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BI-LIPSCHITZITY OF QUASICONFORMAL HARMONIC MAPPINGS IN n-DIMENSIONAL SPACE WITH RESPECT TO k-METRIC

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Abstract. We explore conditions which guarantee bi-Lipschitzity of harmonic quasiconformal maps with respect to k-metric. We prove that harmonic k-quasiconformal maps with nonzero Jacobian between any two domains in $\mathbb{R}^n$ are bi-Lipschitz with respect to k-metric, and prove the converse too.

1. Introduction

We prove results about bi-Lipschitzity of harmonic k-qc mappings $f\colon D_1 \to D_2$, where D_1 and D_2 are arbitrary proper subdomains of $\mathbb{R}^n$, with respect to k-metric.

Similar problems have been studied at the Belgrade Seminar for Complex Analysis. In [**1**], Mateljević proved such a result in n-dimensional space, but only in the case when both D_1 and D_2 are the upper half space in $\mathbb{R}^n$. Also, in the same paper, Proposition 5 gives an estimate in dimension 2 for minimal and maximal moduli of directional derivative at a point, in terms of distance to the boundary, for arbitrary codomain. As a corollary, he proved that every harmonic quasiconformal map of the unit disk is a quasi-isometry with respect to hyperbolic distances. He posed a question if analogue of Proposition 5 holds in higher dimensions. In the case $n = 2$, Manojlović proved in [**2**] that, when D_1 and D_2 are arbitrary domains in the plane, then harmonic quasiconformal maps are bi-Lipschitz with respect to k-metric.

Note that the Lipischitz condition for maps between domains in $\mathbb{R}^n$ was obtained by Mateljević and Vourinen [**6**]. Here a different proof, based on results of Božin and Mateljević [**3**], is given.

Let $B^n(x,r) = \{z \in \mathbb{R}^n : \|z-x\| < r\}$, $\mathbb{S}^{n-1}(x,r) = \partial B^n(x,r)$, and let $B^n, \mathbb{S}^{n-1}$ stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. For a domain $G \subset \mathbb{R}^n$ let $\rho\colon G \to (0,\infty)$ be a continuous function. We say that ρ is a

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metric density if, for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x)ds,$$

exists. In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density $d_\rho\colon G \times G \to [0,\infty)$ defines a metric as follows. For $a, b \in G$, let $d_\rho(a,b) = \inf_\gamma l_\rho(\gamma)$, where the infimum is taken over all locally rectifiable curves in G joining a and b. For a fixed $a, b \in G$, suppose that there exists a d_ρ-length minimizing curve $\gamma\colon [0,1] \to G$ with $\gamma(0) = a$, $\gamma(1) = b$ such that $d_\rho(a,b) = l_\rho(\gamma|[0,t]) + l_\rho(\gamma|[t,1])$, for all $t \in [0,1]$. Then γ is called a geodesic segment joining a and b.

In dimensions $n \geqslant 3$, we do not have a Riemann mapping theorem, and it is natural to look for counterparts of the hyperbolic metric. So-called hyperbolic type metrics have been the subject of many recent papers. One of the most important of these metrics is the quasihyperbolic metric k_G of a domain $G \subset \mathbb{R}$. The quasihyperbolic k-metric $k = k_G$ of G is a particular case of the geodesic metric d_ρ when $\rho(x) = 1/d(x,\partial G)$ [**4, 5**], where $d(x,\partial G)$ is the distance from point x to the boundary of G.

We will consider Euclidean harmonic maps, also called harmonic maps in this paper, i.e., those with zero Laplacian of each coordinate function. Also, we will deal with quasiconformal maps. For a domain D in $\mathbb{R}^n$, a map $f\colon D \to \mathbb{R}^n$ is K-quasiconformal if it is a homeomorphism of D to $f(D)$, and if f belongs to the Sobolev space $W^n_{1,\mathrm{loc}}(D)$ and there exists K, $1 \leqslant K < \infty$, such that $\|Df(x)\|^n \leqslant KJ_f(x)$ a.e. on D, where $\|Df(x)\|$ denote the operator norm of the Jacobian matrix of f at x.

Our main result is that harmonic k-quasiconformal mappings which do not have zero of Jacobian $f\colon D_1 \to D_2$ are bi-Lipschitz. This result is based on two Theorems from [**3**]. We also prove that every harmonic mappings $f\colon D_1 \to D_2$ which is bi-Lipschitz with respect to k-metric is quasiconformal, where D_1 and D_2 are domains in $\mathbb{R}^n$.

