|} = 0. \tag{1.7}$$

*Then all the zeros of $p(z)$ lie in $|z| < \xi$.*

By taking $m = 1$ in Theorem 1.5 we get

COROLLARY 1.1. *Let $p(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ be a polynomial, with*

$$R = \max\left( \left|\frac{a_2}{a_1}\right|, \left|\frac{a_3}{a_2}\right|, \ldots, \left|\frac{a_n}{a_{n-1}}\right| \right).$$

*Then all the zeros of $p(z)$ lie in $|z| < R + |a_1/a_0|$.*

REMARK 1.1. In many cases Corollary 1.1 gives better upper bounds than those obtainable by other known results. For the polynomial

$$p_1(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5,$$

with $|a_0| = 1$, $|a_1| = 2$, $|a_2| = 3$, $|a_3| = 4$, $|a_4| = 5$, $|a_5| = 6$, all the zeros lie in

(i) $|z| < 7$, by Cauchy [**3**], [**10**, Thoerem (27,2)], [**11, 12**],
(ii) $|z| \leqslant 4$, by Kojima [**8, 9**],
(iii) $|z| \leqslant R$, $R > 3.6$, by Govil and Rahman [**6**, Theorem 1],
(iv) $|z| \leqslant R$, $6.9 < R < 7$, by Dehmer [**4**, Theorem 3.2],
(v) $|z| \leqslant R$, $R > 3.8$, by Jain [**7**, Corollary 1],
(vi) $|z| \leqslant R$, $3.9 < R < 4$, by Dehmer and Mowshowitz [**5**, Theorem 2],
(vii) $|z| \leqslant 4$, by Dehmer and Mowshowitz [**5**, Theorem 4],
(viii) $|z| < 3.5$, by Corollary 1.1.

## 2. Lemmas

For the proofs of the theorems we require the following lemmas.

LEMMA 2.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial with complex coefficients such that for certain real $\beta$ and $\alpha$ one has $|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2$, $j = 0, 1, \ldots, n$ and $|a_n| \geqslant |a_{n-1}| \geqslant \cdots \geqslant |a_0|$, then $p(z)$ has all its zeros on or inside the circle*

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

This lemma is due to Govil and Rahman [**6**].

LEMMA 2.2. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial such that for certain $a > 0$*

$$|a_n| \geqslant a|a_{n-1}| \geqslant a^2|a_{n-2}| \geqslant \cdots \geqslant a^{n-1}|a_1| \geqslant a^n|a_0| \tag{2.1}$$

*and for certain real $\beta$ and $\alpha$*

$$|\operatorname{Arg} a_j - \beta| \leqslant \alpha \leqslant \pi/2, \quad j = 0, 1, \ldots, n \tag{2.2}$$

*then $p(z)$ has all its zeros in*

$$|z| \leqslant \frac{1}{a}\left(\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|\right).$$

PROOF OF LEMMA 2.2. Using (2.1) and (2.2) we can say that polynomial $p(z/a)$ satisfies the hypotheses of Lemma 2.1 and therefore has all its zeros in

$$|z| \leqslant \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|.$$

Accordingly the polynomial $p(z) \equiv p\big(\frac{az}{a}\big)$ will have all its zeros in

$$|z| \leqslant \frac{1}{a}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}a^{n-j}|a_j|\right\}. \qquad \square$$

LEMMA 2.3. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial. If $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, \ldots, n$ and $\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0$, $\alpha_n > 0$, then $p(z)$ has all its zeros in*

$$|z| \leqslant 1 + \frac{2}{\alpha_n}\sum_{j=0}^{n}|\beta_j|.$$

This lemma is due to Govil and Rahman [**6**].

