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POSITIVE ENERGY UNITARY IRREDUCIBLE REPRESENTATIONS OF THE SUPERALGEBRA $\mathrm{osp}(1|8,\mathbb{R})$

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Abstract. We continue the study of positive energy (lowest weight) unitary irreducible representations of the superalgebras $\mathrm{osp}(1|2n,\mathbb{R})$. We present the full list of these UIRs. We give a proof of the case $\mathrm{osp}(1|8,\mathbb{R})$.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. This makes the classification of the UIRs of these superalgebras very important. Until recently only those for $D \leqslant 6$ were studied since in these cases the relevant superconformal algebras satisfy [**24**] the Haag–Lopuszanski–Sohnius theorem [**19**]. Thus, such classification was known only for the $D = 4$ superconformal algebras $\mathrm{su}(2,2/N)$ [**17**] (for $N = 1$), [**10–13**] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1,2$) was given in [**23**] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [**8**].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras $\mathrm{osp}(1|2n)$. Initially, the superalgebra $\mathrm{osp}(1|32)$ was put forward for $D = 10$ [**18, 26**]. Later it was realized that $\mathrm{osp}(1|2n)$ would fit any dimension, though they are minimal only for $D = 3,9,10,11$ (for $n = 2,16,16,32$, resp.) [**2,3,16**]. In all cases we need to find at first the UIRs of $\mathrm{osp}(1|2n,\mathbb{R})$ the study of which was started in [**15**] and [**9**]. Later, in [**14**], we finalized the UIR classification of [**15**] as Dobrev–Zhang–Salom (DZS) Theorem. In [**14**] we proved the DZS Theorem for $\mathrm{osp}(1|6)$.

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In the present paper, we prove the DZS Theorem for osp(1|8). For the lack of space, we refer to [**14**, **15**] for extensive literature on the subject.

2. Preliminaries on representations

Our basic references for Lie superalgebras are [**20**–**22**], although in this exposition we follow [**15**].

The even subalgebra of $\mathcal{G} = \mathrm{osp}(1|2n,\mathbb{R})$ is the algebra $\mathrm{sp}(2n,\mathbb{R})$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong \mathrm{su}(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature

$$\chi \;=\; [\, d\,;\, a_1\,, \ldots, a_{n-1}\,] \tag{2.1}$$

where d is the conformal weight, and $a_1, \ldots, a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $\mathrm{su}(n)$ (the simple part of $\mathcal{K}$).

In [**15**] were classified (with some omissions to be spelled out below) the positive energy (lowest weight) UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [**8**, **10**–**13**], resp. The main tool was an adaptation of the Shapovalov form [**25**] on the Verma modules V^χ over the complexification $\mathcal{G}^\mathbb{C} = \mathrm{osp}(1|2n)$ of $\mathcal{G}$.

We recall some facts about $\mathcal{G}^\mathbb{C} = \mathrm{osp}(1|2n)$ (denoted $B(0,n)$ in [**20**, **21**]) as used in [**15**]. The root systems are given in terms of $\delta_1 \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. The even and odd roots systems are [**20**, **21**]

$$\Delta_{\bar 0} = \{\pm\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ \pm 2\delta_i,\ 1 \leqslant i \leqslant n\}, \quad \Delta_{\bar 1} = \{\pm\delta_i,\ 1 \leqslant i \leqslant n\}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [**20**, **21**] $\Pi = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n\}$, or, introducing standard notation for the simple roots,

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \quad \alpha_j = \delta_j - \delta_{j+1},\ j = 1, \ldots, n-1, \quad \alpha_n = \delta_n.$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is

$$\underset{1}{\circ} - \cdots - \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet}\,.$$

The black dot is used to signify that the simple odd root is not nilpotent. In fact, the superalgebras $B(0,n) = \mathrm{osp}(1|2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [**20**, **21**].

