PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114) (2016), 243–249 DOI: 10.2298/PIM150501028Q

A NEW FORMULA FOR THE BERNOULLI NUMBERS OF THE SECOND KIND IN TERMS OF THE STIRLING NUMBERS OF THE FIRST KIND

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Abstract. We find an explicit formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind.

1. Main result

It is well known that the signed Stirling numbers of the first kind $s(n,k)$ for $n \geqslant k \geqslant 1$ may be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1$$

and that the Bernoulli numbers of the second kind b_n for $n \geqslant 0$ may be generated by $\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n$. In combinatorics, the signed Stirling number of the first kind $s(n,k)$ may be defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number $|s(n,k)| = (-1)^{n-k}s(n,k)$. The Bernoulli numbers of the second kind b_n are also called the Cauchy numbers of the first kind, see [**20**,**27**] and closely related references therein.

In [**14**], the following formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind was derived:

$$b_n = \frac{1}{n!}\sum_{k=0}^{n} \frac{s(n,k)}{k+1}.$$

Our main aim is to find a new and explicit formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind. The main result of this paper may be stated as the following theorem.

2010 *Mathematics Subject Classification:* 11B68, 11B73, 11B83.

Key words and phrases: explicit formula; Bernoulli numbers of the second kind; Stirling numbers of the first kind; harmonic number.

Communicated by Gradimir Milovanović.

THEOREM 1.1. *For $n \geqslant 2$, the Bernoulli numbers of the second kind b_n may be computed in terms of the signed Stirling numbers of the first kind $s(n,k)$ by*

$$b_n = \frac{1}{n!}\sum_{k=1}^{n-1}(-1)^k\frac{s(n-1,k)}{(k+1)(k+2)}. \tag{1.1}$$

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on some results elementarily and inductively obtained in [**22**]. These results can be recited as follows.

(1) Corollary 2.3 in [**22**] states that the signed Stirling numbers of the first kind $s(n,k)$ for $1 \leqslant k \leqslant n$ may be computed by

$$s(n,k) = (-1)^{n-k}(n-1)!\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}\sum_{l_{k-1}=1}^{l_{k-2}-1}\frac{1}{l_{k-1}}. \tag{2.1}$$

This formula may be reformulated as

$$(-1)^{n-k}\frac{s(n,k)}{(n-1)!} = \sum_{m=k-1}^{n-1}\frac{1}{m}\left[(-1)^{m-(k-1)}\frac{s(m,k-1)}{(m-1)!}\right].$$

(2) Corollary 2.4 in [**22**] reads that for $1 \leqslant k \leqslant n$ the signed Stirling numbers of the first kind $s(n,k)$ satisfy the recursion

$$s(n+1,k) = s(n,k-1) - ns(n,k). \tag{2.2}$$

This is a recovery of the triangular relation for $s(n,k)$.

(3) Theorem 3.1 in [**22**] tells that the Bernoulli numbers of the second kind b_n for $n \geqslant 2$ may be computed by

$$b_n = (-1)^n\frac{1}{n!}\left(\frac{1}{n+1} + \sum_{k=2}^{n}\frac{a_{n,k} - na_{n-1,k}}{k!}\right), \tag{2.3}$$

where

$$a_{n,2} = (n-1)! \tag{2.4}$$

and, for $n+1 \geqslant k \geqslant 3$,

$$a_{n,k} = (k-1)!(n-1)!\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-3}=1}^{l_{k-4}-1}\frac{1}{l_{k-3}}\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}. \tag{2.5}$$

Observing expressions (2.1) and (2.5), we obtain

$$a_{n,k} = (-1)^{n+k-1}(k-1)!s(n,k-1), \quad n+1 \geqslant k \geqslant 2. \tag{2.6}$$

See [**22**, (2.18)] and [**23**, (6.7)]. By this and recursion (2.2), it follows that

$$\begin{aligned} a_{n,k} - na_{n-1,k} &= (-1)^{n+k-1}(k-1)![s(n,k-1) + ns(n-1,k-1)] \\ &= (-1)^{n+k-1}(k-1)![s(n-1,k-1) + s(n-1,k-2)]. \end{aligned}$$

