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NEW IMMERSION THEOREMS FOR GRASSMANN MANIFOLDS $G_{3,n}$

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Abstract. A Gröbner basis for the ideal determining mod 2 cohomology of Grassmannian $G_{3,n}$, obtained in a previous paper, is used, along with the method of obstruction theory, to establish some new immersion results for these manifolds.

1. Introduction

The theory of Gröbner bases is one of the most powerful tools for deciding whether a certain polynomial in two or more variables belongs to a given ideal. An example where this problem is of particular interest is the mod 2 cohomology algebra of Grassmann manifold $G_{k,n} = O(n+k)/O(n) \times O(k)$. By Borel's description, this algebra is just the polynomial algebra on the Stiefel–Whitney classes $w_1, w_2, \ldots, w_k$ of the canonical vector bundle γ_k over $G_{k,n}$ modulo the ideal $J_{k,n}$ generated by the dual classes $\bar{w}_{n+1}, \bar{w}_{n+2}, \ldots, \bar{w}_{n+k}$.

A reduced Gröbner basis for the ideal $J_{2,n}$ has been obtained in [**6**]. Based on that result for odd n, some new immersions of Grassmannians $G_{2,2l+1}$ were established.

In [**9**] reduced Gröbner bases for all ideals $J_{k,n}$ were determined. We use this result and the method of modified Postnikov towers (MPT) developed by Gitler and Mahowald [**3**] to get new immersion results. In the following, $\operatorname{imm}(G_{3,n})$ stands for the immersion dimension of Grassmannians $G_{3,n}$

$$\operatorname{imm}(G_{3,n}) := \min\{d \mid G_{3,n} \text{ immerses into } \mathbb{R}^d\}.$$

Some lower bounds for $\operatorname{imm}(G_{3,n})$ were established by Oproiu in [**5**], where he used the method of the Stiefel–Whitney classes, and from the general result of Cohen [**1**] one has an upper bound for $\operatorname{imm}(G_{3,n})$. In [**7**] it is shown that $\operatorname{imm}(G_{3,n}) = 6n-3$ if n is a power of two.

Now we have the following new immersion results.

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THEOREM 1.1. *If $n \geqslant 3$ and $n \equiv 1 \pmod 8$, then $G_{3,n}$ immerses into $\mathbb{R}^{6n-6}$.*

This theorem improves Cohen's result whenever $\alpha(3n) < 6$ (as usual, $\alpha(3n)$ denotes the number of ones in the binary expansion of $3n$). For example, if $n = 1 + 2^r + \sum_{j=1}^{s} 2^{r+2j-1} = 1 + 2^r + 2^{r+1} \cdot \frac{2^{2s}-1}{3}$ for some $r \geqslant 3$ and $s \geqslant 0$, we have that $3n = 3 + 2^r + 2^{r+2s+1}$, so $\alpha(3n) = 4$. When $s = 0$, i.e., $n = 2^r + 1$ ($r \geqslant 3$), by Theorem 1.1 and Oproiu's result, we have that $6 \cdot 2^r - 3 \leqslant \operatorname{imm}(G_{3,2^r+1}) \leqslant 6 \cdot 2^r$.

THEOREM 1.2. *If $n \equiv 6 \pmod 8$, then $G_{3,n}$ immerses into $\mathbb{R}^{6n-5}$.*

The best improvement of Cohen's general result obtained from Theorem 1.2 is in the case $n = 2 + \sum_{j=1}^{s} 2^{2j}$, $s \geqslant 1$. Then $3n = 2 + 2^{2s+2}$ and so we are able to decrease the upper bound for $\operatorname{imm}(G_{3,n})$ by 3. For example, by this theorem and Oproiu's result, we have that $29 \leqslant \operatorname{imm}(G_{3,6}) \leqslant 31$.

THEOREM 1.3. *If $n \geqslant 3$ and $n \equiv 2 \pmod 8$, then $G_{3,n}$ immerses into $\mathbb{R}^{6n-7}$.*

Again, there is a number of cases in which Theorem 1.3 improves previously known results. In particular, when $n = 2^r + 2$, $r \geqslant 3$, we have an improvement by 3. In this case, using Oproiu's result and this theorem, we have $6 \cdot 2^r - 3 \leqslant \operatorname{imm}(G_{3,2^r+2}) \leqslant 6 \cdot 2^r + 5$.

In this paper we present only a proof of Theorem 1.1. The other theorems may be proved by using the same techniques.