The corresponding to Π positive root system is

$$\Delta^+_{\bar 0} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ 2\delta_i,\ 1 \leqslant i \leqslant n\}, \quad \Delta^+_{\bar 1} = \{\delta_i,\ 1 \leqslant i \leqslant n\} \tag{2.2}$$

We record how the elementary functionals are expressed through the simple roots:

$$\delta_k = \alpha_k + \cdots + \alpha_n.$$

From the point of view of representation theory, more relevant is the restricted root system, such that

$$\bar\Delta^+ = \bar\Delta^+_{\bar 0} \cup \Delta^+_{\bar 1}, \quad \bar\Delta^+_{\bar 0} \equiv \{\alpha \in \Delta^+_{\bar 0} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar 1}\} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n\}$$

The superalgebra $\mathcal{G} = \mathrm{osp}(1|2n,\mathbb{R})$ is a split real form of $\mathrm{osp}(1|2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram

$$\underset{1}{\circ} - \cdots - \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ}\,.$$

Naturally, for the B_n positive root system we drop the roots $2\delta_i$

$$\Delta^+_{\mathrm{B}_n} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ \delta_i,\ 1 \leqslant i \leqslant n\} \cong \bar{\Delta}^+$$

This shall be used essentially below.

Besides (2.1), we shall use the Dynkin-related labelling:

$$(\Lambda, \alpha_k^\vee) = -a_k, \quad 1 \leqslant k \leqslant n,$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k,\alpha_k)$, and the minus signs are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [**22**]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [**13**, **15**]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is

$$a_n = -2d - a_1 - \cdots - a_{n-1}.$$

The actual Dynkin labelling is given by $m_k = (\rho - \Lambda,\ \alpha_k^\vee)$ where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (2.2)

$$\begin{aligned}
\rho \doteq \rho_{\bar{0}} - \rho_{\bar{1}} &= (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n,\\
\rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n,\\
\rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n).
\end{aligned}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho,\ \alpha_i^\vee) = 1$, $i = 1, \ldots, n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$, the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ (for all k) the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$ (and of B_n).

Having in hand the values of Λ on the basis, we can recover them for any element of $\mathcal{H}^*$. We shall need only $(\Lambda,\ \beta^\vee)$ for all positive roots β as given in [**15**]

$$\begin{aligned}
(\Lambda, (\delta_i - \delta_j)^\vee) &= (\Lambda, \delta_i - \delta_j) = -a_i - \cdots - a_{j-1}\\
(\Lambda, (\delta_i + \delta_j)^\vee) &= (\Lambda, \delta_i + \delta_j) = 2d + a_1 + \cdots + a_{i-1} - a_j - \cdots - a_{n-1}\\
(\Lambda, \delta_i^\vee) &= (\Lambda, 2\delta_i) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1}\\
(\Lambda, (2\delta_i)^\vee) &= (\Lambda, \delta_i) = d + \tfrac{1}{2}(a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1})
\end{aligned} \tag{2.3}$$

To introduce Verma modules we use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$ where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that

$$Zv_0 = 0, \quad Z \in \mathcal{G}^-; \qquad Hv_0 = \Lambda(H)v_0, \quad H \in \mathcal{H}.$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\,v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [**22**] (which generalizes the BGG-criterion [**1**] to the super case) to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [**15**]

$$(\rho - \Lambda,\, \beta^\vee) = m_\beta, \qquad \beta \in \Delta^+, \quad m_\beta \in \mathbb{N}. \tag{2.4}$$

If a condition from (2.4) is fulfilled, then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β: $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the *singular vector* $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions

$$\begin{aligned} &Xv_s^{m,\beta} = 0, \quad X \in \mathcal{G}^-, \\ &Hv_s^{m,\beta} = \Lambda'(H)v_0, \quad H \in \mathcal{H}, \ \Lambda' = \Lambda + m\beta. \end{aligned}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [**4**, **13**]

$$v_s^{m,\beta} = P^{m,\beta}v_0, \quad P^{m,\beta} \in U(\mathcal{G}^+).$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)P^{m,\beta}v_0$.

Note that the Casimirs of $\mathcal{G}^{\mathbb{C}}$ take the same values on V^Λ and $V^{\Lambda'}$.