LEMMA 2.4. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ $(> 1)$, with*

$$\operatorname{Re} a_j = \alpha_j, \qquad \operatorname{Im} a_j = \beta_j, \quad j = 0, 1, \ldots, n. \tag{2.3}$$

*If for certain $a > 0$*

$$\alpha_n \geqslant a\alpha_{n-1} \geqslant a^2\alpha_{n-2} \geqslant \cdots \geqslant a^{n-1}\alpha_1 \geqslant a^n\alpha_0 \geqslant 0, \quad \alpha_n > 0 \tag{2.4}$$

*then $p(z)$ has all its zeros in*

$$|z| \leqslant \frac{1}{a}\left\{1 + \frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|\right\}.$$

PROOF OF LEMMA 2.4. Using (2.4) and (2.3) we can say that the polynomial $p(z/a)$ satisfies the hypotheses of Lemma 2.3 and therefore has all its zeros in

$$|z| \leqslant 1 + \frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|.$$

Accordingly the polynomial $p(z) \equiv p(az/a)$ will have all its zeros in

$$|z| \leqslant \frac{1}{a}\left\{1 + \frac{2}{\alpha_n}\sum_{j=0}^{n} a^{n-j}|\beta_j|\right\}. \qquad \square$$

LEMMA 2.5. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients such that for certain $a > 0$*

$$|a_n| \geqslant a|a_{n-1}| \geqslant a^2|a_{n-2}| \geqslant \cdots \geqslant a^{n-1}|a_1| \geqslant a^n|a_0|.$$

*Then $p(z)$ has all its zeros in $|z| \leqslant K_1/a$, where $K_1$ is the greatest positive root of the trinomial equation $K^{n+1} - 2K^n + 1 = 0$.*

This lemma is due to Govil and Rahman [**6**].

LEMMA 2.6. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, be a polynomial with complex coefficients. Then for every positive real number $r$, all the zeros of $P(z)$ lie in the disc*

$$|z| \leqslant \max\left\{r, \sum_{j=0}^{n-1} |a_j/a_n| \frac{1}{r^{n-j-1}}\right\}.$$

This lemma is due to Aziz and Mohammad [**2**].

LEMMA 2.7. *Let $p(z)$ be a polynomial of degree $n$. Then*

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|.$$

This lemma is due to Bernstein [**13**].

On applying Lemma 2.7 to the polynomial $p(rz)$, we get

LEMMA 2.8. *Let $p(z)$ be a polynomial of degree $n$ ($\geqslant 1$). Then for every positive $r$ we have $\max_{|z|=r} |p'(z)| \leqslant (n/r) \max_{|z|=r} |p(z)|$.*

By repeated application of Lemma 2.8, we get

LEMMA 2.9. *Let $p(z)$ be a polynomial of degree $n(\geqslant 1)$. Then for every positive $r$ we have*

$$\max_{|z|=r} |p^k(z)| \leqslant \frac{n(n-1)\ldots(n-k+1)}{r^k} \max_{|z|=r} |p(z)|, \quad k = 1, 2, \ldots, n.$$

## 3. Proofs of the theorems

PROOF OF THEOREM 1.1. We have

$$p'(z) = \sum_{j=0}^{n-1}(j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_j z^j, \text{ say .} \tag{3.1}$$

Now

$$t|a_{j+1}| \geqslant |a_j|, \quad j = 1, 2, \ldots, n-1, \text{ (by (1.1)).} \tag{3.2}$$

Further $\frac{n-1}{n} \geqslant \frac{j}{j+1}$, $j = 1, 2, \ldots, n-1$, which, by (3.2), makes it possible to write $t\frac{n-1}{n}(j+1)|a_{j+1}| \geqslant j|a_j|$, $j = 1, 2, \ldots, n-1$, i.e., $|b_j| \geqslant \frac{n}{t(n-1)}|b_{j-1}|$, $j = 1, 2, \ldots, n-1$, (by (3.1)). We can now apply Lemma 2.2 to the polynomial $p'(z)$, of degree $(n-1)$ and say that all the zeros of $p'(z)$ lie in

$$|z| \leqslant \frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\right\}.$$

