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PELLANS SEQUENCE AND ITS DIOPHANTINE TRIPLES

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Abstract. We introduce a novel fourth order linear recurrence sequence $\{S_n\}$ using the two periodic binary recurrence. We call it "*pellans sequence*" and then we solve the system

$$ab + 1 = S_x, \quad ac + 1 = S_y \quad bc + 1 = S_z$$

where $a < b < c$ are positive integers. Therefore, we extend the order of recurrence sequence for this variant diophantine equations by means of pellans sequence.

1. Introduction

A *Diophantine m-tuple* is a set of $\{a_1, a_2, \ldots, a_n\}$ of positive rational numbers or integers such that $a_i a_j + 1 = \square$ for all $1 \leqslant i, j \leqslant n$. Diophantus investigated first the problem of finding *rational* quadruples, and found the example $\{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\}$. Then, Fermat found the first integer quadruples as $\{1, 3, 8, 120\}$. In [**6**], the general form of this set was found by Hoggatt and Bergum as follows

$$\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\},$$

where $\{F_n\}_{n\geqslant 0}$ denotes the Fibonacci sequence. Therefore, there exist infinitely many quadruples. The famous theorem of Dujella [**4**] states that there are only finitely many quintuples.

A variant of the problem is obtained if one replaces the squares by the terms of given binary recurrences. This type of problem was started by Luca and Szalay. Luca and Szalay replaced the squares by the terms of Fibonacci and Lucas sequence and found that there is no Fibonacci diophantine triple and the only Lucas diophantine triple is $\{1, 2, 3\}$. Similarly, Alp, Irmak and Szalay put the terms of balancing sequences instead of the squares and they did not found any triples. For details, see [**1, 8, 9**]. Moreover, Fuchs, Luca and Szalay [**5**] investigated the general case for binary sequence and they gave sufficient and necessary conditions

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to have finitely many triples. For an integer $A \geqslant 3$, Irmak and Szalay [**7**] showed that there is no diophantine triples for the sequence $\{u_n\}$ where $\{u_n\}$ satisfies the relation $u_n = Au_{n-1} - u_{n-2}$ with the initial conditions $u_0 = 0$ and $u_1 = 1$.

Up to now, the authors have studied for the special cases of binary recurrence. One way for this type diophantine problems is to extend the problem to recurrent sequences of larger orders. In this paper, we define a fourth order recurrence sequence which we call pellans sequence since odd terms of the sequence are Pell numbers and even terms of the sequence are balancing numbers or half of the terms of even Pell numbers. Afterwards, we investigate its diophantine triples. Now, it is suitable to give the definitions of Pell, Pell–Lucas and Balancing numbers. The terms of Pell sequence $\{P_n\}_{n\geqslant 0}$ satisfy the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ with initial conditions $P_0 = 0$ and $P_1 = 1$. The terms of Pell–Lucas sequence $\{Q_n\}_{n\geqslant 0}$ satisfy the same recurrence relation with Pell sequence together with initial conditions $Q_0 = 2$ and $Q_1 = 2$. The terms of Balancing sequence $\{B_n\}_{n\geqslant 0}$ are defined by $B_n = 6B_{n-1} - B_{n-2}$ together with the initial conditions $B_0 = 0$ and $B_1 = 1$.

Definition 1.1. For a nonnegative integer n, the pellans sequence $\{S_n\}_{n\geqslant 0}$ is defined by $S_{2n} = \frac{1}{2}P_{2n}$ and $S_{2n+1} = P_{2n+1}$.

The first few terms of the pellans sequence are $0, 1, 1, 5, 6, 29, 35, \ldots$. It is obvious that the terms of pellans sequence $\{S_n\}_{n\geqslant 0}$ satisfy $S_n = 6S_{n-2} - S_{n-4}$ with $S_0 = 0$, $S_1 = 1$, $S_2 = 1$ and $S_3 = 5$.

Our main result is the following

Theorem 1.1. *There is no integer solution for the system*

$$ab + 1 = S_x, \quad ac + 1 = S_y, \quad bc + 1 = S_z \tag{1.1}$$

where $0 < a < b < c$ are integers and the sequence $\{S_n\}$ is the pellans sequence.

2. Preliminaries

Now, we present required properties to prove Theorem 1.1.

Lemma 2.1. *Let m and n are positive integers. Then*

(1) $\gcd(S_n, S_m) = S_{\gcd(n,m)}$.

(2) $P_{2n+1} - 1 = \begin{cases} P_nQ_{n+1}, & n \text{ is even} \\ P_{n+1}Q_n, & n \text{ is odd} \end{cases}$ *and* $P_{2n} - 2 = \begin{cases} P_{n+1}Q_{n-1} & n \text{ is even} \\ P_{n-1}Q_{n+1} & n \text{ is odd} \end{cases}$ *where $\{Q_n\}$ is the Pell–Lucas sequence.*

(3) $2B_n = P_{2n} = P_nQ_n$.

(4) $(B_n - 1)(B_n + 1) = B_{n-1}B_{n+1}$.

(5) $B_{2n+1} - 1 = B_nC_{n+1}$ *where $\{C_n\}$ is associated sequence of $\{B_n\}$.*

(6) $C_n = Q_{2n}$.