Certainly, (2.4) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (2.4) is fulfilled, and let us denote $\tilde{I}^\Lambda \equiv \bigcup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

Further we shall use also the following notion. The singular vector v_1 is called *descendant* of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 = P_{12}v_2$. Clearly, in this case we have: $I^1 \subset I^2$ where I^k is the submodule generated by v_k.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [**1**, **22**]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$. (If V^Λ is irreducible, then $L_\Lambda = V^\Lambda$.)

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if $\operatorname{rank}\mathcal{G} \leqslant 2$, or when (2.4) is fulfilled for all simple roots (and, as a consequence, for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition 2.1. [**1**, **6**, **7**] Let V^Λ be a reducible Verma module. A vector $v_{\rm ssv} \in V^\Lambda$ is called a *subsingular vector* if $v_{\rm su} \notin \tilde{I}^\Lambda$ and $Xv_{\rm su} \in \tilde{I}^\Lambda$, for all $X \in \mathcal{G}^-$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [**15**] it was established that from (2.4) follows that the Verma module $V^{\Lambda(\chi)}$ is reducible if one of the following relations holds (following the order of (2.3

(2.5a) $$\mathbb{N} \ni m^-_{ij} = j - i + a_i + \cdots + a_{j-1}$$

(2.5b) $$\mathbb{N} \ni m^+_{ij} = 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d$$

(2.5c) $$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d$$

(2.5d) $$\mathbb{N} \ni m_{ii} = n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d\ .$$

Further we shall use the fact from [**15**] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (2.5d) is fulfilled also (2.5c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j. Naturally, $X^-_j \in \mathcal{G}^-$ corresponds to $-\alpha_j$.

Further, we notice that all reducibility conditions in (2.5a) are fulfilled. In particular, for the simple roots from those condition, (2.5a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, \ldots, n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)v^i_s$, where $\Lambda_i = \Lambda + m^-_i\alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ, we shall consider also the factor-modules $F^\Lambda_c = V^\Lambda/I^\Lambda_c \supset F^\Lambda$. We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c, i.e., $(X^+_i)^{1+a_i}|\Lambda_c\rangle = 0$, $i = 1, \ldots, n-1$.

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (2.5b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (2.5b) are obtained for $m^+_{ij} = 1$ and they are

$$d_{ij} \equiv n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j),$$

the corresponding root being $\delta_i + \delta_j$.

The maximal d coming from the different possibilities in (2.5c,d), resp., are obtained for $m_i = 1$, $m_{ii} = 1$, resp., and they are:

$$d_i \equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}), \quad d_{ii} = d_i - \tfrac{1}{2},$$

the corresponding roots being δ_i, $2\delta_j$, resp.

There are some orderings between these maximal reduction points [**15**]:

(2.6)
$$\begin{gathered}
d_1 > d_2 > \cdots > d_n,\\
d_{i,i+1} > d_{i,i+2} > \cdots > d_{in},\\
d_{1,j} > d_{2,j} > \cdots > d_{j-1,j},\\
d_i > d_{jk} > d_\ell, \qquad i \leqslant j < k \leqslant \ell.
\end{gathered}$$

Obviously the first reduction point is

$$d_1 = n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}).$$

3. Unitarity

The first results on the unitarity were given in [**15**], and then improved in [**14**]. Thus, the statement below should be called Dobrev–Zhang–Salom Theorem.

THEOREM DZS 1. *All positive energy unitary irreducible representations of the superalgebras* $\mathrm{osp}(1|\,2n,\mathbb{R})$ *characterized by the signature* χ *in* (2.1) *are obtained for real* d *and are given as follows:*

