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RINGS IN WHICH THE POWER OF EVERY ELEMENT IS THE SUM OF AN IDEMPOTENT AND A UNIT

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Abstract. A ring R is uniquely π-clean if the power of every element can be uniquely written as the sum of an idempotent and a unit. We prove that a ring R is uniquely π-clean if and only if for any $a \in R$, there exists an integer m and a central idempotent $e \in R$ such that $a^m - e \in J(R)$, if and only if R is Abelian; idempotents lift modulo $J(R)$; and R/P is torsion for all prime ideals $P \supseteq J(R)$. Finally, we completely determine when a uniquely π-clean ring has nil Jacobson radical.

1. Introduction

An attractive problem in ring theory is to determine when a ring is generated additively by idempotents and units. An element of a ring is uniquely clean if it can be uniquely written as the sum of an idempotent and a unit. A ring R is uniquely clean if every element in R is uniquely clean. Many results on such rings can be found in [**3, 5, 6**]. Following Zhou [**6**], a ring R is uniquely π-clean if some power of every element in R is uniquely clean. This is a natural generalization of uniquely clean rings. The motivation of this paper is to develop explicit characterizations of such rings.

In Section 2, we explore the structures of uniquely π-clean rings, and prove that a ring R is uniquely π-clean if and only if for any $a \in R$, there exists an $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in J(R)$, if and only if for any $a \in R$, there exists an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J(R)$, and $J(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ f or all } m \in \mathbb{N}\}$. This extends Lee and Zhou's theorem as well.

In Section 3, we characterize uniquely π-cleanness by means of certain prime ideals. It is shown that a ring R is uniquely π-clean if and only if R is Abelian; every idempotent lifts modulo $J(R)$; and R/P is torsion for all prime ideals P containing

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the Jacobson radical $J(R)$. Furthermore, we consider a type of radical-like ideal $J^*(R)$, and characterize uniquely π-clean ring R by using such a special one.

Finally, we completely determine when a uniquely π-clean ring has nil Jacobson radical. Recall that an element $a \in R$ is uniquely nil-clean provided that there exists a unique idempotent $e \in R$ such that $a - e \in N(R)$ [**3**]. We say that $a \in R$ is uniquely π-nil-clean provided that $a^n \in R$ is uniquely nil-clean for some $n \in \mathbb{N}$. A ring R is uniquely π-nil-clean if every element in R is uniquely π-nil-clean. A ring R is periodic if for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. In the last section, we characterize uniquely π-nil-clean rings. We prove that a ring R is uniquely π-nil-clean if and only if R is uniquely π-clean and $J(R)$ is nil, if and only if R is an Abelian periodic ring, if and only if for any $a \in R$ there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely nil clean, if and only if for any $a \in R$, there exists some $m \in \mathbb{N}$ and a unique idempotent $e \in R$ such that $a^m - e \in P(R)$, where $P(R)$ is the prime radical of R. Here, an element $a \in R$ is uniquely nil clean if there exists a unique idempotent $e \in R$ such that $a - e \in R$ is nilpotent [**3, 5**].

Throughout the paper, all rings are associative with an identity. We use $J(R)$ and $P(R)$ to denote the Jacobson radical and prime radical of a ring R. $N(R)$ stands for the set of all nilpotent elements in R.

2. Structure Theorems

The aim of this is to explore the structures of uniquely π-clean rings. Recall that a ring R is an exchange ring if for any $a \in R$ there exists an idempotent $e \in aR$ such that $1 - e \in (1-a)R$. A ring R is an exchange ring if and only if, for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \big(\bigoplus_{i \in I} A_i'\big)$. The class of exchange rings is very large. For instances, regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero, etc. We begin with

Lemma 2.1. *Every uniquely π-clean ring is an Abelian exchange ring.*

Proof. Let R be uniquely π-clean, let $e \in R$ be an idempotent, and let $r \in R$. Then $x := 1 - \big(e + er(1-e)\big) \in R$ is an idempotent. By hypothesis, $x \in R$ is uniquely clean. One easily checks that

$$x = e + \big(1 - 2e - er(1-e)\big) = \big(e + er(1-e)\big) + \big(1 - 2(e + er(1-e))\big).$$

Further,

$$e = e^2 \in R, \quad \big(1 - 2e - er(1-e)\big)^{-1} = \big(1 - er(1-e)\big)(1-2e),$$
$$\big(e + er(1-e)\big) = \big(e + er(1-e)\big)^2, \quad \big(1 - 2(e + er(1-e))\big)^2 = 1.$$

By the uniqueness, we get $e = e + er(1-e)$, and then $er = ere$. Likewise, $re = ere$. Thus, $er = re$, and therefore R is Abelian.