(7) *If n is odd integer, then $B_n - 1 = \frac{1}{2}P_{n-1}Q_{n+1}$.*

Proof. The third and sixth identities can be found in [**10**]. For the fourth one, we refer to [**2**]. Since the terms of the sequence $\{S_n\}$ are balancing and Pell numbers and special cases of the Lucas sequence, we can write easily the first identity by means of [**3**]. In order to prove the seventh one, we use the Binet formulas for the Pell and associated Pell sequences. Since n is an odd integer, we get

$$\begin{aligned}\frac{1}{2}P_{n-1}Q_{n+1} &= \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}(\alpha^{n+1}+\beta^{n+1})\\ &= \frac{1}{2}\Big(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}+2\Big) = \frac{\kappa^n-\tau^n}{\kappa-\tau}+1 = B_n+1\end{aligned}$$

where α and β are the roots of the equation $x^2-2x-1=0$ and we use the facts $\kappa=\alpha^2$ and $\tau=\beta^2$. The remaining identities can be proven by using Binet formulas. □

Lemma 2.2. *For all integers $n \geqslant 3$, the following inequalities hold*

$$\begin{aligned}\alpha^{n-1.19} &< P_n < \alpha^{n-1.16}\\ \alpha^{n-0.1} &< Q_n < \alpha^{n+0.1}\\ \alpha^{2n-1.97} &< B_n < \alpha^{2n-1.96}\end{aligned}$$

where α is the dominant root of the equation $x^2-2x-1=0$.

Proof. The third inequality is from [**1**]. For $n \geqslant 3$, we have

$$P_n \geqslant \frac{\alpha^n-|\beta^3|}{\alpha-\beta} \geqslant \alpha^n\frac{1-\big(|\beta|/\alpha\big)^3}{\alpha-\beta} \geqslant \alpha^{n-1.19},$$

which gives the lower bound for the sequence $\{P_n\}$. Similarly,

$$P_n \leqslant \frac{\alpha^n+|\beta^3|}{\alpha-\beta} \leqslant \alpha^n\frac{1+\big(|\beta|/\alpha\big)^3}{\alpha-\beta} \leqslant \alpha^{n-1.16}$$

gives the upper bound. The bounds for the sequence $\{Q_n\}$ can be proven in a similar way. □

3. Proof of Theorem 1.1

Since $0<a<b<c$, then $1\cdot 2+1 \leqslant ab+1 = S_x$. We get that $3 \leqslant x$. From now on, the proof is split into two parts.

Case 1. Assume that $z \leqslant 280$.

In this case, we run a computer search to detect system (1.1) for each case. Note that the balancing case has been already solved in [**1**]. Observe that we have

$$a = \sqrt{\frac{(S_x-1)(S_y-1)}{S_z-1}}, \qquad 3 \leqslant x < y < z \leqslant 280.$$

Going through all the eligible values for x, y and z, and we found no integer solution.

Case 2. Assume that $z > 280$.

In this case, we distinguish four main cases depending on the integers y and z.

(1) *Both y and z are even*

If x is an even integer, then the case corresponds to [**1**] since the pellans sequence turns to balancing sequence. Assume that x is odd. The proof mainly depends on the indices y and z apart from the case $k = 2$ and $l = 1$ in [**1**]. If we take $S_x = P_{2x-1}$ in the case $k = 2$, $l = 1$ in [**1**], then we can follow the similar way to complete the proof of theorem. Hence, we omit this case.

(2) *Both y and z are odd*

Since both y and z are odd integers, then the terms of pellans sequence turn to Pell numbers. Now, we give a lemma which gives a relation between the indices y and z.

LEMMA 3.1. *The system*

$$ab + 1 = S_x, \quad ac + 1 = P_y, \quad bc + 1 = P_z \tag{3.1}$$

satisfy $z \leqslant 2y - 2$.

PROOF. The last two equations of (1.1) give $\sqrt{P_z} < c < P_y$. By Lemma 2.2, we have

$$\alpha^{(z-1.19)/2} < \sqrt{P_z} < c < P_y < \alpha^{y-1.16}$$

which yields $z \leqslant 2y - 2$. $\square$

Put $q_1 = \gcd(S_y - 1, S_z - 1)$. Since y and z are odd integers, we replace the pellans sequence by Pell sequence according to the definition of pellans sequence. Applying the second and fourth identities of Lemma 2.1, we get

$$\begin{aligned} q_1 &= \gcd(P_y - 1, P_z - 1) = \gcd\left(P_{\frac{y-i}{2}} Q_{\frac{y+i}{2}}, P_{\frac{z-j}{2}} Q_{\frac{z+j}{2}}\right) \\ &\leqslant P_{\gcd\left(\frac{y-i}{2}, \frac{z-j}{2}\right)} Q_{\gcd\left(\frac{y-i}{2}, \frac{z+j}{2}\right)} Q_{\gcd\left(\frac{y+i}{2}, \frac{z-j}{2}\right)} Q_{\gcd\left(\frac{y+i}{2}, \frac{z+j}{2}\right)} \end{aligned}$$

where $i, j \in \{\pm 1\}$. Let $\gcd\left(\frac{z+\mu_1 j}{2}, \frac{y+\mu_2 i}{2}\right) = \frac{z+\mu_1 j}{2t}$ for some $\mu_1, \mu_2 \in \{\pm 1\}$. Suppose that $t \geqslant 5$ for all quadruples $(i, j, \mu_1, \mu_2) \in \{\pm 1\}^4$. Since $c \mid q_1$, then Lemma 2.2 implies that

$$\alpha^{\frac{z-1.19}{2}} < \sqrt{P_z} < c \leqslant q_1 < \alpha^{\frac{z+1}{10} - 1.16 + 2\frac{z-1}{10} + 0.2 + \frac{z+1}{10} + 0.1}.$$

When we compare the exponents of α, we arrive at a contradiction.