$$\begin{aligned}
& d \geqslant n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) = d_1, \quad a_1 \neq 0,\\
& d \geqslant n - \tfrac{3}{2} + \tfrac{1}{2}(a_2 + \cdots + a_{n-1}) = d_{12}, \quad a_1 = 0,\ a_2 \neq 0,\\
& d = n - 2 + \tfrac{1}{2}(a_2 + \cdots + a_{n-1}) = d_2 > d_{13}, \quad a_1 = 0,\ a_2 \neq 0,\\
& d \geqslant n - 2 + \tfrac{1}{2}(a_3 + \cdots + a_{n-1}) = d_2 = d_{13}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
& d = n - \tfrac{5}{2} + \tfrac{1}{2}(a_3 + \cdots + a_{n-1}) = d_{23} > d_{14}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
& d = n - 3 + \tfrac{1}{2}(a_3 + \cdots + a_{n-1}) = d_3 = d_{24} > d_{15}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
& \vdots\\
& d \geqslant n - 1 - \kappa + \tfrac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
& \qquad\qquad \kappa = \tfrac{1}{2}, 1, \ldots, \tfrac{1}{2}(n-1),\\
& d = n - \tfrac{3}{2} - \kappa + \tfrac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
& \vdots\\
& d = n - 1 - 2\kappa + \tfrac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
& \vdots\\
& d \geqslant \tfrac{1}{2}(n-1), \quad a_1 = \cdots = a_{n-1} = 0\\
& d = \tfrac{1}{2}(n-2), \quad a_1 = \cdots = a_{n-1} = 0\\
& \vdots\\
& d = \tfrac{1}{2}, \quad a_1 = \cdots = a_{n-1} = 0\\
& d = 0, \quad a_1 = \cdots = a_{n-1} = 0
\end{aligned}$$

where the last case is the trivial one-dimensional irrep.

The theorem was partially proved [**15**], while in [**14**] was given a sketch of a proof, and the case $n = 3$ was proved. We are going to give a proof for $\mathrm{osp}(1|8)$.

4. The case of $\mathrm{osp}(1|8)$

For $n = 4$ formula (2.6) simplifies to

$$\begin{array}{c}
d_1 > d_{12} > d_2 > d_{23} > d_3 > d_{34} > d_4\\
\hookrightarrow > d_{13} > \nearrow \hookrightarrow > d_{24} > \nearrow\\
\hookrightarrow > d_{14} > \nearrow
\end{array}$$

In the case of $\mathrm{osp}(1|8)$ Theorem DZS reads:

THEOREM 4.1. *All positive energy unitary irreducible representations of the superalgebras* osp(1|8, ℝ) *characterized by the signature* χ *in* (2.1) *are obtained for real* d *and are given as follows*

$$\begin{aligned}
&d \geqslant 3 + \tfrac{1}{2}(a_1 + a_2 + a_3) = d_1, \quad a_1 \neq 0,\\
&d \geqslant \tfrac{5}{2} + \tfrac{1}{2}(a_2 + a_3) = d_{12}, \quad a_1 = 0,\ a_2 \neq 0,\\
&d = 2 + \tfrac{1}{2}(a_2 + a_3) = d_2 > d_{13}, \quad a_1 = 0,\ a_2 \neq 0,\\
&d \geqslant 2 + \tfrac{1}{2}a_3 = d_2 = d_{13}, \quad a_1 = a_2 = 0,\ a_3 \neq 0\\
&d = \tfrac{3}{2} + \tfrac{1}{2}a_3 = d_{23} > d_{14}, \quad a_1 = a_2 = 0,\ a_3 \neq 0\\
&d = 1 + \tfrac{1}{2}a_3 = d_3 > d_{24}, \quad a_1 = a_2 = 0,\ a_3 \neq 0\\
&d \geqslant \tfrac{3}{2} = d_{23} = d_{14}, \quad a_1 = a_2 = a_3 = 0\\
&d = 1 = d_3 = d_{24}, \quad a_1 = a_2 = a_3 = 0\\
&d = \tfrac{1}{2} = d_{34}, \quad a_1 = a_2 = a_3 = 0,\\
&d = 0 = d_4, \quad a_1 = a_2 = a_3 = 0
\end{aligned}$$

where the last case is the trivial one-dimensional irrep.