For any $a \in R$, then we can find some $m \in \mathbb{N}$ such that $a^m \in R$ is clean. Write $a^m = f + v$, where $f = f^2, v \in U(R)$. Then $a^m - f^m = v$, and so $a - f \in U(R)$.

This implies that R is strongly clean. In view of [**9**, Theorem 30.2], every clean ring is an exchange ring. Therefore R is an exchange ring, as asserted. $\square$

A ring R is strongly clean if for any $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in U(R)$ and $ea = ae$. As a consequence of Lemma 2.1, every uniquely π-clean ring is strongly clean. A ring R is uniquely clean provided that every element in R can be uniquely written as the sum of an idempotent and a unit. It is easy to verify that $\mathbb{Z}/3\mathbb{Z}$ is not uniquely clean as $2 = 0 + 2 = 1 + 1$, while $\mathbb{Z}/3\mathbb{Z}$ is uniquely π-clean. Let $R = \bigoplus_{p \text{ is prime}} \mathbb{Z}/(p+1)\mathbb{Z}$. Then R is strongly clean. For any $1 \leqslant m \leqslant \left[\log_2 p\right], 2^m \in \mathbb{Z}/(p+1)\mathbb{Z}$ is not uniquely clean. Thus, R is not uniquely π-clean. Therefore, we conclude that $\{$uniquely clean rings$\} \subsetneq \{$uniquely π-clean rings$\} \subsetneq \{$strongly clean rings$\}$.

THEOREM 2.1. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *R is Abelian;*
(2) *Every idempotent lifts modulo $J(R)$;*
(3) *$R/J(R)$ is uniquely π-clean.*

PROOF. Suppose R is uniquely π-clean. In view of Lemma 2.1, R is an Abelian exchange ring. This proves (1) and (2), in terms of [**9**, Theorem 30.2]. For any $\bar{a} \in R/J(R)$, then $a \in R$ is uniquely π-clean. Thus, we have some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. This implies that $a^n = e + u$, $e = e^2 \in R$, $u \in U(R)$. Hence, $\bar{a}^n = \overline{e} + \overline{u}$. Write $\bar{a}^n = \overline{f} + \overline{v}$, $\overline{f} = \overline{f}^2 \in R/J(R)$, $\overline{v} \in U\big(R/J(R)\big)$. Clearly, every unit lifts modulo $J(R)$. So we may assume that $f = f^2 \in R, v \in U(R)$. As a result, there exists some $r \in J(R)$ such that $a^n = e + u = f + (v + r)$. By the uniqueness, we get $e = f$. Therefore $R/J(R)$ is uniquely π-clean.

Conversely, assume that (1)–(3) hold. For any $a \in R$, we have $\bar{a} \in R/J(R)$, and so there exists some $n \in \mathbb{N}$ such that $\bar{a}^n \in R$ is uniquely clean. By hypothesis, idempotents lift modulo $J(R)$. In addition, units lift modulo $J(R)$. Thus, $a^n = e + u$, $e = e^2 \in R$, $u \in U(R)$. Write $a^n = f + v$, $f = f^2$, $v \in U(R)$. Then $\bar{a}^n = \overline{f} + \overline{v}$. By the uniqueness, we get $\overline{e} = \overline{f}$, i.e., $e - f \in J(R)$. This infers that $f(1-e) = (e-f)(e-1) \in J(R)$. As every idempotent in R is central, $f(1-e) \in R$ is an idempotent, thus, $f(1-e) = 0$. It follows that $f = fe$. Likewise, $e = ef$. Consequently, $e = f$, and therefore R is uniquely π-clean. $\square$