Substituting this into (2.3) reveals that

$$\begin{aligned}
b_n &= \frac{(-1)^n}{n!}\biggl(\frac{1}{n+1}+\sum_{k=2}^{n}\frac{(-1)^{n+k-1}[s(n-1,k-1)+s(n-1,k-2)]}{k}\biggr)\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\sum_{k=2}^{n}\frac{(-1)^{k-1}[s(n-1,k-1)+s(n-1,k-2)]}{k}\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\Biggl[\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-1)}{k}+\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-2)}{k}\Biggr]\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\Biggl[\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-1)}{k}+\sum_{k=1}^{n-1}\frac{(-1)^{k}s(n-1,k-1)}{k+1}\Biggr]\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\frac{(-1)^{n-1}}{n}+\frac{1}{n!}\sum_{k=2}^{n-1}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}(-1)^{n-1}\Bigl(\frac{1}{n}-\frac{1}{n+1}\Bigr)+\frac{1}{n!}\sum_{k=2}^{n-1}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}\sum_{k=2}^{n}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}\sum_{k=2}^{n}(-1)^{k-1}\frac{s(n-1,k-1)}{k(k+1)}.
\end{aligned}$$

Notice that in the above argument, we use the convention $s(n,0)=0$ for $n\in\mathbb{N}$ and the fact $s(n,n)=1$ for $n\geqslant 0$. The proof of the formula (1.1) in Theorem 1.1 is complete.

3. Remarks

In this section, we show some new findings by several remarks.

Remark 3.1. The idea in Theorem 1.1 and its proof ever implicitly thrilled through in [**23**, Remark 6.7].

Remark 3.2. Making use of relation (2.6) in [**22**, Theorem 2.1] leads to

$$\Bigl(\frac{1}{\ln x}\Bigr)^{(n)}=\frac{1}{x^n}\sum_{k=1}^{n}(-1)^k k!s(n,k)\Bigl(\frac{1}{\ln x}\Bigr)^{k+1},\quad n\in\mathbb{N}.$$

This recovers the first formula in [**13**, Lemma 2].

By the way, the formulas (3.4) and (3.5) in [**22**, Corollary 3.1] recover the second formula in [**13**, Lemma 2].

Remark 3.3. In [**22**, Remark 2.2], it was conjectured that the sequence $a_{n,k}$ for $n\in\mathbb{N}$ and $2\leqslant k\leqslant n+1$ is increasing with respect to n while it is unimodal with respect to k for given $n\geqslant 4$. This conjecture may be partially confirmed as follows.

From (2.5), the increasing monotonicity of the sequence $a_{n,k}$ with respect to n follows straightforwardly.

It is clear that the sequence $(k-1)!$ is increasing with k and the sequence

$$\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-3}=1}^{l_{k-4}-1}\frac{1}{l_{k-3}}\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}$$

is decreasing with k. Since $a_{n,n+1}=n!$, see the equation (2.4) or [**22**, (2.8)], we obtain that

$$a_{n,2}<a_{n,n+1},\quad n\geqslant 2. \tag{3.1}$$

In [**23**, Theorem 2.1], the integral representation

$$s(n,k)=\binom{n}{k}\lim_{x\to0}\frac{d^{n-k}}{dx^{n-k}}\biggl\{\biggl[\int_0^\infty\biggl(\int_{1/e}^1 t^{xu-1}dt\biggr)e^{-u}du\biggr]^k\biggr\} \tag{3.2}$$

was created for $1\leqslant k\leqslant n$. Hence,

$$\begin{aligned}
s(n,n-1)&=n\lim_{x\to0}\frac{d}{dx}\biggl\{\biggl[\int_0^\infty\biggl(\int_{1/e}^1 t^{xu-1}dt\biggr)e^{-u}du\biggr]^{n-1}\biggr\}\\
&=n(n-1)\lim_{x\to0}\biggl[\int_0^\infty\biggl(\int_{1/e}^1 t^{xu-1}dt\biggr)e^{-u}du\biggr]^{n-2}\\
&\quad\times\lim_{x\to0}\biggl[\int_0^\infty\biggl(\int_{1/e}^1 t^{xu-1}\ln t\,dt\biggr)ue^{-u}du\biggr]\\
&=n(n-1)\biggl[\int_0^\infty\biggl(\int_{1/e}^1 \frac1t\,dt\biggr)e^{-u}du\biggr]^{n-2}\int_0^\infty\biggl(\int_{1/e}^1 \frac{\ln t}{t}\,dt\biggr)ue^{-u}du\\
&=-\frac12 n(n-1).
\end{aligned}$$

As a result, by (2.6), it follows that $a_{n,n}=-(n-1)!s(n,n-1)=\frac{n-1}{2}n!\geqslant a_{n,n+1}$, $n\geqslant 3$. Combining this with (3.1) shows that the sequence $a_{n,k}$ for given $n\geqslant4$ has at least one maximum with respect to $2<k<n+1$.