2. Background

In this section we give some background results which will be used in our main proofs.

THEOREM 2.1. [**6**] *Let D_1 and D_2 be two domains in R^n and let ρ_1 and ρ_2 be two densities, $ds = \rho_1(z)|dz|$, and $ds = \rho_2(w)|dw|$ where $|dz|$, and $|dw|$ stand for Euclidean metric, and $\Lambda_f(z), \lambda_f(z)$ are respectively the maximum and the minimum of modulus of eigenvalues of the Jacobian matrix at z, and suppose that $f\colon D_1 \to D_2$ is a C^1 quasiconformal mapping*

(A) *If there is a positive constant c_1 such that at every point z, $\rho_2(f(z))\Lambda_f(z) \leqslant c_1\rho_1(z)$, $z \in D_1$, then $d_{\rho_2}(f(z_1), f(z_2)) \leqslant c_1 d_{\rho_1}(z_1, z_2)$.*

(B) *If $f(D_1) = D_2$, and there is a positive constant c_2 such that at every point z, $\lambda_f(z)\rho_2(f(z)) \geqslant c_2\rho_1(z)$, $z \in D_1$, then $d_{\rho_2}(f(z_1), f(z_2)) \geqslant c_2 d_{\rho_1}(z_1, z_2)$, $z_1, z_2 \in D_1$.*

For convenience, we give a proof of this known result.

PROOF. Part (A). Suppose that γ is geodesic with parametrization

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \dots, \gamma_n(t)),$$

and derivative

$$\gamma'(t) = (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t), \dots, \gamma_n'(t)).$$

Let $\gamma_*(t) = f(\gamma(t))$; then $\gamma_*'(t) = D_f(\gamma(t))\gamma'(t)$, and $\|\gamma_*'(t)\| \leqslant \Lambda_f \|\gamma'(t))\|$. We have

$$d_{\rho_1}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \rho_1(z)|dz| \leqslant \int_0^1 \rho_1(\gamma(t))\|\gamma'(t)\|dt.$$

Letting $w_1 = f(z_1)$ and $w_2 = f(z_2)$, we can write

$$d_{\rho_2}(w_1, w_2) = \inf_{\gamma_*} \int_{\gamma_*} \rho_2(w)|dw| \leqslant \int_0^1 \rho_2(\gamma_*(t))\|\gamma_*'(t))\|dt.$$

Using change of variable,

$$d_{\rho_2}(w_1, w_2) \leqslant \int_0^1 \rho_2(f(\gamma(t)))\Lambda_f\|\gamma'(t))\|dt,$$

and by (A), we get

$$d_{\rho_2}(w_1, w_2) \leqslant c_1 \int_0^1 \rho_1(\gamma(t))||\gamma'(t)||dt \leqslant c_1 \int_{\gamma} \rho_1(z)|dz| \leqslant c_1 d_{\rho_1}(z_1, z_2).$$

Then $d_{\rho_2}(f(z_1), f(z_2)) \leqslant c_1 d_{\rho_1}(z_1, z_2)$.

Part (B). Let g be an inverse function of f. We have $f(z_1) = w_1 \to z_1 = g(w_1)$ and $f(z_2) = w_2 \to z_2 = g(w_2)$.

Let $\gamma(t) = g(\gamma_*(t))$; then $\gamma'(t) = D_g(\gamma_*(t))\gamma_*'(t)$, and thus $\|\gamma'(t)\| \leqslant \Lambda_g\|\gamma_*(t)\|$. Here $\Lambda_g = \frac{1}{\lambda_f}$, because $D_g(w) = [D_f(z)]^{-1}$. It follows that

$$d_{\rho_1}(g(w_1), g(w_2)) = \inf_{\gamma} \int_{\gamma} \rho_1(z)|dz| \leqslant \int_0^1 \rho_1(\gamma(t))\|D_g(\gamma_*(t))\gamma_*(t))\|dt.$$

By assumption in (B), we get

$$\begin{aligned} d_{\rho_1}(g(w_1), g(w_2)) &\leqslant \int_0^1 \rho_1(g(\gamma_*(t)))\frac{1}{\lambda_f}\|\gamma_*'(t)\|dt \leqslant \frac{1}{c_2}\int_0^1 \rho_2(\gamma_*(t))\|\gamma_*'(t)\|dt \\ &\leqslant \frac{1}{c_2}\int_{\gamma_*} \rho_2(w)|dw| \leqslant \frac{1}{c_2} d_{\rho_2}(w_1, w_2). \end{aligned} \tag{2.1}$$

Then $c_2 d_{\rho_1}(z_1, z_2) \leqslant d_{\rho_2}(f(z_1), f(z_2))$. □

In the following two theorems from [**3**, Theorems 4.1 and 4.2], nonzero Jacobian families are defined as closed families of harmonic maps with nonzero Jacobians (see [**3**]).