REMARK 1.1. The detailed proofs of all three theorems can be found in [**8**]. Actually, this paper is an abridged version of [**8**]. In addition to these proofs, the preprint [**8**] contains the already mentioned result from [**7**] and the construction of Gröbner bases for $J_{3,n}$. This construction is not included in this paper, since these Gröbner bases are obtained in full generality in [**9**].

2. Gröbner bases

Throughout this section, we denote by $\mathbb{N}_0$ the set of all nonnegative integers and the set of all positive integers is denoted by $\mathbb{N}$.

Let $G_{k,n}$ be the Grassmann manifold of unoriented k-dimensional vector subspaces in $\mathbb{R}^{n+k}$, and $w_1, w_2, \dots, w_k$ the Stiefel–Whitney classes of the canonical bundle γ_k over $G_{k,n}$. It is known that the cohomology algebra $H^*(G_{k,n}; \mathbb{Z}_2)$ is isomorphic to the quotient $\mathbb{Z}_2[w_1, w_2, \dots, w_k]/J_{k,n}$ of the polynomial algebra $\mathbb{Z}_2[w_1, w_2, \dots, w_k]$ by the ideal $J_{k,n}$ generated by the polynomials $\bar{w}_{n+1}, \dots, \bar{w}_{n+k}$. These are obtained from the equation

$$(1 + w_1 + w_2 + \cdots + w_k)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1.$$

For $k = 3$ (which is the case from now on), one has

$$\bar{w}_r = \sum_{a+2b+3c=r} \binom{a+b+c}{a} \binom{b+c}{b} w_1^a w_2^b w_3^c, \qquad r \in \mathbb{N},$$

where it is understood that $a, b, c \in \mathbb{N}_0$.

Let $n \geqslant 3$ be a fixed integer. We define a set of polynomials in $\mathbb{Z}_2[w_1, w_2, w_3]$.

Definition 2.1. For $m, l \in \mathbb{N}_0$, let

$$g_{m,l} := \sum_{a+2b+3c=n+1+m+2l} \binom{a+b+c-m-l}{a}\binom{b+c-l}{b} w_1^a w_2^b w_3^c.$$

As before, it is understood that $a, b, c \in \mathbb{N}_0$. Note that $g_{0,0} = \bar{w}_{n+1}$.

The following theorem is a special case of Theorem 14 from [**9**].

Theorem 2.1. *The set $G = \{g_{m,l} \mid m + l \leqslant n + 1,\ m, l \in \mathbb{N}_0\}$ is the reduced Gröbner basis for the ideal $J_{3,n} = (\bar{w}_{n+1}, \bar{w}_{n+2}, \bar{w}_{n+3})$ with respect to the grlex ordering on monomials with $w_1 > w_2 > w_3$.*

The polynomials $g_{m,l} \in G$ for $l = n + 1$ and $l = n$ are calculated directly from Definition 2.1:

$$g_{0,n+1} = w_3^{n+1}, \quad g_{0,n} = w_1 w_3^n, \quad g_{1,n} = w_2 w_3^n.$$

Now, the following equalities may be obtained by using the well-known formula $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, $a, b \in \mathbb{Z}$ (it is understood that $\binom{a}{b} = 0$ if b is negative), along with some convenient index shifting

$$\begin{aligned} g_{m+2,l} &= g_{m,l+1} + w_2 g_{m,l} + w_1 g_{m+1,l}, \\ g_{m+1,l+1} &= w_3 g_{m,l} + w_1 g_{m,l+1}, \\ g_{m,l+2} &= w_3 g_{m+1,l} + w_2 g_{m,l+1}. \end{aligned}$$

We list a few elements from G which will be needed in the following section. One may obtain them by the repeated use of the previous equalities

$$\begin{aligned} g_{1,n-2} &= w_1^2 w_2 w_3^{n-2} + w_1 w_3^{n-1} + w_2^2 w_3^{n-2}, \\ g_{2,n-3} &= w_1^2 w_2^2 w_3^{n-3} + w_2^3 w_3^{n-3} + w_3^{n-1}, \\ g_{3,n-3} &= w_1 w_2^3 w_3^{n-3} + w_2^2 w_3^{n-2}, \\ g_{3,n-4} &= w_1^2 w_2^3 w_3^{n-4} + w_1 w_2^2 w_3^{n-3} + w_2^4 w_3^{n-4} + w_2 w_3^{n-2}, \\ g_{5,n-4} &= w_2^5 w_3^{n-4} + w_1 w_3^{n-1}, \\ g_{4,n-5} &= w_1^2 w_2^4 w_3^{n-5} + w_2^5 w_3^{n-5}, \\ g_{6,n-5} &= w_2^6 w_3^{n-5} + w_3^{n-1}. \end{aligned}$$

In the following, we will also use the fact that the set $\{w_1^a w_2^b w_3^c \mid a+b+c \leqslant n\}$ is a vector space basis for $H^*(G_{3,n}; \mathbb{Z}_2)$ (see, e.g., [**9**, Proposition 13]).