COROLLARY 2.1. *Every corner of a uniquely π-clean ring is uniquely π-clean.*

PROOF. Let R be uniquely π-clean, and let $e = e^2 \in R$. In light of Theorem 2.1, $e \in R$ is central. For any $eae \in eRe$, then $eae + 1 - e \in R$ is uniquely π-clean. So we have some $n \in \mathbb{N}$ such that $(eae + 1 - e)^n \in R$ is uniquely clean. Thus, $(eae + 1 - e)^n = f + u$, $f = f^2 \in R$, $u \in U(R)$, and so $(eae)^n = efe + eue$ is clean in eRe. Write $(eae)^n = g + v$, $g = g^2 \in eRe$, $v \in U(eRe)$. Then $(eae + 1 - e)^n = (eae)^n + 1 - e = g + (v + 1 - e)$, where $g = g^2 \in R$. Write $vw = wv = e$. Then $(v + 1 - e)^{-1} = w + 1 - e$, and so $v + 1 - e \in U(R)$. Thus, $g = f = ege = efe$, as required. $\square$

Lemma 2.1 shows that every uniquely π-clean ring is an Abelian exchange ring. We now exhibit an exchange-like property of such rings.

THEOREM 2.2. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *R is Abelian;*
(2) *For any $a \in R$, there exists an $n \in \mathbb{N}$ and a unique idempotent $e \in a^n R$ such that $1 - e \in (1 - a^n)R$.*

PROOF. Suppose that R is uniquely π-clean. In view of Lemma 2.1, every idempotent in R is central. For any $a \in R$, there exists some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Write $a^n = f + v$, where $f = f^2$, $v \in U(R)$. Set $g = 1 - f$. Then $g = g^2 \in R$. Obviously, we get

$$(a^n - g)v = \big(f + v - v(1 - f)v^{-1}\big)v = v^2 + fv - v + vf = a^{2n} - a^n.$$

Thus $g - a^n \in (a^n - a^{2n})R$, and so $g \in a^n R$ and $1 - g \in (1 - a^n)R$.

If there exists an idempotent $h \in a^n R$ such that $1 - h \in (1 - a^n)R$. Write $h = a^n x, xh = x$. Then $xa^n x = x$. It is easy to verify that $xa^n = x(a^n x)a^n = a^n x(xa^n) = a^n(xa^n)x = a^n x$. Write $1 - h = (1 - a^n)y$, $y(1 - h) = y$. Likewise, $y(1 - a^n) = (1 - a^n)y$. One directly checks that $\big(a^n - (1 - h)\big)^{-1} = x - y$, i.e., $a^n - (1 - h) \in U(R)$, By the uniqueness, we get $1 - h = f$. Hence, $g = 1 - f = h$, as desired.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exists an $n \in \mathbb{N}$ and a unique idempotent $e \in a^n R$ such that $1 - e \in (1 - a^n)R$. As in the preceding discussion, we get $a^n - (1 - e) \in U(R)$. Write $a^n = f + v$, where $f = f^2$, $v \in U(R)$. Set $g = 1 - f$. Then $g = g^2 \in R$. Further, we have $g \in a^n R$ and $1 - g \in (1 - a^n)R$. By the uniqueness, we obtain $g = e$. Thus, $f = 1 - e$, hence the result. □

COROLLARY 2.2. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *Every idempotent in R is central.*
(2) *For any $a \in R$, there exists an $n \in \mathbb{N}$ and a unique idempotent $e \in Ra^n$ such that $1 - e \in R(1 - a^n)$.*

PROOF. Obviously, a ring R is uniquely π-clean if and only if so is the opposite ring R^{op}. Applying Theorem 2.2 to R^{op}, we complete the proof. □

A ring R is local if it has only one maximal right ideal. A ring R is potent if for any $a \in R$ there exists some $n \in \mathbb{N}$ such that $a^n = a$. We note that every potent ring is commutative.