Assume that $t \leqslant 4$ and that

$$\frac{z + \mu_1 i}{2k} = \frac{y + \mu_2 j}{2l}$$

holds for suitable positive integers k and l such that $\gcd(k, l) = 1$.

Suppose for the moment $l > k$. Then, we get $z = y + 1$ since $y < z$ together with $y + \mu_2 j > z + \mu_1 i$. As both y and z are odd integers, the equation $z = y + 1$ is impossible.

Now, assume that $k = l$. Since $z + \mu_1 i = y + \mu_2 j$, we obtain that $z = y + 2$. By Lemma 2.1,

$$P_z - 1 = P_{\frac{z+i}{2}} Q_{\frac{z-i}{2}}, \qquad P_{z-2} - 1 = P_{\frac{z-2-i}{2}} Q_{\frac{z-2+i}{2}}.$$

hold for some $i \in \{\pm 1\}$. If $z \equiv 3 \pmod 4$, then we have the following, by Lemma 2.1

$$\begin{aligned} q_1 &= \gcd(P_z - 1, P_{z-2} - 1) = \gcd\left(P_{\frac{z+1}{2}} Q_{\frac{z-1}{2}}, P_{\frac{z-3}{2}} Q_{\frac{z-1}{2}}\right) \\ &= P_{\gcd\left(\frac{z+1}{2}, \frac{z-3}{2}\right)} Q_{\frac{z-1}{2}} = P_2 Q_{\frac{z-1}{2}} = 2Q_{\frac{z-1}{2}}. \end{aligned}$$

Since $q_1 = 2Q_{\frac{z-1}{2}}$, then we have $c \,|\, Q_{\frac{z-1}{2}}$ or $c \,|\, 2Q_{\frac{z-1}{2}}$. If $c \,|\, Q_{\frac{z-1}{2}}$, then

$$Q_{\frac{z-1}{2}} = c_1 c > c_1 \sqrt{P_z},$$

and applying Lemma 2.2, we obtain

$$c_1 \alpha^{\frac{z-1.19}{2}} < c_1 \sqrt{P_z} < Q_{\frac{z-1}{2}} < \alpha^{\frac{z-1}{2}+0.1}.$$

So $c_1 < \alpha^{0.2} < 1.2$ which gives $c_1 = 1$. Then, we obtain $c = Q_{\frac{z-1}{2}}$. When we put $c = Q_{\frac{z-1}{2}}$ in (3.1), we have $a = P_{\frac{z-3}{2}}$ and $b = P_{\frac{z+1}{2}}$. From the first equation in (3.1), one can easily see that $ab + 1 = P_{\frac{z-3}{2}} P_{\frac{z+1}{2}} + 1 = P^2_{\frac{z-1}{2}} = S_x$. If x is odd, then we obtain $P_x = P^2_{\frac{z-1}{2}}$. When we apply the upper and lower bounds for the sequence $\{P_n\}$, we deduce

$$\begin{aligned} P_x = P^2_{\frac{z-1}{2}} &\Rightarrow \alpha^{x-1.19} < \alpha^{z-3.32} \Rightarrow 2.13 < z - x \\ P_x = P^2_{\frac{z-1}{2}} &\Rightarrow \alpha^{z-1-2.38} < \alpha^{x-1.16} \Rightarrow z - x < 2.22. \end{aligned}$$

There are no two odd integers x and y such that $2.13 < z - x < 2.22$. Similarly, we get $1.35 < z - x < 1.4$ when x is even. This is impossible, too.

In the sequel, if $2Q_{\frac{z-1}{2}} = c_1' c > c_1' \sqrt{P_z}$, then $c_1' < 2Q_{\frac{z-1}{2}} / \sqrt{P_z} < 2\alpha^{0.2} < 2.4$. If $c_1' = 2$, we get $c = Q_{\frac{z-1}{2}}$ which is the same situation as above. If $c_1' = 1$, then we obtain that $a = P_{\frac{z-3}{2}}/2$ and $b = P_{\frac{z+1}{2}}/2$. So, $P^2_{\frac{z-1}{2}} + 5 = 4S_x$. If x is an odd integer, then $S_x = P_x$ which yields that $3.48 < z - x < 3.8$. But, this is impossible. Similarly, if x is an even integer, then $S_x = B_{\frac{x}{2}}$. Using the upper and lower bounds for the sequence, we get $2.7 < z - x < 3$ which is not possible.