3. Immersions

As before, let w_i be the i-th Stiefel–Whitney class of the canonical vector bundle γ_3 over $G_{3,n}$ ($n \geqslant 3$) and let r be the (unique) integer such that $2^{r+1} < 3n < 2^{r+2}$, i.e., $\frac{2}{3} \cdot 2^r < n < \frac{4}{3} \cdot 2^r$. It is well known (see [**5**, p. 183]) that for the stable normal bundle ν of $G_{3,n}$ one has

$$w(\nu) = (1 + w_1^4 + w_2^2 + w_1^2 w_2^2 + w_3^2)(1 + w_1 + w_2 + w_3)^{2^{r+1}-n-3}. \tag{3.1}$$

For $n \leqslant 2^r - 3$, by the result of Stong [**10**] $\mathrm{ht}(w_1) = 2^r - 1$ and by the result of Dutta and Khare [**2**] $\mathrm{ht}(w_2) \leqslant 2^r - 1$. Also, $w_3^{2^r} = 0$ since $3 \cdot 2^r > 3 \cdot (2^r - 3) \geqslant$

$3n = \dim(G_{3,n})$ and we have that $(1+w_1+w_2+w_3)^{2^r} = 1$. This means that in this case ($\frac{2}{3}\cdot 2^r < n \leqslant 2^r - 3$) formula (3.1) simplifies to

$$w(\nu) = (1+w_1^4+w_2^2+w_1^2w_2^2+w_3^2)(1+w_1+w_2+w_3)^{2^r-n-3}. \tag{3.2}$$

In order to shorten the upcoming calculations, we give two equalities concerning the action of the Steenrod algebra $\mathcal{A}_2$ on $H^*(G_{3,n};\mathbb{Z}_2)$ which can be obtained by using the basic properties of $\mathcal{A}_2$ and formulas of Wu and Cartan. It is understood that a, b and c are nonnegative integers.

$$\begin{aligned} Sq^1(w_1^aw_2^bw_3^c) &= (a+b+c)w_1^{a+1}w_2^bw_3^c + bw_1^aw_2^{b-1}w_3^{c+1},\\ Sq^2(w_1^aw_2^bw_3^c) &= \tbinom{a+b+c}{2}w_1^{a+2}w_2^bw_3^c + b(a+c)w_1^{a+1}w_2^{b-1}w_3^{c+1}\\ &\quad + (b+c)w_1^aw_2^{b+1}w_3^c + \tbinom{b}{2}w_1^aw_2^{b-2}w_3^{c+2}. \end{aligned}$$

In the rest of the paper, it is understood that n is a fixed integer such that $n \geqslant 3$ and $n \equiv 1 \pmod 8$.

Lemma 3.1. *If ν is the stable normal bundle of $G_{3,n}$, then*

(a) $w_i(\nu) = 0$ *for* $i \geqslant 3n-8$*;*
(b) $w_1(\nu) = w_2(\nu) = 0$*;*
(c) $w_4(\nu) = w_2^2$*.*

Proof. As above, let $r \geqslant 3$ be the integer such that $2^{r+1} < 3n < 2^{r+2}$. If $n \geqslant 2^r$, then n must be $\geqslant 2^r+1$. So we have that $2^{r+1} \leqslant 2n-2$. The top class in expression (3.1), $(w_1^2w_2^2+w_3^2)w_3^{2^{r+1}-n-3}$, is in degree $6+3\cdot(2^{r+1}-n-3) \leqslant 6+3\cdot(n-5) = 3n-9$ and (a) follows in this case.

If $n < 2^r$, then we actually have that $n < 2^r-2$ (since $n \equiv 1 \pmod 8$), so formula (3.2) holds. The top class there is in degree $6+3\cdot(2^r-n-3)$ and, since $3n > 2^{r+1}$, we have that $2^r < \frac{3}{2}n$, implying $6+3\cdot(2^r-n-3) < 6+3\cdot\frac{n-6}{2} < 6+3\cdot(n-6) = 3n-12$. This proves (a).