THEOREM 2.2. *For every nonzero Jacobian closed family of k-quasiconformal harmonic maps, there is a constant $c > 0$, such that if $f\colon B^n \to \mathbb{R}^n$ is from the family, $d(0, \partial f(B^n)) \geqslant 1$, and $f(0) = 0$, then $J_f(0) \geqslant c$.*

Theorem 2.3. *There is a constant $c > 0$, depending only on k, such that if $f\colon D_1 \to \mathbb{R}^n$ is k-quasiconformal harmonic map, $d(0, \partial f(B^n)) \leqslant 1$, and $f(0) = 0$, then $J_f(0) \leqslant c$.*

We will also need the following well known theorem for qc maps, called local quasi-symmetry (see, for instance, [**4**]).

Theorem 2.4. *If $f\colon B^n \to \mathbb{R}^n$ is a K-quasiconformal map and $\hat{f}\colon \bar{B}^n \to \mathbb{R}^n$ its continuous extension, then for any two points $a, b \in \mathbb{S}^{n-1}$*

$$\frac{d(\hat{f}(0), \hat{f}(a))}{d(\hat{f}(0), \hat{f}(b))} \leqslant c(k, n),$$

for some constant $c(K, n)$ independent of f.

3. Bi-Lipschitzity with respect to *k*-metric

Theorem 3.1. *Suppose that $f\colon D_1 \to D_2$, where $D_1, D_2 \subsetneq \mathbb{R}^n$, is a harmonic quasi-conformal mapping, and that f belongs to a nonzero Jacobian family of harmonic maps, then the following holds for some constant C*

$$\frac{1}{C} J^{\frac{1}{n}}(z) \leqslant \frac{d(f(z), \partial D_2)}{d(z, \partial D_1)} \leqslant C J^{\frac{1}{n}}(z).$$

Proof. Let z_0 be a point in D_1, $r_1 = d(z_0, \partial D_1)$, $r_2 = d(f(z_0), \partial D_2)$.

Let $B(z_0, r_1)$ be the n dimensional ball centered at z_0 of radius r_1 and let $D_3 = f(B(z_0, r_1))$. Also assume that f is K-quasiconformal.

Define $\hat{f}\colon B^n \to \mathbb{R}^n$ by $\hat{f}(z) = \frac{1}{r_2}(f(z_0 + r_1 z) - f(z_0))$. Note that since $D_3 \subseteq D_2$, we have $r_2 = d(f(z_0), \partial D_2) \geqslant d(f(z_0), \partial D_3)$, and hence $d(0, \partial \hat{f}(B^n)) \leqslant 1$. We have

$$J_f(z_0) = \det \begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \dots & \partial f_2/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \dots & \partial f_n/\partial x_n \end{bmatrix}$$

$$J_{\hat{f}}(0) = \Big(\frac{1}{r_2}\Big)^n \det \begin{bmatrix} r_1\partial f_1/\partial x_1 & \dots & r_1\partial f_1/\partial x_n \\ r_1\partial f_2/\partial x_1 & \dots & r_1\partial f_2/\partial x_n \\ \vdots & \ddots & \vdots \\ r_1\partial f_n/\partial x_1 & \dots & r_1\partial f_n/\partial x_n \end{bmatrix} = \Big(\frac{r_1}{r_2}\Big)^n J_f(z_0).$$

Since, $d(0, \partial \hat{f}(B^n)) \leqslant 1$, by Theorem 2.3 $J_{\hat{f}}(0) \leqslant c$, so

$$\frac{r_1^n}{r_2^n} J_f(z_0) \leqslant c$$

$$\frac{1}{c_1} J_f(z_0)^{\frac{1}{n}} \leqslant \frac{r_2}{r_1} = \frac{d(f(z), \partial D_2)}{d(z, \partial D_1)}$$

where $c_1 = c^{1/n}$.