Therefore all the zeros of $p(z)$, with moduli greater than

$$\frac{t(n-1)}{n}\left\{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{n|a_n|}\sum_{j=0}^{n-2}\Big(\frac{n}{t(n-1)}\Big)^{n-1-j}(j+1)|a_{j+1}|\right\},$$

are simple. □

PROOF OF THEOREM 1.2. We have

$$p'(z) = \sum_{j=0}^{n-1}(j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_j z^j, \text{ say.} \tag{3.3}$$

Now

$$t\alpha_{j+1} \geqslant \alpha_j \geqslant 0, \quad j = 1, 2, \ldots, n-1, \text{ (by (1.2)).} \tag{3.4}$$

Further $\frac{n-1}{n} \geqslant \frac{j}{j+1}$, $j = 1, 2, \ldots, n-1$, which, by (3.4), makes it possible us write $\frac{t(n-1)}{n}(j+1)\alpha_{j+1} \geqslant j\alpha_j \geqslant 0$, $j = 1, 2, \ldots, n-1$, i.e.,

$$\operatorname{Re} b_j \geqslant \frac{n}{t(n-1)}\operatorname{Re} b_{j-1} \geqslant 0, \quad j = 1, 2, \ldots, n-1, \text{ (by (3.3)).} \tag{3.5}$$

As $\operatorname{Re} b_{n-1} > 0$, by (3.3) and (1.2), we can say by using (3.5) that the polynomial $p'(z)$ satisfies the hypotheses of Lemma 2.4 and therefore has all its zeros in

$$|z| \leqslant \frac{t(n-1)}{n}\left\{1 + \frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\Big(\frac{n}{t(n-1)}\Big)^{n-j}(j+1)|\beta_{j+1}|\right\}.$$

Accordingly all the zeros of $p(z)$, with moduli greater than

$$\frac{t(n-1)}{n}\left\{1 + \frac{2}{n\alpha_n}\sum_{j=0}^{n-1}\Big(\frac{n}{t(n-1)}\Big)^{n-j}(j+1)|\beta_{j+1}|\right\}$$

are simple. □

PROOF OF THEOREM 1.3. It is similar to the proof of Theorem 1.1 with one change: Lemma 2.5 instead of Lemma 2.2. □

PROOF OF THEOREM 1.4. With $w = ae^{i\alpha_a}$, let us consider the polynomial

$$F(z) = p(wz/n + w) = p(w) + z(w/n)p'(w) + \frac{z^2}{2!}(w/n)^2 p''(w) + \cdots + \frac{z^n}{n!}(w/n)^n p^{(n)}(w). \tag{3.6}$$

Then the polynomial

$$G(z) = z^n F(1/z) = \sum_{k=0}^{n} (w/n)^k \frac{p^{(k)}(w)}{k!} z^{n-k}, \tag{3.7}$$

will have all its zeros, by using Lemma 2.6 with $r = a$, in the disc

$$|z| \leqslant \max\left\{a, \sum_{k=0}^{n-1} |w/n|^{n-k} \frac{|p^{(n-k)}(w)|}{(n-k)!|p(w)|a^{n-k-1}}\right\},$$

and as

$$|p(w)| \geqslant \frac{a^k}{n(n-1)\dots(n-k+1)}|p^{(k)}(w)|, \geqslant \frac{|w|^k|p^{(k)}(w)|}{n^k k!}, \quad k = 1, 2, \dots, n,$$

by (1.4) and Lemma 2.9, we can say that $G(z)$ has all its zeros in the disc

$$|z| \leqslant \max\left\{a, \sum_{k=0}^{n-1} \frac{1}{a^{n-k-1}}\right\} = \frac{a}{nR_a}, \text{ (by (1.5))}.$$

Therefore by (3.7), we can say that $F(z)$ has all its zeros in $|z| \geqslant \frac{nR_a}{a}$, i.e., $F(z)$ does not vanish in $|z| < \frac{nR_a}{a}$ and accordingly, by using (3.6), we can say that $p(z)$ does not vanish in the disc $|z - w| < R_a$. □