LEMMA 2.2. *Let R be a local ring. If R is uniquely π-clean, then $R/J(R)$ is potent.*

PROOF. Suppose that there exists some $a \in R$ such that $a^n - a \notin J(R)$ for all $n \geqslant 2$. Then $a(a^{n-1} - 1) \in U(R)$ as R is a local ring. This implies that $a \in U(R)$ and $a^{n-1} - 1 \in U(R)$ for all $n \geqslant 2$. Since R is uniquely π-clean, we have an $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely clean. But $a^m = 0 + a^m = 1 + (a^m - 1)$, a contradiction. Therefore, for any $a \in R$, there exists some integer $n \geqslant 2$ such that $a^n - a \in J(R)$. That is, $R/J(R)$ is potent. □

LEMMA 2.3. [**6**, Theorem 3.1] *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *R is Abelian;*
(2) *Every idempotent lifts modulo $J(R)$;*
(3) *$R/J(R)$ is potent.*

We have accumlated all the information necessary to prove the following.

THEOREM 2.3. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is uniquely π-clean.*
(2) *For any $a \in R$, there exist an $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in J(R)$.*

PROOF. (1) $\Rightarrow$ (2) In view of Lemma 2.3, $R/J(R)$ is potent. For any $a \in R$, $\bar{a} \in R/J(R)$ is potent, and so $\bar{a}^m \in R/J(R)$ is an idempotent for some $m \in \mathbb{N}$. By using Lemma 2.3 again, we can find a central idempotent $e \in R$ such that $\bar{a}^m = \overline{e}$, and so $a^m - e \in J(R)$.

(2) $\Rightarrow$ (1) If $e \in R$ is an idempotent, then we have a central idempotent $f \in R$ such that $e - f \in J(R)$. As $(e - f)^3 = e - f$, we deduce that $e = f$; hence, every idempotent in R is central. If $e - e^2 \in J(R)$, then we can find a central idempotent $f \in R$ such that $e^m - f \in J(R)$ for some $m \in \mathbb{N}$. As $e - e^2 \in J(R)$, if $m \geqslant 3$, we see that $e - e^m = (e - e^2) + (e - e^2)e + \cdots + (e - e^2)e^{m-2} \in J(R)$. Thus $e - f \in J(R)$, and then idempotents lift modulo $J(R)$.

For any $a \in R$, there exists $m \in \mathbb{N}$ such that $a^m - e \in J(R)$ for a central idempotent. Hence, $\bar{a}^m = \overline{e}$ in $R/J(R)$. Thus, $S := R/J(R)$ is periodic. Thus, S is an Abelian exchange ring. If $x^2 = 0$ and $x \neq 0$ in S, then $x \notin J(S)$. For any $r \in S$, there exists some idempotent $g \in Srx$ such that $1 - g \in S(1 - rx)$. Write $g = crx$ for a $c \in S$. Then $g = g^2 = (crx)g = (cr)gx = (cr)(crx)x = (cr)^2x^2 = 0$, as S is Abelian. Thus, $1 - rx \in S$ is left invertible. Since S is Abelian, it is easy to check that $1 - rx \in U(S)$. This shows that $x \in J(S)$; hence, $x = 0$. This gives a contradiction. Therefore S is reduced.

Let $a \in R$; there exist m, n $(m > n)$ such that $\bar{a}^m = \bar{a}^n$ in S. Choose $k = n(m - n)$. It is easy to verify that $p = \bar{a}^{k+1}$ is potent and $w = \bar{a} - \bar{a}^{k+1} \in N(S)$. Further, $\bar{a} = p + w = p$ is potent, and so S is potent. Applying Lemma 2.3, we complete the proof. $\square$

COROLLARY 2.3. *Let R be a ring. Then R is uniquely clean if and only if*

(1) *R is uniquely π-clean;*
(2) *$J(R) = \{x \in R \mid x - 1 \in U(R)\}$.*

PROOF. Obviously, $J(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. Suppose that $1 - x \in U(R)$. Then we have an idempotent $e \in R$ and an element $u \in J(R)$ such that $x = e + u$ and $ex = xe$ by [**10**, Theorem 20]. Thus, $1 - e = (1 - x) + u \in U(R)$, and so $1 - e = 1$. This implies that $e = 0$, whence $x = u \in J(R)$. Therefore $J(R) = \{x \in R \mid 1 - x \in U(R)\}$.

