PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 101(115) (2017), 135–142 https://doi.org/10.2298/PIM1715135D

ON WEAKLY CLEAN AND WEAKLY EXCHANGE RINGS HAVING THE STRONG PROPERTY

Peter V. Danchev

Abstract. We define two classes of rings calling them *weakly clean rings* and *weakly exchange rings* both equipped with the *strong property*. Although the classes of weakly clean rings and weakly exchange rings are different, their two proper subclasses above do coincide. This extends results due to W. Chen (Commun. Algebra, 2006) and Chin-Qua (Acta Math. Hungar., 2011). We also completely characterize *strongly invo-regular rings*, thus somewhat extending results due to Danchev–McGovern (J. Algebra, 2015). Some other principal results concerning weakly clean and weakly exchange rings are discussed as well.

1. Introduction and Background

Throughout this paper, let all rings be *associative*, containing *identity* element. Our notations and terminology are classical and follow essentially those from [**23**]. For instance, for a ring R, the symbol $J(R)$ is reserved for the Jacobson radical of R. Also, a ring is said to be *abelian* if all its idempotents are central. Nevertheless, before stating our new notions, we will give a brief history of the basic concepts used here.

In their fundamental paper [**1**], Ahn and Anderson introduced in the commutative case the so-called *weakly clean rings* that are rings such that each element is the sum or the difference of a unit and an idempotent; however the definition of *weak cleanness* remains valid even in the general noncommutative variant (cf. [**7**] and [**8**]). This, actually, is a natural generalization of the classical notion of *clean* rings, defined by Nicholson in [**24**] as these rings for which any element is the sum of a unit and an idempotent. If they commute, these rings are called *strongly clean.*

On the other hand, by virtue of [**20**] or [**24**], a ring R is an *exchange ring* if, for every $x \in R$, there exists an idempotent e such that $e \in xR$ and $1-e \in (1-x)R$. It is well known that clean rings are exchange while the converse is untrue; for abelian rings these two sorts of rings, however, coincide. Generalizing this, in [**29**] (see [**6**] or [**7**] too) a ring R is said to be *weakly exchange* if, for each $x \in R$, there exists an

2010 *Mathematics Subject Classification:* 16S34.
Key words and phrases: weakly clean rings, clean rings, exchange rings, weakly exchange rings, strongly invo-regular rings.
Communicated by Zoran Petrović.

idempotent e such that $e \in xR$ and $1-e \in (1-x)R$ or $1-e \in (1+x)R$. By the proof of the necessity of Theorem 2.1 from [**6**], it is well known that weakly clean rings are in general weakly exchange, but the converse manifestly fails. However, for abelian rings, in the cited theorem from [**6**] was shown that weakly exchange rings are precisely the weakly clean rings.

Furthermore, in [**4**], W. Chen introduced the class of *strongly exchange rings* as those R such that, for every $x \in R$, there exist an idempotent $e \in R$ and two elements $a, b \in R$ for which $e = ax = xa$ and $1-e = b(1-x) = (1-x)b$. Moreover, there was obtained that these rings are exactly the strongly clean rings.

The key information about the principally known achievements in this object is the following. In [**24**] it was proven that if e is an arbitrary idempotent, then R is an exchange ring if, and only if, both corners eRe and $(1-e)R(1-e)$ are exchange. As a direct consequence, it follows that if R is exchange, then so is the full matrix ring $\mathbb{M}_n(R)$, $n \in \mathbb{N}$, and vice versa. About the "weakly case", we shall prove below that if R is a weakly exchange ring, then the same is eRe for any idempotent e (compare with Proposition 2.1). In this direction, in [**5**, Theorem 2.2] it was shown that if R is a strongly exchange ring, then so is eRe for an arbitrary idempotent e of R.