Note that, by Theorem 2.4, for any point $w \in \partial D_3$,

$$d(f(z_0), \partial D_3) \geqslant \frac{1}{c(K,n)} d(f(z_0), w).$$

So, since by our construction there is a point w which belongs to both ∂D_2 and ∂D_3, we have

$$d(f(z_0), \partial D_3) \geqslant \frac{1}{c(K,n)} d(f(z_0), \partial D_2)$$

and so we have

$$d(f(z_0), \partial D_3) \geqslant \frac{r_2}{c(K,n)}.$$

Now again, define $\hat{f}\colon B^n \to \mathbb{R}^n$ by $\hat{f}(z) = \frac{c(K,n)}{r_2}(f(z_0 + r_1 z) - f(z_0))$ for $z \in B^n$. Note that $d(0, \partial \hat{f}(B^n)) \geqslant 1$. We have

$$J_f(z_0) = \det \begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \dots & \partial f_2/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \dots & \partial f_n/\partial x_n \end{bmatrix}$$

$$J_{\hat{f}}(0) = \Big(\frac{c(K,n)}{r_2}\Big)^n = \det \begin{bmatrix} r_1\partial f_1/\partial x_1 & \dots & r_1\partial f_1/\partial x_n \\ r_1\partial f_2/\partial x_1 & \dots & r_1\partial f_2/\partial x_n \\ \vdots & \ddots & \vdots \\ r_1\partial f_n/\partial x_1 & \dots & r_1\partial f_n/\partial x_n \end{bmatrix}$$

$$J_{\hat{f}}(0) = c(K,n)^n \Big(\frac{r_1}{r_2}\Big)^n J_f(z_0).$$

By Theorem 2.2, since $d(0, \partial \hat{f}(B^n)) \geqslant 1$, we have $J_{\hat{f}}(0) \geqslant c$, so

$$\frac{r_1^n}{r_2^n} J_f(z_0) \geqslant \frac{c}{c(K,n)^n}.$$

Then

$$c_2 J_f(z_0)^{\frac{1}{n}} \geqslant \frac{r_2}{r_1} = \frac{d(f(z_0), \partial D_2)}{d(z_0, \partial D_1)}$$

where $c_2 = \frac{c(K,n)}{c^{1/n}}$. Finally, set $C = \max(c_1, c_2)$. □

A consequence of Theorem 3.1 is the following:

THEOREM 3.2. *Suppose that $f\colon D_1 \to D_2$, where $D_1, D_2 \subsetneq \mathbb{R}^n$, is a harmonic K-quasiconformal mapping, and that f belongs to a nonzero Jacobian family of harmonic maps. Then f is bi-Lipschitz with respect to k-metric.*

PROOF. From the quasiconformality condition and using that our map is C^1, we have a constant k such that at every point z

$$\Lambda_f(z) \leqslant k J^{\frac{1}{n}}(z), \quad \lambda_f(z) \geqslant \frac{1}{k} J^{\frac{1}{n}}(z)$$

where Λ_f and λ_f are the greatest and smallest moduli of eigenvalues of the Jacobian matrix. By Theorem 3.1, there is a constant C such that

$$\frac{1}{C}J^{\frac{1}{n}}(z) \leqslant \frac{d(f(z),\partial D_2)}{d(z,\partial D_1)} \leqslant CJ^{\frac{1}{n}}(z).$$

The metric densities for k metrics are

$$\rho_1(z) = \frac{1}{d(z,\partial D_1)}, \quad \rho_2(z) = \frac{1}{d(w,\partial D_2)},$$

and so we have

$$\begin{aligned}
\rho_2(f(z))\Lambda_f(z) &= \frac{1}{d(f(z),\partial D_2)}\Lambda_f(z) \leqslant \frac{1}{d(f(z),\partial D_2)}kJ^{\frac{1}{n}}(z)\\
&\leqslant \frac{1}{d(f(z),\partial D_2)}kC\frac{d(f(z),\partial D_2)}{d(z,\partial D_1)} = kC\frac{1}{d(z,\partial D_1)} = kC\rho_1(z),\\
\rho_2(f(z))\lambda_f(z) &= \frac{1}{d(f(z),\partial D_2)}\lambda_f(z) \geqslant \frac{1}{d(f(z),\partial D_2)}\frac{1}{k}J^{\frac{1}{n}}(z)\\
&\geqslant \frac{1}{d(f(z),\partial D_2)}\frac{1}{kC}\frac{d(f(z),\partial D_2)}{d(z,\partial D_1)} = \frac{1}{kC}\frac{1}{d(z,\partial D_1)} = \frac{1}{kC}\rho_1(z).
\end{aligned}$$