Parts (b) and (c) we read off from formula (3.1) (using the fact that $2^{r+1}-n-3 \equiv 4 \pmod 8$)

$$\begin{aligned} w_1(\nu) &= (2^{r+1}-n-3)w_1 = 0,\\ w_2(\nu) &= \tbinom{2^{r+1}-n-3}{2}w_1^2 + (2^{r+1}-n-3)w_2 = 0,\\ w_4(\nu) &= w_1^4+w_2^2+\tbinom{2^{r+1}-n-3}{4}w_1^4 + \tbinom{2^{r+1}-n-3}{3}\tbinom{3}{1}w_1^2w_2\\ &\quad + \tbinom{2^{r+1}-n-3}{2}\tbinom{2}{1}w_1w_3 + \tbinom{2^{r+1}-n-3}{2}w_2^2 = w_2^2, \end{aligned}$$

and the lemma follows. ☐

Lemma 3.2. *For the map $Sq^2\colon H^{3n-6}(G_{3,n};\mathbb{Z}_2) \to H^{3n-4}(G_{3,n};\mathbb{Z}_2)$, we have*

$$\begin{aligned} &Sq^2(w_1^2w_2^2w_3^{n-4}) = w_1^2w_3^{n-2} + w_1w_2^2w_3^{n-3} + w_2^4w_3^{n-4} + w_2w_3^{n-2};\\ &Sq^2(w_1w_2w_3^{n-3}) = w_1^2w_3^{n-2} + w_1w_2^2w_3^{n-3};\\ &Sq^2(w_3^{n-2}) = w_1^2w_3^{n-2} + w_2w_3^{n-2}. \end{aligned}$$

PROOF. We use the Gröbner basis G to calculate:

$$\begin{aligned}Sq^2(w_1^2w_2^2w_3^{n-4}) &= \tbinom{n}{2}w_1^4w_2^2w_3^{n-4} + 2(n-2)w_1^3w_2w_3^{n-3}\\ &\quad + (n-2)w_1^2w_2^3w_3^{n-4} + \tbinom{2}{2}w_1^2w_3^{n-2}\\ &= w_1^2w_2^3w_3^{n-4} + w_1^2w_3^{n-2}\\ &= g_{3,n-4} + w_1w_2^2w_3^{n-3} + w_2^4w_3^{n-4} + w_2w_3^{n-2} + w_1^2w_3^{n-2}.\end{aligned}$$

Since $g_{m,l} = 0$ in $H^*(G_{3,n};\mathbb{Z}_2)$, we obtain the first equality. Also,

$$Sq^2(w_1w_2w_3^{n-3}) = \tbinom{n-1}{2}w_1^3w_2w_3^{n-3} + (n-2)w_1^2w_3^{n-2} + (n-2)w_1w_2^2w_3^{n-3}$$

and using the congruence $n \equiv 1 \pmod 8$, we directly get the second equality. Similarly,

$$Sq^2(w_3^{n-2}) = \tbinom{n-2}{2}w_1^2w_3^{n-2} + (n-2)w_2w_3^{n-2} = w_1^2w_3^{n-2} + w_2w_3^{n-2},$$

and we are done. □

LEMMA 3.3. *The map* $Sq^2\colon H^{3n-4}(G_{3,n};\mathbb{Z}_2) \to H^{3n-2}(G_{3,n};\mathbb{Z}_2)$ *is given by the following equalities:*

$$\begin{aligned}Sq^2(w_1^2w_3^{n-2}) &= w_1w_3^{n-1} + w_2^2w_3^{n-2},\\ Sq^2(w_1w_2^2w_3^{n-3}) &= Sq^2(w_2^4w_3^{n-4}) = Sq^2(w_2w_3^{n-2}) = w_1w_3^{n-1}.\end{aligned}$$

PROOF. The set $\{w_1^2w_3^{n-2}, w_1w_2^2w_3^{n-3}, w_2^4w_3^{n-4}, w_2w_3^{n-2}\}$ is a vector space basis for $H^{3n-4}(G_{3,n};\mathbb{Z}_2)$. We proceed to the calculation.

$$\begin{aligned}Sq^2(w_1^2w_3^{n-2}) &= \tbinom{n}{2}w_1^4w_3^{n-2} + (n-2)w_1^2w_2w_3^{n-2} = w_1^2w_2w_3^{n-2}\\ &= g_{1,n-2} + w_1w_3^{n-1} + w_2^2w_3^{n-2} = w_1w_3^{n-1} + w_2^2w_3^{n-2},\\ Sq^2(w_1w_2^2w_3^{n-3}) &= \tbinom{n}{2}w_1^3w_2^2w_3^{n-3} + 2(n-2)w_1^2w_2w_3^{n-2}\\ &\quad + (n-1)w_1w_2^3w_3^{n-3} + \tbinom{2}{2}w_1w_3^{n-1} = w_1w_3^{n-1},\\ Sq^2(w_2^4w_3^{n-4}) &= \tbinom{n}{2}w_1^2w_2^4w_3^{n-4} + 4\cdot(n-4)w_1w_2^3w_3^{n-3} + nw_2^5w_3^{n-4}\\ &\quad + \tbinom{4}{2}w_2^2w_3^{n-2} = w_2^5w_3^{n-4} = g_{5,n-4} + w_1w_3^{n-1} = w_1w_3^{n-1},\\ Sq^2(w_2w_3^{n-2}) &= \tbinom{n-1}{2}w_1^2w_2w_3^{n-2} + (n-2)w_1w_3^{n-1} + (n-1)w_2^2w_3^{n-2}\\ &= w_1w_3^{n-1}.\end{aligned}$$