PROOF. For $d > d_1$ there are no singular vectors and we have unitarity. At $d = d_1$ there is a singular vector of weight $\delta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ [**5, 15**]:

$$\begin{aligned}
(4.1) \qquad v^1_{\delta_1} = \sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\sum_{k_3=0}^{1} b_{k_1,k_2,k_3}(X_1^+)^{1-k_1}(X_2^+)^{1-k_2}(X_3^+)^{1-k_3}\\
\times X_4^+(X_3^+)^{k_3}(X_2^+)^{k_2}(X_1^+)^{k_1}v_0 \equiv \mathcal{P}^{1,\delta_1}v_0,
\end{aligned}$$

$$b_{k_1,k_2,k_3} = (-1)^{k_1+k_2+k_3}(a_1 + k_1)\frac{2 + a_1 + a_2}{1 + a_1 + a_2 - k_2}\,\frac{3 + a_1 + a_2 + a_3}{3 + a_1 + a_2 + a_3 - k_3}$$

where $H^s = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_s$, and a basis in terms of simple root vectors only is used. This singular vector is nontrivial for $a_1 \neq 0$ and must be eliminated to obtain a UIR. Below $d < d_1$ this vector is not singular but has negative norm and thus there is no unitarity for $a_1 \neq 0$. On the other hand for $a_1 = 0$ and any d vector (4.1) is descendant of the compact root singular vector $X_1^+ v_0$ which is already factored out for $a_1 = 0$.

Thus, below we discuss only the cases with $a_1 = 0$, when we have unitarity for $d > d_{12} = \frac{5}{2} + \frac{1}{2}(a_2 + a_3)$. Then at the next reducibility point $d = d_{12}$, we have a singular vector corresponding to the root $\delta_1 + \delta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$ which is given by

(4.2)

$$\begin{aligned}
v^1_{\delta_1+\delta_2} = \frac{1}{2 + 2a_2 + a_3} \times\\
\Big(-\tfrac{1}{2}(Y_4Y_3X_3^+(X_2^+)^2X_1^+) - \tfrac{1}{4}(Y_4^2(X_3^+)^2(X_2^+)^2X_1^+) + (Y_4^2X_3^+X_{23}^+X_2^+X_1^+)a_2\\
- 2(Y_4Y_2X_{23}^+X_1^+)a_2(a_2+1) - (Y_4^2X_{23}^+X_{23}^+X_1^+)a_2(a_2+1)\\
- (Y_4Y_3X_{23}^+X_2^+X_1^+)(a_3+2) - 2(Y_3Y_2X_2^+X_1^+)(a_2+a_3+1)(a_2+a_3+2)
\end{aligned}$$

$$
\begin{aligned}
&-(Y_3^2(X_2^+)^2X_1^+)(a_2+a_3+1)(a_2+a_3+2)\\
&-4(Y_2^2X_1^+)a_2(a_2+1)(a_2+a_3+1)(a_2+a_3+2)\\
&+(Y_{23}X_2^+X_1^+)(2a_2+1)(a_2+a_3+1)(a_2+a_3+2)+(Y_4Y_2X_3^+X_2^+X_1^+)(2a_2+a_3+2)\\
&+\tfrac{1}{4}(Y_{34}X_3^+(X_2^+)^2X_1^+)2a_2+2a_3+3)+(Y_{24}X_{23}^+X_1^+)a_2(a_2+1)(2a_2+2a_3+3)\\
&+\tfrac{1}{2}(Y_{34}X_{23}^+X_2^+X_1^+)(a_3-2a_2(a_2+a_3+1)+2)\\
&-\tfrac{1}{2}(Y_{24}X_3^+X_2^+X_1^+)(a_3+2a_2(a_2+a_3+2)+2)\\
&+(a_2+1)(a_2+a_3+2)\\
&\quad\times\big(2(Y_4Y_3X_{13}^+X_2^+)-(Y_{34}X_{13}^+X_2^+)-2(Y_4Y_3X_{23}^+X_{12}^+)+(Y_{34}X_{23}^+X_{12}^+)\\
&\qquad+2(Y_4Y_2X_3^+X_{12}^+)-(Y_{24}X_3^+X_{12}^+)-2(Y_4Y_1X_3^+X_2^+)+(Y_{14}X_3^+X_2^+)\big)\\
&+a_2(a_2+1)(a_2+a_3+2)\big(-4(Y_4Y_2X_{13}^+)+2(Y_{24}X_{13}^+)+4(Y_4Y_1X_{23}^+)-2(Y_{14}X_{23}^+)\big)\\
&+(a_2+1)(a_2+a_3+1)(a_2+a_3+2)\times\\
&\big(-4(Y_3Y_2X_{12}^+)+2(Y_{23}X_{12}^+)+4(Y_3Y_1X_2^+)-2(Y_{13}X_2^+)-8(Y_2Y_1)a_2+4a_2Y_{12}\big)\Big)v_0
\end{aligned}
$$