Conversely, assume that (1) and (2) hold. In view of Lemma 2.3, $R/J(R)$ is potent. It follows from $J(R) = \{x \in R \mid x - 1 \in U(R)\}$ that $U\big(R/J(R)\big) = \{\bar{1}\}$.

Write $p = p^n (n \geqslant 2)$ in $R/J(R)$. Then $\big(1 - p^{n-1} + p\big)^{-1} = 1 - p^{n-1} + p^{n-2}$. Hence, $p = p^{n-1}$, and so $p^2 = p^n = p$. This implies that $R/J(R)$ is Boolean. Therefore we complete the proof by Lemma 2.1 and [**10**, Theorem 20]. $\square$

Theorem 2.4. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *For any $a \in R$, there exists an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J(R)$.*
(2) *$J(R) = \{x \in R \mid x^m - 1 \in U(R)$ for all $m \in \mathbb{N}\}$.*

Proof. Suppose that R is uniquely π-clean. Let $a \in R$. In view of Theorem 2.3, there exist an $m \in \mathbb{N}$ and a central idempotent $g \in R$ such that $a^m - g \in J(R)$. If there exists an idempotent $f \in R$ such that $a^m - f \in J(R)$, then $g - f = (a^m - f) - (a^m - g) \in J(R)$. Clearly, $(g - f)^3 = g - f$, and so $(g - f)\big(1 - (g - f)^2\big) = 0$. Thus, $g = f$, i.e., the uniqueness is verified.

Clearly, $J(R) \subseteq \{x \in R \mid x^m - 1 \in U(R)$ for all $m \in \mathbb{N}\}$. If $x \notin J(R)$, then $0 \neq xR \nsubseteq J(R)$. In view of Lemma 2.1, R is an exchange ring, and so there exists an idempotent $0 \neq e \in xR$. Write $e = xr$ for an $r \in R$. Choose $a = exe + 1 - e$. Then we can find some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely clean. In addition, R is Abelian by Lemma 2.1. Obviously, $a^m = 0 + \big(ex^m e + 1 - e\big) = e + \big(e(x^m - 1)e + 1 - e\big)$. If $x^m - 1 \in U(R)$, then $0 = e$, a contradiction. This implies that $x^m - 1 \notin U(R)$. That is, $x \notin \{x \in R \mid x^m - 1 \in U(R)$ for all $m \in \mathbb{N}\}$. Therefore $\{x \in R \mid x^m - 1 \in U(R)$ for all $m \in \mathbb{N}\} \subseteq J(R)$, as required.

Conversely, assume that (1) and (2) hold. Let $x \in N(R)$. Then $x^m - 1 \in U(R)$ for all $m \in \mathbb{N}$. By hypothesis, we get $x \in J(R)$. Therefore, every nilpotent element in R is contained in $J(R)$. Let $e \in R$ be an idempotent, and let $r \in R$. Then $e + er(1 - e) \in R$ is an idempotent. Hence, there exists a unique $f \in R$ such that $\big(e + er(1 - e)\big) - f \in J(R)$. By the preceding discussion, $\big(e + er(1 - e)\big) - e = er(1 - e) \in J(R)$. The uniqueness forces $e = f$. But $\big(e + er(1 - e)\big) - \big(e + er(1 - e)\big) \in J(R)$, and so $e + er(1 - e) = f = e$. This shows that $er = ere$. Likewise, $re = ere$. That is, $er = re$, and then R is Abelian. For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $w := a^m - e \in J(R)$. Then $a^m = (1 - e) + (2e - 1 + w)$. As $(2e - 1)^2 = 1$, we see that $2e - 1 + w \in U(R)$. If there exists an idempotent $f \in R$ such that $a^m - f \in U(R)$, then $e - f = (a^m - f) - (a^m - e) \in U(R)$. One easily checks that $(e + f - 1)(e - f)^2 = 0$, and therefore $e + f - 1 = 0$. Thus, $f = 1 - e$, hence the result. $\square$

Corollary 2.4. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J(R)$.*
(2) *$N(R) \subseteq J(R)$.*

Proof. Suppose that R is uniquely π-clean. (1) is obvious by Theorem 2.4. Let $a \in N(R)$. Then $1 - a^m \in U(R)$ for all $m \in \mathbb{N}$. It follows by Theorem 2.4 that $a \in J(R)$. Therefore $N(R) \subseteq J(R)$.