□

LEMMA 3.4. *The map* $Sq^1\colon H^{3n-3}(G_{3,n};\mathbb{Z}_2) \to H^{3n-2}(G_{3,n};\mathbb{Z}_2)$ *is given by*

$$Sq^1(w_1w_2w_3^{n-2}) = w_2^2w_3^{n-2}, \quad Sq^1(w_2^3w_3^{n-3}) = Sq^1(w_3^{n-1}) = 0.$$

PROOF. We know that the classes $w_1w_2w_3^{n-2}$, $w_2^3w_3^{n-3}$ and w_3^{n-1} form an additive basis for $H^{3n-3}(G_{3,n};\mathbb{Z}_2)$. Using the Gröbner basis G, we have

$$\begin{aligned}Sq^1(w_1w_2w_3^{n-2}) &= nw_1^2w_2w_3^{n-2} + w_1w_3^{n-1} = g_{1,n-2} + w_2^2w_3^{n-2} = w_2^2w_3^{n-2},\\ Sq^1(w_2^3w_3^{n-3}) &= nw_1w_2^3w_3^{n-3} + 3w_2^2w_3^{n-2} = w_1w_2^3w_3^{n-3} + w_2^2w_3^{n-2} = g_{3,n-3} = 0,\\ Sq^1(w_3^{n-1}) &= (n-1)w_1w_3^{n-1} = 0,\end{aligned}$$

and the lemma is proved. □

In the proof of the following lemma, we shall make use of the fact that for any cohomology class u and any nonnegative integers m and k,

$$Sq^m(u^{2^k}) = \begin{cases} (Sq^{\frac{m}{2^k}}u)^{2^k}, & 2^k \mid m \\ 0, & 2^k \nmid m \end{cases}$$

The case $k=1$ is obtained from the Cartan formula and the rest is easily proved by induction on k.

LEMMA 3.5. *For the class $w_1w_2^4w_3^{n-5} \in H^{3n-6}(G_{3,n};\mathbb{Z}_2)$, we have*

(a) $Sq^2Sq^1(w_1w_2^4w_3^{n-5}) = w_3^{n-1}$,
(b) $Sq^2(w_1w_2^4w_3^{n-5}) = 0$,
(c) $(Sq^4 + w_2^2)(w_1w_2^4w_3^{n-5}) = 0$.

PROOF. One has

$$\begin{aligned} Sq^1(w_1w_2^4w_3^{n-5}) &= nw_1^2w_2^4w_3^{n-5} + 4w_1w_2^3w_3^{n-4} \\ &= w_1^2w_2^4w_3^{n-5} = g_{4,n-5} + w_2^5w_3^{n-5} = w_2^5w_3^{n-5}; \end{aligned}$$

and

$$\begin{aligned} Sq^2Sq^1(w_1w_2^4w_3^{n-5}) &= \tbinom{n}{2}w_1^2w_2^5w_3^{n-5} + 5(n-5)w_1w_2^4w_3^{n-4} + nw_2^6w_3^{n-5} + \tbinom{5}{2}w_2^3w_3^{n-3} \\ &= w_2^6w_3^{n-5} = g_{6,n-5} + w_3^{n-1} = w_3^{n-1}. \end{aligned}$$

This proves (a). Also,

$$\begin{aligned} Sq^2(w_1w_2^4w_3^{n-5}) &= \tbinom{n}{2}w_1^3w_2^4w_3^{n-5} + 4(n-4)w_1^2w_2^3w_3^{n-4} \\ &\quad + (n-1)w_1w_2^5w_3^{n-5} + \tbinom{4}{2}w_1w_2^2w_3^{n-3} \end{aligned}$$

and since $n \equiv 1 \pmod 8$, this is obviously equal to zero.