If $z \equiv 1 \pmod 4$, then

$$\begin{aligned} q_1 &= \gcd(P_z - 1, P_{z-2} - 1) = \gcd\left(P_{\frac{z-1}{2}} Q_{\frac{z+1}{2}}, P_{\frac{z-1}{2}} Q_{\frac{z-3}{2}}\right) \\ &= Q_{\gcd\left(\frac{z+1}{2}, \frac{z-3}{2}\right)} P_{\frac{z-1}{2}} = Q_1 P_{\frac{z-1}{2}} = 2P_{\frac{z-1}{2}}. \end{aligned}$$

Assume that $q_1 = 2P_{\frac{z-1}{2}}$. Since $c \,|\, q_1$, then $c \,|\, P_{\frac{z-1}{2}}$ or $c \,|\, 2P_{\frac{z-1}{2}}$. If $c \,|\, P_{\frac{z-1}{2}}$, then $P_{\frac{z-1}{2}} = c_2 c > c_2 \sqrt{P_z}$. Since $P_{\frac{z-1}{2}} / \sqrt{P_z} < 1$, we get $c_2 < P_{\frac{z-1}{2}} / \sqrt{P_z} < 1$ which is impossible. If $c \,|\, 2P_{\frac{z-1}{2}}$, then $2P_{\frac{z-1}{2}} = c_2' c > c_2' \sqrt{P_z}$. Since $2P_{\frac{z-1}{2}} / \sqrt{P_z} < \alpha^{-0.27} < 0.79$, then we obtain $c_2' < 0.79$, which is not possible.

In what follows, assume that $k > l$ with $\frac{k}{l} \geqslant 2$. Here,

$$z = \frac{k}{l} (y + \mu_2 i) - \mu_1 j \geqslant 2y - 3.$$

Together with Lemma 3.1, we deduce that $z = 2y - 3$ or $z = 2y - 2$. Since both y and z are odd integers, $z = 2y - 2$ is impossible. Therefore, there is only one possibility which is $z=2y-3$. Together with Lemma 2.1(4) yields that

$$q_1=\gcd(P_{2y-3}-1,P_y-1)=\gcd(P_{y-1}Q_{y-2},P_y-1).$$

If $y\equiv 1 \pmod 4$, then

$$\begin{aligned}\alpha^{\frac{z-1.19}{2}}&<q_1=\gcd(P_{y-1}Q_{y-2},P_y-1)=\gcd\left(P_{y-1}Q_{y-2},P_{\frac{y-1}{2}}Q_{\frac{y+1}{2}}\right)\\&\leqslant \gcd\left(P_{y-1},P_{\frac{y-1}{2}}\right)\gcd\left(P_{y-1},Q_{\frac{y+1}{2}}\right)\gcd\left(Q_{y-2},P_{\frac{y-1}{2}}\right)\gcd\left(Q_{y-2},Q_{\frac{y+1}{2}}\right)\\&\leqslant P_{\frac{y-1}{2}}Q_2Q_1Q_3<\alpha^{\frac{z+1}{4}+5.14}\end{aligned}$$

yields that $z<24$, which is not possible.

If $y\equiv 3 \pmod 4$, then

$$\begin{aligned}\alpha^{\frac{z-1.19}{2}}&<q_1=\gcd(P_{y-1}Q_{y-2},P_y-1)=\gcd\left(P_{y-1}Q_{y-2},P_{\frac{y+1}{2}}Q_{\frac{y-1}{2}}\right)\\&\leqslant \gcd\left(P_{y-1},P_{\frac{y+1}{2}}\right)\gcd\left(P_{y-1},Q_{\frac{y-1}{2}}\right)\gcd\left(Q_{y-2},P_{\frac{y+1}{2}}\right)\gcd\left(Q_{y-2},Q_{\frac{y-1}{2}}\right)\\&\leqslant P_2Q_{\frac{y-1}{2}}Q_3Q_1<\alpha^{\frac{z+1}{4}+5.14}.\end{aligned}$$

But this is also impossible since $z>280$. In the sequel, assume that $\frac{k}{l}<2$. Note that this condition implies $k\geqslant 3$. Taking any pair $(\mu_1,\mu_2)\neq(\mu_1',\mu_2')$, then we have $z+\mu_1'j=\frac{k}{l}(y+\mu_2 i)-\mu_1 j+\mu_1' j$. The main objective is to find an upper bound for $q_1^{(0)}=\gcd\left(\frac{z+\mu_1'j}{2},\frac{y+\mu_2'i}{2}\right)$. Then

$$\begin{aligned}q_1^{(0)}&=\frac{1}{2}\gcd\left(z+\mu_1'j,y+\mu_2'i\right)\\&\leqslant\frac{1}{2}\gcd\left(k(y+\mu_2 i)-l\left(\mu_1 j-\mu_1' j\right),k\left(y+\mu_2' j\right)\right)\leqslant|k+l|.\end{aligned}$$

Since $2\leqslant l<k\leqslant 4$ and $(k,l)=1$, the cases $(k,l)=(4,3)$, $(3,2)$ hold. Therefore,

$$\alpha^{\frac{z-1.19}{2}}<\sqrt{P_z}<\gcd(P_z-1,P_y-1)\leqslant\alpha^{\frac{z+1}{4}+3(7+0.1)-1.16}.$$

Then we get $z<84$ which is not possible.