Besides, in [**21**] it was demonstrated that if eRe and $(1-e)R(1-e)$ are both clean, then so is R; in particular, if R is clean, then $\mathbb{M}_n(R)$ is clean. However, in [**26**] and [**27**] it was manifestly illustrated that if R is clean, then eRe need not be even weakly clean and hence this corner ring is not clean for some idempotent e which is neither central nor full. As for the matrix situation, let us consider the ring $\mathbb{Z}_{(15)}$, consisting of all rational numbers $\frac{t}{s}$ such that s is not divided by 15, which is commutative weakly clean but not clean. It is readily verified by simple computations that over such a ring the matrix $\left(\begin{smallmatrix} 10 & 0 \\ 0 & 5 \end{smallmatrix}\right)$ is not weakly clean as well. Thus, if R is weakly clean, then $\mathbb{M}_n(R)$ need not be again weakly clean. Whence if both eRe and $(1-e)R(1-e)$ are weakly clean, R is not necessarily weakly clean too. Nevertheless, in the aforementioned Theorem 2.2 from [**5**], it was established that if R is a strongly clean ring, then so is eRe for any idempotent e in R.

The aim of the present work is to strengthen the mentioned above results due to Chen in [**5**] and Chin-Qua in [**6**] in a wider context. Specifically, we will put a few new definitions and will prove that some of them are tantamount. Certain additional facts concerning these notions are obtained as well. For some others we will obtain a complete characterization; e.g. for the class of strongly invo-regular rings (Theorem 2.2), the class of which is independent of the class of weakly nil-clean rings as defined in [**15**] and [**2**].

The article is organized as follows: in the second section we state and prove our main results which provide a comprehensive description of the already defined new ring classes and which are distributed into three subsections. And in the subsequent third section we end with several unanswered questions of interest.

2. Main Results

The major tools here are the following:

DEFINITION 2.1. A ring R is called *weakly clean with the strong property* if, for any $r \in R$, there are a unit u and an idempotent e such that $ue = eu$ and either $r = u + e$ or $r = u - e$.

It can be readily seen that an element r is weakly clean with the strong property exactly when r or $-r$ is strongly clean.

DEFINITION 2.2. A ring R is called *weakly exchange with the strong property* if, for any $x \in R$, there are an idempotent e and elements $a, b \in R$ such that $e = ax = xa$ and $1 - e = b(1 - x) = (1 - x)b$ or $1 - e = b(1 + x) = (1 + x)b$.

The key instrument in our exploration will be the commutativity between certain elements in the rings. This will help us to construct appropriate units and idempotents which are rather necessary to demonstrate some critical equivalencies.

2.1. A class of weakly clean rings. The following statement improves Theorem 2.1 from [**6**] and Theorem 2.2 in [**5**] as well as Theorem 2.1 of [**8**]. It actually shows that Definitions 2.1 and 2.2 are tantamount.

THEOREM 2.1. *A ring R is weakly exchange with the strong property if, and only if, it is weakly clean with the strong property.*

PROOF. $\Rightarrow$ For each $x \in R$ there exist $e^2 = e \in R$ and $a, b \in R$ such that $e = ax = xa$ and either $1 - e = b(1 - x) = (1 - x)b$ or $1 - e = b(1 + x) = (1 + x)b$. Since $ax = xa$, we have that $ea = axa = ae$ and $e = ee = eax = xae = xea$. Set $a_1 = ea$. Then $e = a_1x = xa_1$ and $ea_1 = a_1e = a_1$. Analogously, putting $b_1 = (1 - e)b = b(1 - e)$, we deduce that either $1 - e = (1 - e)^2 = (1 - e)b(1 - x) = b_1(1 - x) = (1 - x)b_1$ or by symmetry $1 - e = b_1(1 + x) = (1 + x)b_1$, so in both cases $(1 - e)b_1 = b_1(1 - e) = b_1$. Therefore, $(a_1 - b_1)(x - (1 - e)) = a_1x - a_1(1 - e) - b_1x + b_1(1 - e) = e - b_1x - a_1 + a_1e + b_1 = e - b_1x + b_1 = e + b_1(1 - x) = e + 1 - e = 1$. Analogically, one checks that $(x - (1 - e))(a_1 - b_1) = 1$. Hence, $x - 1 + e$ is a unit. Moreover, $x(1 - e) = x - xe = x - xax = x - ex = (1 - e)x$. Consequently, we can represent $x = (x - 1 + e) + (1 - e)$, where by what we have just argued above the first term is a unit, while the second is obviously an idempotent. Clearly, $(x - 1 + e)(1 - e) = (x - 1)(1 - e) = x(1 - e) - (1 - e) = (1 - e)x - (1 - e) = (1 - e)(x - 1) = (1 - e)(x - 1 + e)$, as required, by taking into account that $e(1 - e) = (1 - e)e = 0$.