PROOF OF THEOREM 1.5. Using (1.6) we get

$$|a_k| \leqslant r^{k-m}|a_m|, \quad m \leqslant k \leqslant n. \tag{3.8}$$

Now for $|z| > r$, we have

$$\begin{aligned} |p(z)| &\geqslant |a_0||z|^n\left\{1 - \frac{1}{|a_0|}\sum_{k=m}^{n}\frac{|a_k|}{|z|^k}\right\}, \\ &\geqslant |a_0||z|^n\left\{1 - \frac{|a_m|r^{-m}}{|a_0|}\sum_{k=m}^{n}\left(\frac{r}{|z|}\right)^k\right\}, \text{ (by (3.8))}, \\ &> |a_0||z|^n\left\{1 - \frac{|a_m|r^{-m}}{|a_0|}\sum_{k=m}^{\infty}\left(\frac{r}{|z|}\right)^k\right\}, \\ &= |a_0||z|^n\left\{\frac{|z|^m - r|z|^{m-1} - (|a_m|/|a_0|)}{(|z|^m - r|z|^{m-1})}\right\}. \end{aligned} \tag{3.9}$$

Further if $g(x) = x^m - rx^{m-1} - \frac{|a_m|}{|a_0|}$, then

$$g(r) = -\frac{|a_m|}{|a_0|} < 0, \tag{3.10}$$

and as $\xi$ is unique positive root of the equation (1.7), we can say that

$$x^m - rx^{m-1} - \frac{|a_m|}{|a_0|} \geqslant 0 \quad \text{for} \quad x \geqslant \xi, \tag{3.11}$$

with

$$\xi > r, \ \text{(by (3.10))}. \tag{3.12}$$

Now by (3.9), (3.11) and (3.12) we can say that $|p(z)| > 0$ for $|z| \geqslant \xi$. □

## References


1. A. Aziz, Q. G. Mohammad, *On the zeros of certain class of polynomials and related analytic functions*, J. Math. Anal. Appl. **75** (1980), 495–502.
2. ———, *Zero-free regions for polynomials and some generalizations of Eneström-Kakeya theorem*, Canad. Math. Bull. **27**(3) (1984), 265–272.
3. A. L. Cauchy, *Exercises de Mathématique*, Oeuvres **9**(2) 1829, p.122.
4. M. Dehmer, *On the location of zeros of complex polynomials*, J. Ineq. Pure and Appl. Maths. **7**(1) (2006).
5. M. Dehmer, A. Mowshowitz, *Bounds on the moduli of polynomial zeros*, Appl. Maths. Comp. **218** (2011), 4128–4137.
6. N. K. Govil, Q. I. Rahman, *On the Eneström-Kakeya theorem*, Tôhoku Math. J. **20** (1968), 126–136.
7. V. K. Jain, *On Cauchy's bound for zeros of a polynomial*, Bull. Math. Soc. Sci. Math. Roumanie **50**(98) (2007), 273–279.
8. T. Kojima, *On a theorem of Hadamard's and its application*, Tôhoku Math. J. **5** (1914), 58.
9. ———, *On the limits of the roots of an algebraic equation*, Tôhoku Math. J. **11** (1917), 119–127.
10. M. Marden, *Geometry of polynomials*, Math. Surveys no. 3, Amer. Math. Soc., Providence, Rhode Island, 1966.
11. G. V. Milovanovic, D. S. Mitrinovic, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore–New Jersey–London–Hong Kong, 1994.
12. Q. I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials, Critical Points, Zeros and Extremal Properties*, Oxford University Press, Oxford, 2002.
13. A. C. Schaeffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc. **47** (1941), 565–579


V. K. Jain, former Professor
Mathematics Department
IIT Kharagpur
India
vinayjain.kgp@gmail.com

(Received 29 10 2015)
(Revised 17 11 2015)