Finally, for (c) we use the Cartan formula and we get

$$(Sq^4 + w_2^2)(w_1w_2^4w_3^{n-5}) = w_1^2Sq^3(w_2^4w_3^{n-5}) + w_1Sq^4(w_2^4w_3^{n-5}) + w_1w_2^6w_3^{n-5}.$$

Now, since $n-5$ is divisible by 4, $w_2^4w_3^{n-5} = \big(w_2w_3^{\frac{n-5}{4}}\big)^4$ and so $Sq^3(w_2^4w_3^{n-5}) = 0$ and

$$Sq^4(w_2^4w_3^{n-5}) = \Big(Sq^1\Big(w_2w_3^{\frac{n-5}{4}}\Big)\Big)^4 = \Big(\big(1+\tfrac{n-5}{4}\big)w_1w_2w_3^{\frac{n-5}{4}} + w_3^{\frac{n-5}{4}+1}\Big)^4 = w_3^{n-1}$$

where the latter equality holds because $\frac{n-5}{4}$ is an odd integer (since $n \equiv 1 \pmod 8$). We conclude that

$$(Sq^4 + w_2^2)(w_1w_2^4w_3^{n-5}) = w_1w_3^{n-1} + w_1w_2^6w_3^{n-5} = w_1g_{6,n-5} = 0,$$

and the proof of the lemma is completed. □

LEMMA 3.6. *For the classes $w_1w_2^2w_3^{n-3}, w_2w_3^{n-2} \in H^{3n-4}(G_{3,n};\mathbb{Z}_2)$, we have*

(a) $Sq^1(w_1w_2^2w_3^{n-3}) = w_2^3w_3^{n-3} + w_3^{n-1}$, $\quad Sq^1(w_2w_3^{n-2}) = w_3^{n-1}$;
(b) $Sq^2(w_1w_2^2w_3^{n-3} + w_2w_3^{n-2}) = 0$.

PROOF. (a) We have

$$\begin{aligned} Sq^1(w_1w_2^2w_3^{n-3}) &= nw_1^2w_2^2w_3^{n-3} + 2w_1w_2w_3^{n-2} = w_1^2w_2^2w_3^{n-3} \\ &= g_{2,n-3} + w_2^3w_3^{n-3} + w_3^{n-1} = w_2^3w_3^{n-3} + w_3^{n-1}, \\ Sq^1(w_2w_3^{n-2}) &= (n-1)w_1w_2w_3^{n-2} + w_3^{n-1} = w_3^{n-1}. \end{aligned}$$

(b) Similarly,

$$\begin{aligned} Sq^2(w_1w_2^2w_3^{n-3}+w_2w_3^{n-2}) = {} & \tfrac{n}{2}w_1^3w_2^2w_3^{n-3}+2(n-2)w_1^2w_2w_3^{n-2}+(n-1)w_1w_2^3w_3^{n-3} \\ & + \tbinom{2}{2}w_1w_3^{n-1} + \tbinom{n-1}{2}w_1^2w_2w_3^{n-2} + (n-2)w_1w_3^{n-1} + (n-1)w_2^2w_3^{n-2} = 0, \end{aligned}$$

and we are done. □

LEMMA 3.7. *For the class $w_1w_3^{n-2} \in H^{3n-5}(G_{3,n};\mathbb{Z}_2)$, we have*

$$Sq^2(w_1w_3^{n-2}) = w_1w_2w_3^{n-2}.$$

PROOF. We simply calculate:

$$Sq^2(w_1w_3^{n-2}) = \tbinom{n-1}{2}w_1^3w_3^{n-2} + (n-2)w_1w_2w_3^{n-2} = w_1w_2w_3^{n-2},$$

and the lemma is proved. □

PROOF OF THEOREM 1.1. It is well known that the Grassmann manifold $G_{k,n}$ is orientable if and only if $n+k$ is even, and therefore, $G_{3,n}$ is orientable (the orientability of $G_{3,n}$ can also be deduced from Lemma 3.1 (b)). We shall use the theorem of Hirsch [**4**] which states that a smooth orientable compact m-manifold M^m immerses into $\mathbb{R}^{m+l}$ if and only if the classifying map $f_\nu\colon M^m \to BSO$ of the stable normal bundle ν of M^m lifts up to $BSO(l)$.

$$\begin{tikzcd} & BSO(l) \arrow[d, "p"] \\ M^m \arrow[ur, dashed] \arrow[r, "f_\nu"] & BSO \end{tikzcd}$$

The dimension of $G_{3,n}$ is $3n$, and hence, we need to lift $f_\nu\colon G_{3,n} \to BSO$ up to $BSO(3n-6)$. The $3n$-MPT for the fibration $p\colon BSO(3n-6) \to BSO$ is given in Diagram 1 (K_m stands for the Eilenberg–MacLane space $K(\mathbb{Z}_2, m)$).