$\Leftarrow$ Given $x \in R$, we may write $x = u + f$ or $x = u - f$ with u a unit and f an idempotent such that $uf = fu$. Clearly, $ux = u(u + f) = u^2 + uf = u^2 + fu = (u + f)u = xu$ or $ux = u(u - f) = u^2 - uf = u^2 - fu = (u - f)u = xu$, so that in both cases u and x commute. Similarly, by the same reason, $u^{-1}x = xu^{-1}$.

Now, setting $e = u(1 - f)u^{-1}$, it follows that $e^2 = u(1 - f)u^{-1}.u(1 - f)u^{-1} = u(1 - f)^2u^{-1} = u(1 - f)u^{-1} = e$. In the case when $x = u + f$, we have that $(x - e)u = (u + f - u(1 - f)u^{-1})u = u^2 + fu - u(1 - f) = u^2 + fu - u + uf = u^2 + 2uf - u = x^2 - x$. So, $e = x + (x - x^2)u^{-1} = x(1 + (1 - x)u^{-1}) = (1 + (1 - x)u^{-1})x = (1 + (1 - u - f)u^{-1})x = (1 - f)u^{-1}x = u^{-1}ex$, because by what we have observed above $xu^{-1} = u^{-1}x$. Likewise, $1 - e = 1 - x - (x - x^2)u^{-1} = (1 - xu^{-1})(1 - x) = (1 - x)(1 - xu^{-1})$ since we again use that $xu^{-1} = u^{-1}x$.

Next, when $x = u - f$, we have $(x+e)u = (u - f + u(1-f)u^{-1})u = u^2 - fu + u - uf = u^2 - 2uf + u = x^2 + x$, whence $x + e = (x^2+x)u^{-1}$. Thus, $e = (x^2+x)u^{-1} - x = x((1+x)u^{-1} - 1) = ((1+x)u^{-1} - 1)x = ((1+u-f)u^{-1} - 1)x = (1-f)u^{-1}x = u^{-1}ex$ because as we have already seen $u^{-1}x = xu^{-1}$. Also, $1 - e = 1 + x - (x^2+x)u^{-1} = (1+x)(1-xu^{-1}) = (1-xu^{-1})(1+x)$ since $u^{-1}x = xu^{-1}$. $\square$

REMARK 2.1. Actually, since $uf = fu$, it follows that $e = u(1-f)u^{-1} = (1-f)uu^{-1} = 1 - f$ is again an idempotent and some things can be considerably simplified.

As a valuable consequence to our main characterization theorem, we yield:

COROLLARY 2.1. *If R is a weakly exchange ring with the strong property, then so is eRe for any idempotent e of R.*

In particular, if $\mathbb{M}_n(R)$ is a weakly exchange ring having the strong property for $n \geqslant 1$, then the same holds for the ring R.