where the root vector X_{jk}^+ corresponds to the compact root $\delta_j-\delta_{k+1}=\alpha_j+\alpha_{j+1}+\cdots+\alpha_k$, Y_k corresponds to the odd (noncompact) root $\delta_k=\alpha_k+\alpha_{k+1}+\cdots+\alpha_n$, (thus $Y_4\equiv X_4^+$), Y_{jk} corresponds to the even noncompact root $\delta_j+\delta_k$. In (4.2) it is more convenient to use a PBW type of basis with the compact roots $X_{\dots}^+$ to the right of the noncompact roots $Y_{\dots}$. The norm of (4.2) is

$$
\begin{aligned}
64a_2(a_2+1)^2(a_2+2)(a_2+a_3+1)(a_2+a_3+2)^2(a_2+a_3+3)&\\
\times(-2d+a_2+a_3+4)(-2d+a_2+a_3+5)/(2a_2+a_3+2)^2.&
\end{aligned}
$$

For $d=d_{12}$, $a_1=0$, $a_2\neq 0$ the singular vector (4.2) is nontrivial and gives rise to a invariant subspace which must be factored out for unitarity. For $d<\frac{5}{2}+\frac{1}{2}(a_2+a_3)$, the vector (4.2) is not singular, but has negative norm and there is no unitarity for $a_2\neq 0$, except at the isolated unitary point $d=2+\frac{1}{2}(a_2+a_3)=d_2>d_{13}$ where vector (4.2) has zero norm and can not spoil the unitarity. For that value of d there is a singular vector $v^1_{\delta_2}$ of weight $\delta_2=\alpha_2+\alpha_3+\alpha_4$ [**5, 15**]:

$$
\begin{aligned}
v^1_{\delta_2}=\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}b_{k_1,k_2}&(X_2^+)^{1-k_1}(X_3^+)^{1-k_2}\\
&\times X_4^+(X_3^+)^{k_2}(X_2^+)^{k_1}v_0\equiv\mathcal{P}^{1,\delta_2}v_0,\\
b_{k_1,k_2}=(-1)^{k_1+k_2}&\frac{a_2+k_1}{1+a_2+a_3-k_2}
\end{aligned}
$$

which has to be factored out for unitarity for $a_2\neq 0$, while for $a_2=0$ it is descendant of the compact vector $X_2^+v_0$.

Overall no further unitarity is possible for $a_2\neq 0$, thus below we consider only the cases $a_1=a_2=0$. Then the singular vectors above are descendants of compact root singular vectors $X_1^+v_0$ and $X_2^+v_0$, thus, there is no obstacle for unitarity for $d>2+\frac{1}{2}a_3=d_2=d_{13}$ (for $a_1=a_2=0$). The next reducibility point is $d=d_{13}=d_2$.

The singular vector for $d = d_{13}$ and $m = 1$ has weight $\delta_1 + \delta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$:

$$\begin{aligned}
v^1_{\delta_1+\delta_3} = \\
\Big(& -4a_1(Y_4Y_3X_3^+X_{12}^+) - 2a_1(Y_4^2(X_3^+)^2X_{12}^+) - 2(a_1+a_2+1)(Y_4^2X_3^+X_{23}^+X_1^+) \\
& + 4a_1(a_1+a_2+1)(Y_4^2X_3^+X_{13}^+) + 4(a_3+1)(Y_4Y_3X_{23}^+X_1^+) - 8a_1(a_3+1)(Y_4Y_3X_{13}^+) \\
& - 4(a_1+a_2+a_3+2)(Y_4Y_2X_3^+X_1^+) + 8a_1(a_1+a_2+a_3+2)(Y_4Y_1X_3^+) \\
& + 8a_3(a_1+a_2+a_3+2)(Y_3Y_2X_1^+) + 4a_3(a_1+a_2+a_3+2)(Y_3^2X_2^+X_1^+) \\
& - 8a_1a_3(a_1+a_2+a_3+2)(Y_3^2X_{12}^+) + 2(a_1(a_3-1) + a_2(a_3-1) - 2)(Y_{34}X_{23}^+X_1^+) \\
& - 4a_1(a_1(a_3-1) + a_2(a_3-1) - 2)(Y_{34}X_{13}^+) \\
& + 2(a_1+a_2+2)(a_1+a_2+a_3+2)(Y_{24}X_3^+X_1^+) \\
& - 4a_1(a_1+a_2+2)(a_1+a_2+a_3+2)(Y_{14}X_3^+) \\
& - 4(a_1+a_2+2)a_3(a_1+a_2+a_3+2)(Y_{23}X_1^+) \\
& - (a_1+a_2+2a_3+2)(Y_{34}X_3^+X_2^+X_1^+) + 2a_1(a_1+a_2+2a_3+2)(Y_{34}X_3^+X_{12}^+) \\
& + 8a_1(a_1+a_2+2)a_3(a_1+a_2+a_3+2)Y_{13} - 16a_1a_3(a_1+a_2+a_3+2)(Y_3Y_1) \\
& + 2(Y_4Y_3X_3^+X_2^+X_1^+) + Y_4^2(X_3^+)^2X_2^+X_1^+\Big)v_0.
\end{aligned}$$

For $a_1 = a_2 = 0$ it is descendant of the compact root singular vector $X_1^+v_0$. However, there is a subsingular vector

$$\begin{aligned}
v^{ss}_{2,13} = \\
\Big(& 2a_3(Y_{23}Y_1) - 2a_3(Y_{13}Y_2) + 2a_3(Y_3(Y_{12})) - 4a_3(Y_3Y_2Y_1) + 2(Y_4Y_3Y_2X_{13}^+) \\
& - Y_{34}Y_2X_{13}^+ + Y_{24}Y_3X_{13}^+ - Y_4Y_{23}X_{13}^+ - 2(Y_4Y_3Y_1X_{23}^+) + Y_{34}Y_1X_{23}^+ - Y_{14}Y_3X_{23}^+ \\
& + Y_4Y_{13}X_{23}^+ + 2(Y_4Y_2Y_1X_3^+) - Y_{24}Y_1X_3^+ + Y_{14}Y_2X_3^+ - Y_4(Y_{12})X_3^+\Big)v_0
\end{aligned}$$

with the norm $-16a_3(a_3+3)(-2d+a_3+2)(-2d+a_3+3)(-2d+a_3+4)$. This vector must be factorized in order to obtain UR at $d = d_2 = d_{13}$. But below this value, the vector $v^{ss}_{2,13}$ above has negative norm if $a_3 \neq 0$ and there is no unitarity, except at the isolated unitary point $d = \frac{3}{2} + \frac{1}{2}a_3 = d_{23} > d_{14}$. At that value of d there is a singular vector of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3$:

$$\begin{aligned}
v^1_{\delta_2+\delta_3} = \Big(& 2(a_3+1)(Y_4Y_3X_{23}^+) - 2(a_3+1)(Y_4Y_2X_3^+) + 2a_3(a_3+1)(Y_3^2X_2^+) \\
& - (a_3+1)(Y_{34}X_{23}^+) + (a_3+1)(Y_{24}X_3^+) - \tfrac{1}{2}(2a_3+1)(Y_{34}X_3^+X_2^+) \\
& -2a_3(a_3+1)Y_{23} + 4a_3(a_3+1)(Y_3Y_2) + Y_4Y_3X_3^+X_2^+ + \tfrac{1}{2}(Y_4^2(X_3^+)^2X_2^+)\Big)v_0
\end{aligned} \tag{4.3}$$

with the norm $16a_3(a_3+1)^2(a_3+2)(-2d+a_3+2)(-2d+a_3+3)$. For $a_3 \neq 0$, singular vector (4.3) should be factored for unitarity, while for $a_3 = 0$ it is descendant of the compact singular vectors.