(3) *y is even and z is odd*

By the definition of the pellans sequence, we get that $S_z=P_z$ and $S_y=B_{\frac{y}{2}}$ since y is an even integer and z is an odd integer. Now, we give a lemma which implies a relation between the integers y and z.

LEMMA 3.2. *The system*

$$ab+1=S_x,\quad ac+1=B_{\frac{y}{2}},\quad bc+1=P_z \tag{3.2}$$

satisfies $z\leqslant 2y-3$.

PROOF. The last two equations yields $\sqrt{P_z}<c<B_{\frac{y}{2}}$. By Lemma 2.2, we have

$$\alpha^{\frac{z-1.19}{2}}<\sqrt{P_z}<c<B_{\frac{y}{2}}<\alpha^{y-1.96},$$

which yields $z\leqslant 2y-3$. □

Put $q_2 = \gcd(S_y - 1, S_z - 1) = \gcd\left(B_{\frac{y}{2}} - 1, P_z - 1\right)$. System (3.2) gives that

$$\sqrt{P_z} < q_2 = \gcd\left(B_{\frac{y}{2}} - 1, P_z - 1\right).$$

After applying the properties in Lemma 2.1, we obtain the following for some $i \in \{\pm 1\}$

$$\begin{aligned} q_2 &= \gcd\left(B_{\frac{y}{2}} - 1, P_z - 1\right) \leqslant \gcd\left(B_{\frac{y}{2}-1} B_{\frac{y}{2}+1}, P_{\frac{z-i}{2}} Q_{\frac{z+i}{2}}\right) \\ &\leqslant \gcd\left(B_{\frac{y}{2}-1}, P_{\frac{z-i}{2}}\right) \gcd\left(B_{\frac{y}{2}-1}, Q_{\frac{z+i}{2}}\right) \gcd\left(B_{\frac{y}{2}+1}, P_{\frac{z-i}{2}}\right) \\ &\quad \times \gcd\left(B_{\frac{y}{2}+1}, Q_{\frac{z+i}{2}}\right) \\ &\leqslant \gcd\left(\frac{1}{2} P_{y-2}, P_{\frac{z-i}{2}}\right) \gcd\left(\frac{1}{2} P_{y-2}, Q_{\frac{z+i}{2}}\right) \gcd\left(\frac{1}{2} P_{y+2}, P_{\frac{z-i}{2}}\right) \\ &\quad \times \gcd\left(\frac{1}{2} P_{y+2}, Q_{\frac{z+i}{2}}\right) \\ &\leqslant \prod_{j \in \{\pm 2\}} P_{\gcd(y+j, \frac{z-i}{2})} Q_{\gcd(y-j, \frac{z+i}{2})}. \end{aligned}$$

Let $\gcd\left(y + \eta_1 j, \frac{z+\eta_2 i}{2}\right) = \frac{z+\eta_2 i}{2t}$ for some $\eta_1, \eta_2, i \in \{\pm 1\}^3$ and $j \in \{\pm 2\}$. Firstly, suppose that $t \geqslant 4$. Lemma 2.2 leads to

$$\alpha^{\frac{z-1.19}{2}} < \alpha^{\frac{z-1}{8} - 1.16 + \frac{z+1}{8} - 1.16 + \frac{z-1}{8} + 0.1 + \frac{z+1}{8} + 0.1}.$$

When we compare the exponents of α, we arrive at a contradiction.

Now, assume that $t = 3$. Since $\gcd\left(y + \eta_1 j, \frac{z+\eta_2 i}{2}\right) = \frac{z+\eta_2 i}{6}$, then one of the equations $\frac{z+\eta_2 i}{6} = y + \eta_1 j$, $\frac{z+\eta_2 i}{3} = y + \eta_1 j$, $\frac{z+\eta_2 i}{3} = \frac{y+\eta_1 j}{2}$ and $\frac{z+\eta_2 i}{6} = \frac{y+\eta_1 j}{5}$ holds where $\eta_1, \eta_2, i \in \{\pm 1\}^3$ and $j \in \{\pm 2\}$. If $y + \eta_1 j = \frac{z+\eta_2 i}{6}$, then Lemma 3.2 yields that $z = 6y + 6\eta_1 j - \eta_2 i \leqslant 2y - 3$. Thus $2z + 6 \leqslant 4y \leqslant \eta_2 i - 6\eta_1 j - 3 \leqslant 10$, contradicting the fact $z > 280$. If $y + \eta_1 j = \frac{z+\eta_2 i}{3}$, then we get $z = 3y + 3\eta_1 j - \eta_2 i$. Together with Lemma 3.2, we obtain that $y \leqslant \eta_2 i - 3\eta_1 j - 3 \leqslant 4$. Therefore $z \leqslant 5$, which is impossible.