Remark 3.4. By integral representation (3.2) and direct computation, we can recover that

$$\begin{aligned}
s(n,1)&=\binom{n}{1}\lim_{x\to0}\frac{d^{n-1}}{dx^{n-1}}\int_0^\infty\biggl(\int_{1/e}^1 t^{xu-1}dt\biggr)e^{-u}du\\
&=n\lim_{x\to0}\int_0^\infty\biggl[\int_{1/e}^1 t^{xu-1}(\ln t)^{n-1}dt\biggr]u^{n-1}e^{-u}du\\
&=n\int_0^\infty\biggl[\int_{1/e}^1 \frac{(\ln t)^{n-1}}{t}\,dt\biggr]u^{n-1}e^{-u}du\\
&=(-1)^{n+1}\int_0^\infty u^{n-1}e^{-u}du\\
&=(-1)^{n+1}(n-1)!
\end{aligned}$$

and

$$
\begin{aligned}
s(n,2) &= \binom{n}{2} \lim_{x\to 0} \frac{d^{n-2}}{dx^{n-2}} \left\{ \left[\int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du \right]^2 \right\} \\
&= \binom{n}{2} \lim_{x\to 0} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^\infty \left[\int_{1/e}^1 t^{xu-1} (\ln t)^k dt \right] u^k e^{-u} du \\
&\quad \times \int_0^\infty \left[\int_{1/e}^1 t^{xu-1} (\ln t)^{n-k-2} dt \right] u^{n-k-2} e^{-u} du \\
&= \binom{n}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} \int_0^\infty \left[\int_{1/e}^1 \frac{(\ln t)^k}{t} dt \right] u^k e^{-u} du \\
&\quad \times \int_0^\infty \left[\int_{1/e}^1 \frac{(\ln t)^{n-k-2}}{t} dt \right] u^{n-k-2} e^{-u} du \\
&= (-1)^n \binom{n}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{k!}{k+1} \frac{(n-k-2)!}{n-k-1} \\
&= (-1)^n (n-2)! \binom{n}{2} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)} \\
&= (-1)^n \frac{n!}{2} \sum_{k=0}^{n-2} \frac{1}{(k+1)(n-k-1)} \\
&= (-1)^n \frac{(n-1)!}{2} \sum_{k=0}^{n-2} \left(\frac{1}{k+1} + \frac{1}{n-k-1} \right) \\
&= (-1)^n (n-1)! H(n-1),
\end{aligned}
$$

where $H(n) = \sum_{k=1}^{n} \frac{1}{k}$ is the n-th harmonic number. Consequently, we find a relation

$$s(n,2) = (-1)^n (n-1)! H(n-1), \quad n \in \mathbb{N} \tag{3.3}$$

or, equivalently,

$$H(n) = \frac{(-1)^{n+1} s(n+1,2)}{n!}, \quad n \in \mathbb{N} \tag{3.4}$$

between the n-th harmonic number $H(n)$ and the signed Stirling numbers of the first kind $s(n,2)$. Relation (3.3), or say, (3.4), may also be deduced by considering (2.5) and (2.6).

We point out that relation (3.3), or say, (3.4) recovers [**2**, p. 275, (6.58)].

For more information on the n-th harmonic numbers $H(n)$, please refer to [**1**, **9**, **10**, **12**, **24**, **26**] and closely related references therein.

Remark 3.5. For more information on some new results for the Bernoulli numbers, the Cauchy numbers, and the Stirling numbers of the first and second kinds, please refer to [**3–8**, **11**, **15–18**, **20–23**, **25**, **27**] and closely related references therein.

REMARK 3.6. This paper is a slightly revised and corrected version of the preprint [**19**].

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(Received 27 10 2014)