Conversely, assume that (1) and (2) hold. Let $e \in R$, and let $x \in R$. Then $ex(1 - e) \in J(R)$. By hypothesis, we have some $m \in \mathbb{N}$ such that the expressions

$\big(e + ex(1-e)\big)^m = \big(e + ex(1-e)\big) + 0 = e + ex(1-e)$ are unique. This implies that $ex(1-e) = 0$, and so $ex = exe$. Likewise, $xe = exe$. Therefore R is Abelian. This yields the result by Theorem 2.3. $\square$

COROLLARY 2.5. *Let R be a local ring. Then the following statements are equivalent:*

(1) *R is uniquely π-clean.*
(2) $U(R) = \{x \in R \mid$ *There is an* $m \in \mathbb{N}$ *such that* $x^m - 1 \in J(R)\}$.
(3) $J(R) = \{x \in R \mid x^m - 1 \in U(R)$ *for all* $m \in \mathbb{N}\}$.

PROOF. (1) $\Rightarrow$ (3) is clear from Theorem 2.4.

(3) $\Rightarrow$ (2) Obviously, $\{x \in R \mid \text{there is an } m \in \mathbb{N} \text{ such that } x^m - 1 \in J(R)\} \subseteq U(R)$. For any $x \in U(R)$, $x \notin J(R)$. By hypothesis, there exists some $m \in \mathbb{N}$ such that $x^m - 1 \notin U(R)$. As R is local, $x^m - 1 \in J(R)$. This implies that $U(R) \subseteq \{x \in R \mid \text{there is an } m \in \mathbb{N} \text{ such that } x^m - 1 \in J(R)\}$, as required.

(2) $\Rightarrow$ (1) For any $x \in R$, we see that either $x \in J(R)$ or $x \in U(R)$. This implies that $\bar{x} = \bar{0}$ or $\bar{x}^m = \bar{1}$ in $R/J(R)$. Thus $R/J(R)$ is potent. In light of Lemma 2.3, R is uniquely π-clean. $\square$

3. Factors of Prime Ideals

The aim of this section is to characterize uniquely π-clean rings by means of prime ideals containing the Jacobson radicals. We use $J\text{-}\operatorname{spec}(R)$ to denote the set $\{P \in \operatorname{Spec}(R) \mid J(R) \subseteq P\}$. Obviously, every maximal ideal is contained in $J\text{-}\operatorname{spec}(R)$. Set

$$J^*(R) = \bigcap \{P \mid P \text{ is a maximal ideal of } R\}.$$

We will see that $J(R) \subseteq J^*(R)$. In general, they are not the same. For instance, $J(R) = 0$ and $J^*(R) = \{x \in R \mid \dim_F(xV) < \infty\}$, where $R = \operatorname{End}_F(V)$ and V is an infinite-dimensional vector space over a field F. Furthermore, we characterize a uniquely π-clean ring R by means of the radical-like ideal $J^*(R)$. A ring R is strongly π-regular if, for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. We have

LEMMA 3.1. [**7**, Corollary 2.8] *Let R be a commutative ring. Then the following statements are equivalent:*

(1) *R is strongly π-regular.*
(2) *R is an exchange ring in which every prime ideal of R is maximal.*

LEMMA 3.2. *Let R be an Abelian exchange ring, and let $x \in R$. Then $RxR = R$ if and only if $x \in U(R)$.*

PROOF. If $x \in U(R)$, then $RxR = R$. Conversely, assume that $RxR = R$. As in the proof of [**4**, Proposition 17.1.9], there exists an idempotent $e \in R$ such that $e \in xR$ such that $ReR = R$. This implies that $e = 1$. Write $xy = 1$. Then $yx = y(xy)x = (yx)^2$. Hence, $yx = y(yx)x$. Therefore $1 = x(yx)y = xy(yx)xy = yx$, and so $x \in U(R)$. This completes the proof. $\square$