Then by Theorem 2.1 the map f is bi-Lipschitz with respect to k-metric. □

Theorem 3.3. *If a bijective harmonic map $f\colon D_1 \to D_2$, where $D_1, D_2 \subset \mathbb{R}^n$, is bi-Lipschitz with respect to k-metric, then it is a quasiconformal mapping.*

Proof. Note that, by elliptic regularity, f is a C^1 map. Let $x, x+\Delta x$ be two points in D_1 where $\|\Delta x\| \to 0$, and suppose that the Jacobian matrix $D_f(x)$ maps unit sphere to ellipsoid with minimal and maximal axes equal to λ_f and Λ_f respectively, and let ρ_1, and ρ_2 be metric density functions in D_1 and D_2 respectively. Assume that

$$\frac{1}{c}d_{\rho_2}(f(x),f(y)) \leqslant d_{\rho_1}(x,y) \leqslant cd_{\rho_2}(f(x),f(y)).$$

We prove that $\frac{\Lambda_f}{\lambda_f} \leqslant c^2$, wherefrom quasiconformality follows. As $\Delta x \to 0$, we have

$$\begin{aligned}
d_{\rho_1}(x,x+\Delta x) &= \rho_1(x)\|\Delta x\|(1+o(1)),\\
d_{\rho_2}(f(x),f(x+\Delta x)) &= \rho_2(x)\|Df\Delta x\|(1+o(1)).
\end{aligned}$$

Note that

$$\Lambda_f = \sup_{e,\|e\|=1}\|D_f(x)e\| \quad \text{and} \quad \lambda_f = \inf_{e,\|e\|=1}\|D_f(x)e\|.$$

Suppose supremum is achieved for vector e_1, and infimum is achieved for e_2 (since matrix multiplication is continuous, and unit sphere is compact, there have to be such vectors e_1 and e_2).

We are going to consider $\Delta x = te_1$, $t \to 0$ and $\Delta x = te_2$, $t \to 0$. Putting $\Delta x = te_1$, $t \to 0$ we have

$$\begin{aligned}
d_{\rho_1}(x,x+te_1) &= \rho_1(x)t(1+o(1)) \text{ as } t \to 0\\
d_{\rho_2}(f(x),f(x+te_1)) &= \rho_2(f(x))\Lambda_f t(1+o(1)) \text{ as } t \to 0.
\end{aligned}$$

Putting $\Delta x = te_2$, $t \to 0$, we have

$$d_{\rho_1}(x, x + te_2) = \rho_1(x)t(1 + o(1)) \text{ as } t \to 0$$
$$d_{\rho_2}(f(x), f(x + te_2)) = \rho_2(f(x))\lambda_f t(1 + o(1)) \text{ as } t \to 0.$$

Using the bi-Lipschitz condition, we get

$$\frac{1}{c}\rho_2(f(x))t\lambda_f(1 + o(1)) \leqslant \rho_1(x)t(1 + o(1)) \leqslant c\rho_2(f(x))t\lambda_f(1 + o(1)),$$
$$\frac{1}{c}\rho_2(f(x))t\Lambda_f(1 + o(1)) \leqslant \rho_1(x)t(1 + o(1)) \leqslant c\rho_2(f(x))t\Lambda_f(1 + o(1)).$$

Letting t tend to zero and dividing by t, we get

$$\frac{1}{c}\rho_2(f(x))\lambda_f \leqslant \rho_1(x) \leqslant c\rho_2(f(x))\lambda_f,$$
$$\frac{1}{c}\rho_2(f(x))\Lambda_f \leqslant \rho_1(x) \leqslant c\rho_2(f(x))\Lambda_f.$$

So $\frac{1}{c}\rho_2(f(x))\Lambda_f \leqslant c\rho_2(f(x))\lambda_f$, wherefrom $\frac{\Lambda_f}{\lambda_f} \leqslant c^2$. □

Note that the proof of previous theorem assumes only that ρ_1 and ρ_2 are positive continous and that f is C^1. So in fact we have proved

THEOREM 3.4. *Suppose that ρ_1, ρ_2 are positive continuous metric densities defined in $\mathbb{R}^n$ domains D_1 and D_2 respectively, and $f\colon D_1 \to D_2$ is C^1 bijection which is bi-Lipschitz with respect metrics d_{ρ_1} and d_{ρ_2}. Then f is a quasiconformal mapping.*

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