Assume that $\frac{z+\eta_2 i}{3} = \frac{y+\eta_1 j}{2} = t$. The facts $z = 3t - \eta_2 i$ and $y = 2t - \eta_1 j$ yield that

$$\begin{aligned} q_2 &< P_{\gcd\left(2t-\eta_1 j-2, \frac{3t-\eta_2 i-i}{2}\right)} Q_{\gcd\left(2t-\eta_1 j-2, \frac{3t-\eta_2 i+i}{2}\right)} \\ &\quad \times P_{\gcd\left(2t-\eta_1 j+2, \frac{3t-\eta_2 i-i}{2}\right)} Q_{\gcd\left(2t-\eta_1 j+2, \frac{3t-\eta_2 i+i}{2}\right)} \\ &< \alpha^{2(16-1.16)+16+0.1+\frac{z+1}{3}+0.1}. \end{aligned} \tag{3.3}$$

As $\alpha^{\frac{z-1.19}{2}} < q_2$, we get $z < 280.53$ together with (3.3). Similarly, for the case $\frac{z+\eta_2 i}{6} = \frac{y+\eta_1 j}{5}$, we find that the upper bound of z is 150. Both cases contradict with the assumption $z > 280$. In the sequel, suppose that $t = 2$. The possibilities are $y + \eta_1 j = \frac{z+\eta_2 i}{4}$ and $\frac{y+\eta_1 j}{3} = \frac{z+\eta_2 i}{4}$. If we continue as above, we get the upper bound as 126.9 for the case $\frac{y+\eta_1 j}{3} = \frac{z+\eta_2 i}{4}$. This is not possible since $z > 280$. The case $y+\eta_1 j = \frac{z+\eta_2 i}{4}$ yields together with Lemma 3.2 that $z = 4y+4\eta_1 j-\eta_2 i \leqslant 2y-3$. Consequently, $z + 3 \leqslant 2y \leqslant \eta_2 i - 4\eta_1 j - 3 \leqslant 6$, which is not possible.

The case $t = 1$ leads to $z = 2(y + \eta_1 j) - \eta_2 i$. Since $2y - 5 \leqslant z \leqslant 2y - 3$ and z is odd integer, then there are two possibilities which are $z = 2y - 3$ and $z = 2y - 5$. Assume that $z = 2y - 3$. When we divide the third equation by the second one in system (3.2), we obtain the following inequality.

$$\alpha^{y-2.25} < \frac{P_{2y-3}}{P_y/2} = \frac{bc+1}{ac+1} < \frac{b}{a}.$$

By multiplying both sides with a^2, we get $a^2\alpha^{y-2.25} < ab < S_x$. If x is even, then $a^2\alpha^{y-2.25} < S_x = B_{\frac{x}{2}} < \alpha^{x-1.96}$. Therefore,

$$a^2 < \alpha^{x-y-1.96+2.25} \leqslant \alpha^{-0.71} < 0.6.$$

But this is not possible since a is positive integer.

Now, assume that x is odd. The inequalities $a^2\alpha^{y-2.25} < S_x = P_x < \alpha^{x-1.16}$ yield

$$a^2 < \alpha^{x-y-1.16+2.25} \leqslant \alpha^{0.09} < 1.09.$$

The only possibility is $a = 1$. Equation system (3.2) yields that $b = P_x - 1$, $c = \frac{P_y}{2} - 1$ and $(P_x - 1)\left(\frac{P_y}{2} - 1\right) = P_{2y-3} - 1$. Since

$$\alpha^{y-2.18} < P_x - 1 = \frac{P_{2y-3} - 1}{P_y/2 - 1} < \alpha^{y-2.17},$$
$$\alpha^{x-1.28} < P_x - 1 = \frac{P_{2y-3} - 1}{P_y/2 - 1} < \alpha^{x-1.16},$$

we get that $-1.03 < x - y < -0.8$ which gives that $y = x + 1$. The equation $(P_x - 1)\left(\frac{P_y}{2} - 1\right) = P_{2y-3} - 1$ yields that $(P_x - 1)\left(\frac{P_{x+1}}{2} - 1\right) = P_{2x-1} - 1$. But this is not possible since $(P_x - 1)\left(\frac{P_{x+1}}{2} - 1\right) < P_{2x-1} - 1$. Similarly to above, we see that the case $z = 2y - 5$ is also impossible. In order to avoid unnecessary repetition, we omit this case.

(4) *y is odd and z is even*

Now, we give a lemma.

Lemma 3.3. *The system*

$$ab + 1 = S_x, \quad ac + 1 = P_y, \quad bc + 1 = B_{\frac{z}{2}}$$

satisfies $z \leqslant 2y - 1$.

Proof. The previous equation system gives $\sqrt{B_{z/2}} < c < P_y$. By Lemma 2.2, we have $\alpha^{(z-1.97)/2} < \sqrt{B_{z/2}} < c < P_y < \alpha^{y-1.16}$, which yields $z \leqslant 2y - 1$. □

Put $q_3 = \gcd(S_y - 1, S_z - 1)$. By the definition of the pellans sequence, we get

$$q_3 = \gcd(S_y - 1, S_z - 1) = \gcd\left(P_y - 1, B_{\frac{z}{2}} - 1\right).$$