In the same range for $a_3 \neq 0$ at $d = d_3 = 1 + \frac{1}{2}a_3$ there is a singular vector of weight $\delta_3 = \alpha_3 + \alpha_4$:

$$v^1_{\delta_3} = \sum_{k=0}^{1}(-1)^k(a_3+k)(X_3^+)^{1-k}X_4^+(X_3^+)kv_0 \equiv \mathcal{P}^{1,\delta_3}v_0 \tag{4.4}$$

which must be factored out for unitarity.

On the other hand, for $a_1 = a_2 = a_3 = 0$ all (sub)singular vectors above are descendants of the compact singular vectors $X_k^+ v_0$, $k = 1,2,3$, and there is no obstacle for unitarity for $d > \frac{3}{2} = d_{23} = d_{14}$. For $a_3 = 0$ and $d = \frac{3}{2}$ there is also a singular vector of weight $\delta_1 + \delta_4$:

$$v^1_{\delta_1+\delta_4} = \big(-4(Y_4Y_2X_1^+) - 2(Y_4^2X_{23}^+X_1^+) + 2(Y_4Y_3X_2^+X_1^+) + Y_4^2X_3^+X_2^+X_1^+ - 3(Y_{34}X_2^+X_1^+) + 6(Y_{24}X_1^+)\big)v_0$$

but it is also descendant of compact singular vectors. Finally, for $d = \frac{3}{2}$ there is a subsingular vector of weight $\delta_1 + \delta_2 + \delta_3 + \delta_4$:

$$v^{ss}_{\delta_1+\delta_2+\delta_3+\delta_4} = \sum_{i,j,k,\ell=1}^{4} \epsilon^{ijk\ell} Y_iY_jY_kY_\ell\, v_0 \tag{4.5}$$

where $\epsilon^{ijk\ell}$ is the totally antisymmetric symbol so that $\epsilon^{1234} = 1$. The norm of the vector (4.5) is $2304(-1+d)d(-3+2d)(-1+2d)$. Thus, for $\frac{3}{2} > d > 1$ there is no unitarity since then the vector (4.5) has negative norm. In all cases there will be no unitarity for $d \leqslant 1$, except possibly when $a_1 = a_2 = a_3 = 0$ to which we restrict below. At $d = d_3 = d_{24} = 1$ there are the singular vector (4.4) and the singular vector of weight $\delta_2 + \delta_4 = \alpha_2 + \alpha_3 + 2\alpha_4$:

$$v^1_{\delta_2+\delta_4} = \big(-(a_3+2)(Y_{34}X_2^+) + 2(Y_4Y_3X_2^+) + Y_4^2X_3^+X_2^+\big)v_0$$

both of which are descendants of compact singular vectors. At $d = d_3 = d_{24} = 1$, there is also a subsingular vector

$$v^{ss}_{\delta_2+\delta_3+\delta_4} = (Y_2Y_3Y_4 - Y_4Y_3Y_2) = \frac{1}{3}\sum_{i,j,k=2}^{4} \epsilon^{ijk} Y_iY_jY_k\, v_0$$

of the norm $144d(d-1)(2d-1)$. It is not an obstacle for unitarity for $d = 1$, but for $d < 1$. Thus, there is no unitarity for $d < 1$ except at the isolated unitary point $d = \frac{1}{2} = d_{34}$. At that point all (sub)singular vectors above are descendants of compact singular vectors. Yet there is the singular vector

$$v^1_{\delta_3+\delta_4} = \big(\tfrac{1}{2}Y_4^2X_3^+ - 2Y_3Y_4 + Y_{34}\big)\, v_0$$

with the norm $8d(2d-1)$. It is not an obstacle for unitarity for $d = \frac{1}{2}$, but for $d < \frac{1}{2}$. Thus, there is no unitarity for $d < \frac{1}{2}$ except at the isolated point $d = d_4 = 0 = a_1 = a_2 = a_3$ where we have the trivial one-dimensional UIR since all possible states are descendants of factored out singular vectors. □

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