The table of k-invariants is the following one.

$k_1^1:\ (Sq^2+w_2)w_{3n-5} = 0$
$k_2^1:\ (Sq^2+w_2)Sq^1w_{3n-5} + Sq^1w_{3n-3} = 0$
$k_3^1:\ (Sq^4+w_4)w_{3n-5} + Sq^2w_{3n-3} = 0$
$k_1^2:\ (Sq^2+w_2)k_1^1 + Sq^1k_2^1 = 0$

According to Lemma 3.1 (a), $f_\nu^*(w_{3n-5}) = w_{3n-5}(\nu) = 0$ and $f_\nu^*(w_{3n-3}) = w_{3n-3}(\nu) = 0$, so there is a lifting $g_1\colon G_{3,n} \to E_1$ of f_ν.

$$\begin{tikzcd}
 & BSO(3n-6) \arrow[d, "q_3"] & \\
 & E_2 \arrow[r, "k_1^2"] \arrow[d, "q_2"] & K_{3n-3} \\
 & E_1 \arrow[r, "k_1^1\times k_2^1\times k_3^1"] \arrow[d, "q_1"] & K_{3n-4}\times K_{3n-3}\times K_{3n-2} \\
G_{3,n} \arrow[r, "f_\nu"] \arrow[ur, dashed, "g"] \arrow[uur, dashed, "h"] & BSO \arrow[r, "w_{3n-5}\times w_{3n-3}"] & K_{3n-5}\times K_{3n-3}
\end{tikzcd}$$

Diagram 1.

Let us remark here that for every lifting $g\colon G_{3,n}\to E_1$ of f_ν, one has

$$Sq^2(g^*(k_1^1)) = Sq^1(g^*(k_2^1)). \tag{3.3}$$

This is obtained by applying g^* to the relation $(Sq^2+w_2)k_1^1 = Sq^1k_2^1$ in $H^*(E_1;\mathbb{Z}_2)$ (which produces the k-invariant k_1^2) and using Lemma 3.1 (b).

We have a lifting $g_1: G_{3,n}\to E_1$ and in order to make the next step (to lift f_ν up to E_2), we need to modify g_1 (if necessary) to a lifting g such that $g^*(k_1^1)=g^*(k_2^1)=g^*(k_3^1)=0$. By choosing a map $\alpha\times\beta\colon G_{3,n}\to K_{3n-6}\times K_{3n-4}=\Omega(K_{3n-5}\times K_{3n-3})$ (i.e., classes $\alpha\in H^{3n-6}(G_{3,n};\mathbb{Z}_2)$ and $\beta\in H^{3n-4}(G_{3,n};\mathbb{Z}_2)$), we get another lifting $g_2\colon G_{3,n}\to E_1$ (induced by g_1, α and β) as the composition:

$$G_{3,n}\xrightarrow{\triangle} G_{3,n}\times G_{3,n}\xrightarrow{(\alpha\times\beta)\times g_1} K_{3n-6}\times K_{3n-4}\times E_1\xrightarrow{\mu} E_1$$

where $\triangle$ is the diagonal mapping and $\mu\colon \Omega(K_{3n-5}\times K_{3n-3})\times E_1\to E_1$ is the action of the fibre in the principal fibration $q_1: E_1\to BSO$. By looking at the relations that produce the k-invariants k_1^1, k_2^1 and k_3^1 and using Lemma 3.1, we conclude that the following equalities hold (see [**3**, p. 95]):

$$\begin{aligned}
g_2^*(k_1^1) &= g_1^*(k_1^1)+(Sq^2+w_2(\nu))(\alpha) = g_1^*(k_1^1)+Sq^2\alpha;\\
g_2^*(k_2^1) &= g_1^*(k_2^1)+(Sq^2+w_2(\nu))Sq^1\alpha+Sq^1\beta = g_1^*(k_2^1)+Sq^2Sq^1\alpha+Sq^1\beta;\\
g_2^*(k_3^1) &= g_1^*(k_3^1)+(Sq^4+w_4(\nu))(\alpha)+Sq^2\beta = g_1^*(k_3^1)+(Sq^4+w_2^2)(\alpha)+Sq^2\beta.
\end{aligned}$$