The properties in Lemma 2.1 yield that

$$\begin{aligned}
q_3 &\leqslant \gcd\left(P_{\frac{y-i}{2}} Q_{\frac{y+i}{2}}, B_{\frac{z}{2}-1} B_{\frac{z}{2}+1}\right) \\
q_3 &\leqslant \gcd\left(P_{\frac{y-i}{2}}, B_{\frac{z}{2}-1}\right) \gcd\left(P_{\frac{y-i}{2}}, B_{\frac{z}{2}+1}\right) \gcd\left(Q_{\frac{y+i}{2}}, B_{\frac{z}{2}-1}\right) \gcd\left(Q_{\frac{y+i}{2}}, B_{\frac{z}{2}+1}\right) \\
&\leqslant \gcd\left(P_{\frac{y-i}{2}}, P_{z-2}\right) \gcd\left(P_{\frac{y-i}{2}}, P_{z+2}\right) \gcd\left(Q_{\frac{y+i}{2}}, P_{z-2}\right) \gcd\left(Q_{\frac{y+i}{2}}, P_{z+2}\right) \\
&= \prod_{j \in \left\{\pm 2\right\}} \gcd\left(P_{\frac{y-i}{2}}, P_{z+j}\right) \gcd\left(Q_{\frac{y+i}{2}}, P_{z-j}\right) \\
&= \prod_{j \in \{\pm 2\}} P_{\gcd\left(\frac{y-i}{2}, z+j\right)} Q_{\gcd\left(\frac{y+i}{2}, z-j\right)}
\end{aligned}$$

for some $i \in \{\pm 1\}$. Let $\gcd\left(\frac{y+\xi_1 i}{2}, z+\xi_2 j\right) = \frac{z+\xi_2 j}{w}$ where $\xi_1, \xi_2 \in \{\pm 1\}$. First assume that $w \geqslant 8$. Then we have

$$\alpha^{\frac{z-1.97}{2}} < \sqrt{B_{\frac{z}{2}}} < c < q_3 \leqslant \alpha^{\left(\frac{z-j}{8}-1.16\right)+\left(\frac{z+j}{8}-1.16\right)+\left(\frac{z-j}{8}+0.1\right)+\left(\frac{z-j}{8}+0.1\right)}.$$

The above formula leads to $\frac{z}{2} - 0.99 < \frac{z}{2} - 2.12$ which is an absurdity.

Now, assume that $w \leqslant 7$. Further assume that $\frac{y+\xi_1 i}{2k} = \frac{z+\xi_2 j}{l}$ holds for a suitable positive integer l coprime to k. If $k \geqslant l$, then according to $y < z$,

$$y + \xi_2 j < z + \xi_2 j \leqslant \frac{y + \xi_1 i}{2}$$

yields that $y \leqslant 5$. The inequality $z \leqslant 2y - 1$ implies that $z \leqslant 9$ which is impossible.

Assume that $k < l$. First we analyze the case $2 \leqslant \frac{l}{2k}$. Then,

$$z = \frac{l}{2k}(y + \xi_1 i) - \xi_2 j \geqslant 2(y-1) - 2 = 2y - 4,$$

which together with $z \leqslant 2y - 1$ implies the following possibilities.

If $z = 2y - 4$, then by Lemma 2.1 we have the following for some $i \in \{\pm 1\}$

$$\begin{aligned}
\alpha^{\frac{z-1.97}{2}} &< \gcd(P_y - 1, B_{\frac{z}{2}} - 1) = \gcd(P_y - 1, B_{y-2} - 1) \\
&= \gcd\left(P_{\frac{y-i}{2}} Q_{\frac{y+i}{2}}, \frac{1}{2} P_{y-3} Q_{y-1}\right) < \gcd\left(P_{\frac{y-i}{2}} Q_{\frac{y+i}{2}}, P_{y-3} Q_{y-1}\right) \\
&\leqslant P_{\gcd\left(\frac{y-i}{2}, y-3\right)} Q_{\gcd\left(\frac{y-i}{2}, y-1\right)} Q_{\gcd\left(\frac{y+i}{2}, y-3\right)} Q_{\gcd\left(\frac{y+i}{2}, y-1\right)}.
\end{aligned}$$

If $i = 1$, then

$$\begin{aligned}
\alpha^{\frac{z-1.97}{2}} &= \alpha^{\frac{2y-5.97}{2}} \\
&\leqslant P_{\gcd\left(\frac{y-1}{2}, y-3\right)} Q_{\gcd\left(\frac{y-1}{2}, y-1\right)} Q_{\gcd\left(\frac{y+1}{2}, y-3\right)} Q_{\gcd\left(\frac{y+1}{2}, y-1\right)} \\
&\leqslant P_2 Q_{\frac{y-1}{2}} Q_4 Q_2 < \alpha^{\frac{y}{2}+7.14}
\end{aligned}$$

leads to $y \leqslant 20$. Then $z \leqslant 36$ which is a contradiction.

If $i=-1$, then

$$\begin{aligned}\alpha^{\frac{z-1.97}{2}} &= \alpha^{\frac{2y-5.97}{2}}\\ &\leqslant P_{\gcd\left(\frac{y+1}{2},y-3\right)}Q_{\gcd\left(\frac{y+1}{2},y-1\right)}Q_{\gcd\left(\frac{y-1}{2},y-3\right)}Q_{\gcd\left(\frac{y-1}{2},y-1\right)}\\ &\leqslant P_4Q_2Q_4Q_{\frac{y-1}{2}} < \alpha^{\frac{y}{2}+7.14}.\end{aligned}$$

When we compare the exponents of α, we arrive at a contradiction.