PROOF. Suppose $x \in eRe$. Since $x \in R$, there exist an idempotent $f \in R$ and elements $a, b \in R$ such that $f = ax = xa$ and either $1 - f = b(1-x) = (1-x)b$ or $1 - f = b(1+x) = (1+x)b$. But $x = ere$ for some $r \in R$, so that $xe = ex = x$ and $fe = axe = ax = xa = exa = ef = f$. Hence $(ef)^2 = ef$ and $ef = efe$ lies in eRe. Furthermore, $xeae = xae = exae = eaxe = eax = eaex$. It follows now that $ef = eax = eaxe = exae = xeae = eaex$. Also, $e - ef = e - efe = e(1-f)e = eb(1 \pm x)e = eb(e \pm xe) = eb(e^2 \pm ex) = ebe(e \pm x)$. On the other hand, $(e \pm x)ebe = (e^2 \pm xe)be = (e \pm ex)be = e(1 \pm x)be = eb(1 \pm x)e = eb(e \pm xe) = eb(e \pm ex) = eb(e^2 \pm ex) = ebe(e \pm x)$. Whence $e - ef = ebe(e \pm x) = (e \pm x)ebe$, as required.

Since $R \cong E_{11}\mathbb{M}_n(R)E_{11}$ for the idempotent matrix E_{11} with $(1,1)$-entry 1 and the other entries 0, the second part follows now immediately. $\square$

As a direct consequence, we derive the following assertion. However, remark that by using another manipulation, we will also give a direct and more transparent proof (due to Šter), based on an idea from [**17**] and presented in a clearer form.

COROLLARY 2.2. *If R is a weakly clean ring with the strong property, then eRe is a weakly clean ring with the strong property for any idempotent e of R.*

In particular, if $\mathbb{M}_n(R)$ is a weakly clean ring having the strong property for $n \geqslant 1$, then the same is valid for the ring R.

PROOF. We assert that if $a \in eRe$ is strongly clean in R, then it is strongly clean in eRe. To that purpose, assume that $a \in eRe$, and suppose that $a = g + u$, where g is a nilpotent, u is a unit and all a, g, u commute. Therefore, $(1-g)a = (1-g)u$, so that $1 - g = u^{-1}(1-g)a$. Thus $1 - g \in Ra$, whence $1 - g \in Re$. Similarly, $1 - g = au^{-1}(1-g)$, hence $1 - g \in eR$. Finally, $1 - g \in eRe$. Hence g commutes with e, and so u also commutes with e. Now, since $ae = ea = a$, it plainly follows that $a = ge + ue$ is a strongly clean decomposition of a in eRe, where $(ge)^2 = ge$ is an idempotent and ue is a unit which has the inverse $u^{-1}e$. This substantiates our assertion after all.

Furthermore, by what we have already shown above, given $a \in eRe$, if a (respectively, $-a$) is strongly clean in R (i.e., a is weakly clean in R with the strong property), then a (respectively, $-a$) is strongly clean in eRe (i.e., a is weakly clean in eRe with the strong property), as required.

The second part follows as above in Corollary 2.1. $\square$

Moreover, concerning the converse implication, namely that if eRe and $(1-e)R(1-e)$ are weakly clean with the strong property, then so will eventually be R, it cannot be happen because the matrix situation "R being strongly clean implies $\mathbb{M}_n(R)$ is strongly clean" is still not settled. In this aspect, it is worth noticing that we established in [**13**] that if eRe and $(1-e)R(1-e)$ are both commutative nil-clean rings, then R is a nil-clean ring. In addition, if R is a commutative nil-clean ring, then $\mathbb{M}_n(R)$ is nil-clean for any $n \geqslant 1$ (see [**3**] as well).

As usual, for an integer $i \geqslant 2$, let $\mathbb{Z}_{(i)}$ be the subring of the ring $\mathbb{Q}$ of all rationals with dominators not divided by i. As a crucial example of a weakly clean ring having the strong property is the commutative weakly clean non-clean ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ (see [**1**] too). Likewise, as noted above, the commutative weakly clean ring $\mathbb{Z}_{(15)}$, which is not clean, is an other valuable example of a weakly clean ring having the strong property. Moreover, any strongly clean ring, being a clean ring equipped with the strong property, is also a weakly clean ring having the strong property; e.g., the triangular (upper or lower) matrix ring $\mathbb{T}_n(R)$ is a non-commutative example of a strongly clean ring for all $n \in \mathbb{N}$, whenever R is a commutative clean ring (cf. [**28**] and [**19**]).