First, we need to prove that $g_1^*(k_1^1)$ is in the image of $Sq^2\colon H^{3n-6}(G_{3,n};\mathbb{Z}_2)\to H^{3n-4}(G_{3,n};\mathbb{Z}_2)$. Let us assume, to the contrary, that $g_1^*(k_1^1)$ is not in this image. The classes $w_1^2w_3^{n-2}$, $w_1w_2^2w_3^{n-3}$, $w_2^4w_3^{n-4}$ and $w_2w_3^{n-2}$ form a vector space basis for $H^{3n-4}(G_{3,n};\mathbb{Z}_2)$ and from Lemma 3.2 we conclude that the sum of all basis elements and the sum of any two basis elements are in the image of Sq^2. This means that $g_1^*(k_1^1)$ is either a basis element or a sum of three distinct basis elements. Now, by looking at Lemma 3.3, we see that $Sq^2(g_1^*(k_1^1))\in\{w_1w_3^{n-1}, w_1w_3^{n-1}+w_2^2w_3^{n-2}\}$ and from formula (3.3) we have that $Sq^2(g_1^*(k_1^1)) = Sq^1(g_1^*(k_2^1))$. But according to Lemma 3.4 and the fact that the set $\{w_1w_3^{n-1}, w_2^2w_3^{n-2}\}$ is a vector space basis for $H^{3n-2}(G_{3,n};\mathbb{Z}_2)$, $Sq^1(g_1^*(k_2^1))$ cannot belong to $\{w_1w_3^{n-1}, w_1w_3^{n-1}+w_2^2w_3^{n-2}\}$. This contradiction proves that we can find a class $\alpha \in H^{3n-6}(G_{3,n};\mathbb{Z}_2)$ such that $Sq^2\alpha = g_1^*(k_1^1)$.

Since $\{w_1w_3^{n-1}, w_2^2w_3^{n-2}\}$ is a basis for $H^{3n-2}(G_{3,n};\mathbb{Z}_2)$, by Lemma 3.3, there is a class $\beta \in H^{3n-4}(G_{3,n};\mathbb{Z}_2)$ such that $Sq^2\beta = g_1^*(k_3^1)+(Sq^4+w_2^2)(\alpha)$, and so we have a lifting $g_2\colon G_{3,n}\to E_1$ (induced by g_1 and these classes α and β) such that $g_2^*(k_1^1)=g_2^*(k_3^1)=0$.

There is one more obstruction for lifting f_ν up to E_2: $g_2^*(k_2^1)\in H^{3n-3}(G_{3,n};\mathbb{Z}_2)$. Since $g_2^*(k_1^1)=0$, by equality (3.3) we have that $Sq^1(g_2^*(k_2^1))=0$ and according to Lemma 3.4, $g_2^*(k_2^1)$ must be in the subgroup of $H^{3n-3}(G_{3,n};\mathbb{Z}_2)$ generated by $w_2^3w_3^{n-3}$ and w_3^{n-1}. Observe the classes $\alpha' := w_1w_2^4w_3^{n-5}\in H^{3n-6}(G_{3,n};\mathbb{Z}_2)$ and $\beta' := w_1w_2^2w_3^{n-3}+w_2w_3^{n-2}\in H^{3n-4}(G_{3,n};\mathbb{Z}_2)$. By Lemma 3.5 (a), $Sq^2Sq^1\alpha' = w_3^{n-1}$ and according to Lemma 3.6 (a), $Sq^1\beta' = w_2^3w_3^{n-3}$. This means that we can choose the coefficients $a,b\in\{0,1\}$ such that $Sq^2Sq^1(a\alpha')+Sq^1(b\beta') = g_2^*(k_2^1)$. Finally, from Lemma 3.5, parts (b) and (c), and Lemma 3.6(b), we conclude that for the lifting $g\colon G_{3,n}\to E_1$ induced by g_2 and the classes $a\alpha'$ and $b\beta'$, all obstructions vanish, i.e., $g^*(k_1^1)=g^*(k_2^1)=g^*(k_3^1)=0$.

Therefore, the lifting g lifts up to E_2, i.e., there is a map $h\colon G_{3,n}\to E_2$ such that $q_1\circ q_2\circ h = q_1\circ g = f_\nu$.

For the final step, we observe that the set $\{w_1w_2w_3^{n-2}, w_2^3w_3^{n-3}, w_3^{n-1}\}$ is a vector space basis for $H^{3n-3}(G_{3,n};\mathbb{Z}_2)$. By looking at the relation that produces the k-invariant k_1^2 and according to Lemma 3.6(a), Lemma 3.7 and Lemma 3.1(b), one sees that the indeterminacy of k_1^2 is all of $H^{3n-3}(G_{3,n};\mathbb{Z}_2)$. Hence, the lifting $h\colon G_{3,n}\to E_2$ can be chosen such that $h^*(k_1^2)=0$. This completes the proof of the theorem. □

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