Assume that $z=2y-2$. Then

$$\begin{aligned}\alpha^{\frac{z-1.97}{2}} = \alpha^{\frac{2y-3.97}{2}} &< \prod_{j\in\{\pm 2\}} P_{\gcd\left(\frac{y-i}{2},2y-2+j\right)}Q_{\gcd\left(\frac{y+i}{2},2y-2-j\right)}\\ &\leqslant P_6P_4Q_6Q_4 < \alpha^{17.88}\end{aligned}$$

yields that $z\leqslant 37$. But, this is impossible.

Now, assume that $\frac{l}{2k}<2$. This implies that $l\geqslant 2$. If $l=2$, then $k=1$ as $k<l$. Together with $y<z$ and $\frac{y+\xi_1 i}{2}=\frac{z+\xi_2 j}{2}$, we get $z=y+1$. If $y\equiv 1 \pmod 4$, then $z\equiv 1 \pmod 4$. So,

$$\begin{aligned}\alpha^{\frac{z-1.19}{2}} &< c < q_3 = \gcd\Big(P_y-1,\frac{P_z}{2}-1\Big)\\ &< \gcd(P_y-1,P_{y+1}-2) = \gcd\Big(P_{\frac{y-1}{2}}Q_{\frac{y+1}{2}},P_{\frac{y+1-2}{2}}Q_{\frac{y+1+2}{2}}\Big)\\ &= P_{\frac{y-1}{2}}Q_1.\end{aligned}$$

The inequality $\frac{z-1.19}{2}<\frac{y-1}{2}-1.16+0.8$ yields that $-0.19<-3.16$ since $z=y+1$. But this is false.

If $y\equiv 3 \pmod 4$, then $z\equiv 0 \pmod 4$. Therefore,

$$\begin{aligned}\alpha^{\frac{z-1.19}{2}} &< c < q_3 = \gcd\Big(P_y-1,\frac{P_z}{2}-1\Big)\\ &< \gcd(P_y-1,P_{y+1}-2) = \gcd\Big(P_{\frac{y+1}{2}}Q_{\frac{y-1}{2}},P_{\frac{y+1+2}{2}}Q_{\frac{y+1-2}{2}}\Big)\\ &= Q_{\frac{y-1}{2}}P_1.\end{aligned}$$

When we compare the exponents of α, we get $\frac{z-1.19}{2}<\frac{y-1}{2}+0.1$. As $z=y+1$, we arrive at a contradiction.

In the sequel, assume that $l\geqslant 3$. Taking any pair $(\xi_1',\xi_2')\neq(\xi_1,\xi_2)$, we have

$$z+\xi_2' j = \frac{l}{2k}(y+\xi_1 i)-\xi_2 j+\xi_2' j.$$

When we evaluate the upper bound for $q_3^{(0)}=\gcd\big(\frac{y+\xi_1' i}{2},z+\xi_2' j\big)$, we have

$$\begin{aligned}q_3^{(0)} &= \gcd\Big(\frac{y+\xi_1' i}{2},z+\xi_2' j\Big) = \gcd\Big(\frac{y+\xi_1' i}{2},\frac{l}{2k}(y+\xi_1 i)-\xi_2 j+\xi_2' j\Big)\\ &\leqslant \gcd\big(l(y+\xi_1' i),l(y+\xi_1 i)-2k(\xi_2 j-\xi_2' j)\big)\\ &\leqslant \big|l(\xi_1' i-\xi_1 i)-2k(\xi_2 j-\xi_2' j)\big|.\end{aligned}$$

The three cases $\xi_1' \neq \xi_1, \xi_2' \neq \xi_2$ and $\xi_1' \neq \xi_1, \xi_2' = \xi_2$ and $\xi_1' = \xi_1, \xi_2' \neq \xi_2$ imply that $q_3^{(0)} \leqslant 2(2k+l)$. Then

$$\begin{aligned}\alpha^{\frac{z-1.97}{2}} &< q_3 = \gcd\left(P_y - 1, B_{\frac{z}{2}} - 1\right)\\ &\leqslant \prod_{j\in\{\pm 2\}} P_{\gcd\left(\frac{y-i}{2}, z+j\right)} Q_{\gcd\left(\frac{y+i}{2}, z-j\right)} < \alpha^{\frac{z+2}{l}+2(2k+l)+2\cdot 0.1-2\cdot 1.16}.\end{aligned}$$

But none of these pairs satisfies the inequality $\frac{l}{2k} < 2$. The eligible pairs are

$$\begin{aligned}(l,k) = P(3,1), (3,2), (4,3), (5,2), (5,3), (5,4),\\ (6,5), (7,2), (7,3), (7,4), (7,5), (7,6).\end{aligned}$$

The previous argument provides the upper bounds

$$\begin{aligned}z < 57.19,\ 81.19,\ 77.46,\ 57.55,\ 70.89,\ 84.22,\\ 93.6, 59.23,\ 70.43,\ 81.63,\ 92.83,\ 104.03,\end{aligned}$$

respectively. But this is impossible since $z > 280$. Hence, the proof of Theorem 1.1 is completed.

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