2.2. Weakly exchange rings. Here we will treat the corner problem for weakly exchange rings in symmetry to Corollary 2.1, which was not considered in [**7**] and [**11**].

PROPOSITION 2.1. *If R is a weakly exchange ring and e is an arbitrary idempotent of R, then eRe is also a weakly exchange ring. In particular, if $\mathbb{M}_n(R)$ is a weakly exchange ring for $n \geqslant 1$, then so is the ring R.*

PROOF. Given $x \in eRe$, we write $x = ere$ for some $r \in R$. Thus $xe = ex = x$. Since $x \in R$, there is an idempotent $f \in xR$, say $f = xa$ for some $a \in R$, such that either $1 - f \in (1 - x)R$ or $1 - f \in (1 + x)R$. Observing that $ef = exa = xa = f$, it follows that $(fe)^2 = fefe = f^2e = fe = efe$ lies in eRe. But, finally, this enables us that $e - fe = (1 - f)e \in (1 \pm x)Re = (e \pm xe)Re = (e^2 \pm xe)Re = (e \pm x)eRe$, as required.

The second part follows analogously to Corollary 2.1 listed above. $\square$

However, the eventual truthfulness of the converse part is not known yet, that is, if eRe and $(1 - e)R(1 - e)$ are both weakly exchange rings, is R also a weakly exchange ring? In addition, if R is weakly exchange, is $\mathbb{M}_n(R)$ weakly exchange as well?

2.3. Strongly invo-regular rings. Referring to [**18**], a ring R is said to be *nil-clean* if, for each $r \in R$, there are a nilpotent q and an idempotent e such that $r = q + e$. If $qe = eq$, then R is called *strongly nil-clean*. Generalizing this notion, in [**15**] and more generally in [**2**] it was introduced the concept of a *weakly nil-clean* ring R as such a ring that for every $r \in R$, $r = q + e$ or $r = q - e$. Additionally, if $qe = eq$ holds, we will say that R is *weakly nil-clean with the strong property* (cf. [**10**]). In [**15**] were totally characterized commutative weakly nil-clean rings; in [**2**] this was done for abelian weakly nil-clean rings; in [**10**] the same holds of weakly nil-clean rings having the strong property; in [**12**] this was finally produced for arbitrary weakly nil-clean rings.

Our next definition generalizes reduced weakly nil-clean rings that, owing to [**15**], are weakly boolean rings, i.e., rings for which $r^2 = r$ or $r^2 = -r$ for any $r \in R$.

Definition 2.3. We shall say that a ring R is *strongly invo-regular* if, for each $r \in R$, there is an involution $v \in R$ (that is, $v^2 = 1$) with $r^2 = rv$.

Apparently, strongly invo-regular rings are strongly regular, while the reverse implication is false. It is long known that strongly regular rings are subdirect products of division rings. Also, as noticed above, reduced weakly nil-clean rings are strongly invo-regular, whereas the converse is untrue as the following construction manifestly demonstrates. In [**15**] it was shown that the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil-clean, because of the problematical elements $(1, -1)$ and $(-1, 1)$. However, it is obviously strongly invo-regular, since $(1, -1)$ and $(-1, 1)$ are both involutions; in fact, $(1, -1)^2 = (-1, 1)^2 = (1, 1)$.

We are now ready to describe strongly invo-regular rings in terms of some finite fields.

Theorem 2.2. *Any strongly invo-regular ring can be embedded as a subring of the direct product of (finitely or infinitely many) copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and vice versa.*

Proof. Since for all elements $r \in R$ we have $r^2 = rv$ for some involution v depending on r, it follows that $r^3 = r$. Therefore, a classical theorem of Jacobson can be applied to get that R is commutative. Moreover, it is well known that $J(R)$ is the intersection of all maximal ideals M in R and hence, R being strongly regular implies that $J(R) = \cap_{M \in Max(R)} M = \{0\}$. Whence R is reduced. On the other side, the map $R \to \prod_{M \in Max(R)} (R/M)$, defined in the standard way, is a homomorphism with kernel $\cap_{M \in Max(R)} M = J(R)$. That is, we have an injection $R/J(R) \to \prod_{M \in Max(R)} (R/M)$. With this observation at hand, it now follows that $R \cong R/J(R)$ is embeddable inside of $\prod_{M \in Max(R)} (R/M)$. However, R/M is always a field, provided M is a maximal ideal of R, and since strongly invo-regular rings are obviously closed under homomorphic images, R/M being simultaneously a strongly invo-regular ring and a field must be isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ because either $v = 1$ or $v = -1$, and then $r^2 = r$ reduces to $r = 0$ or $r = 1$ as well as $r^2 = -r$ reduces to $r = 0$ or $r = -1$.

Conversely, it is self-evident that each element $r \in \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ can be expressed as $r^2 = rv$ for some appropriate involution v, as required. $\square$

Remark 2.2. This theorem is an expansion of the classical result that a ring is boolean if and only if it is embeddable in $\prod_\lambda \mathbb{Z}_2$. Utilizing [**22**], every element of a strongly invo-regular ring is a sum of two idempotents. It is also straightforward that $r = r^3$ amounts to $r = r^2v$, i.e., to $r^2 = rv$, where $v = -r^2 + r + 1$ is an involution.

On the other side, concerning the above embedding of R into the direct product of fields, it could be an isomorphism in some cases; in fact, if M and N are two different maximal ideals in a ring R, then the "co-maximal" property $M + N = R$ is always true because $M + N$ is an ideal of R which properly contains both M and N. This leads us to the surprising fact that the above map becomes in this case a surjection. Thus, with the aid of the Chinese Remainder Theorem, we deduce that $MN = M \cap N$ and $R/(M \cap N) \cong (R/M) \times (R/N)$, as desired.

3. Open Problems

We close with the following questions of some interest and importance:

PROBLEM 3.1. Suppose that R is a ring such that, for every $x \in R$, there exist an idempotent $e = ax = xa$, for some element $a \in R$, and a nilpotent $q \in R$ such that $1 - e = (1 - e)(1 + q)(1 - x) = (1 - x)(1 - e)(1 + q)$. Is this ring strongly nil-clean?

Notice that such a ring has to be weakly nil-clean in the sense of [**16**].

Mimicking the idea from [**15**], we will say that a ring R is *uniquely weakly clean* if, for each $x \in R$, there exists a unique idempotent e of R such that $x - e$ or $x + e$ is a unit.

So, we once again come to the following problem (see also [**8**] or Problem 6 from [**7**]).

PROBLEM 3.2. Characterize uniquely weakly clean rings.

For a nice characterization of uniquely clean rings the interested reader may see [**25**] and [**4**], respectively.

A ring R is said to be *fully (nil-) clean* if every element is the sum of a unit (nilpotent) and a full idempotent (note that f is a full idempotent in R provided that $RfR = R$).

PROBLEM 3.3. If R is a fully (nil-) clean ring and e is an arbitrary idempotent, is it true that eRe is fully (nil-) clean, and conversely? In particular, does it follow that $\mathbb{M}_n(R)$ is fully (nil-) clean?

PROBLEM 3.4. Characterize *invo-regular rings* that are rings R such that for each element $r \in R$ there exists an involution v with the property $r = rvr$.

Clearly these rings are unit-regular.

PROBLEM 3.5. Characterize those rings R such that for every $r \in R$ there exists a nilpotent q with the property $r(1 + q)r = r + q$.

When R is commutative, it is pretty easy to see that R is nil-clean because $R/N(R)$ is boolean, where $N(R)$ is the nil-radical of R (see [**9**] as well).

Acknowledgements. The author would like to express his sincere thanks to Dr. Janez Šter for the valuable communication on the present topic. He is also very grateful to the referee for the helpful suggestions given in the report.

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Department of Mathematics
Plovdiv University
Plovdiv 4000
Bulgaria
pvdanchev@yahoo.com

(Received 10 08 2015)
(Revised 21 04 2016)