Herstein's theorem says that a ring R is periodic if and only if for any $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n = a^{n+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. We recall that a ring R is torsion, provided that for any nonzero $a \in R$ there exists $m \in \mathbb{N}$ such that $a^m = 1$. With this information we now derive

THEOREM 3.1. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *R is Abelian;*
(2) *Every idempotent lifts modulo $J(R)$;*
(3) *R/P is torsion for all $P \in J\text{-}\operatorname{spec}(R)$.*

PROOF. Suppose R is uniquely π-clean. In view of Lemmas 2.1 and 2.3, R is an Abelian exchange ring, and $R/J(R)$ is potent. Let $P \in J\text{-}\operatorname{spec}(R)$. Then $R/J(R)/P/J(R) \cong R/P$ is prime; hence, $P/J(R)$ is a prime ideal of $R/J(R)$. As every potent ring is commutative, $R/J(R)$ is a commutative π-regular ring. It follows from Lemma 3.1 that $P/J(R)$ is a maximal ideal of $R/J(R)$. We infer that P is a maximal ideal of R.

Clearly, $\overline{R} := R/P$ is an Abelian exchange ring. Since P is maximal, R/P is simple. For any $0 \neq x \in \overline{R}$, we have $\overline{R}x\overline{R} = \overline{R}$. By Lemma 3.2, $x \in U(R/P)$. Hence, R/P is a division ring. On the other hand, $R/P \cong R/J(R)/P/J(R)$ is potent. Thus, we have some $m \in \mathbb{N}$ such that $x^{m+1} = x$, and so $x^m = 1$. This implies that R/P is torsion, as required.

Conversely, assume that (1)–(3) hold. Assume that R is not uniquely π-clean. Set $S = R/J(R)$. In view of Theorem 2.3, S is not periodic. By using Herstein's theorem, there exists some $a \in S$ such that $a^m \neq a^{m+1}f(a)$ for any $m \in \mathbb{N}$ and any $f(x) \in \mathbb{Z}[x]$. Let $\Omega = \{I \lhd S \mid \bar{a}^m \neq \bar{a}^{m+1}f(\bar{a}) \text{ in } S/I \text{ for any } m \in \mathbb{N} \text{ and any } f(x) \in \mathbb{Z}[x]\}$. Then Ω is an nonempty inductive. By using Zorn's lemma, there exists an ideal Q of S which is maximal in Ω. If Q is not prime, then there exist two ideals K and L of R such that $K, L \nsubseteq Q$ and $KL \subseteq Q$. By the maximality of Q, we can find some $s, t \in \mathbb{N}$ and some $f(x), g(x) \in \mathbb{Z}[x]$ such that $\bar{a}^s = \bar{a}^{s+1}f(\bar{a})$ in $R/(K+Q)$ and $\bar{a}^t = \bar{a}^{t+1}g(\bar{a})$ in $R/(L+Q)$. Thus, $a^s - a^{s+1}f(a) \in K+Q$ and $a^t - a^{t+1}g(a) \in L+Q$, and so $\big(a^s - a^{s+1}f(a)\big)\big(a^t - a^{t+1}g(a)\big) \in (K+Q)(L+Q) \subseteq KL + Q \subseteq Q$. In S/Q, we have $\bar{a}^{s+t} = \bar{a}^{s+t+1}h(\bar{a})$ for some $h(x) \in \mathbb{Z}[x]$. This contradicts the choice of Q. Hence, $Q \in J\text{-}\operatorname{spec}(R)$. By hypothesis, R/Q is torsion, and so R/Q is periodic, which is impossible. Therefore R is uniquely π-clean. $\square$

COROLLARY 3.1. *A ring R is uniquely clean if and only if*

(1) *R is uniquely π-clean.*
(2) *$R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R.*

PROOF. Suppose R is uniquely clean. Then R is uniquely π-clean. (2) is proved by [**3**, Theorem 2.1].

Conversely, assume that (1) and (2) hold. For all maximal ideals M of R, $1_{R/M}$ is not the sum of two units in R/M. In view of Lemma 2.1, R is an Abelian exchange ring, and so it is clean. Let $x \in R$. Write $x = e_1 + u_1 = e_2 + u_2$, $e_1 = e_1^2$, $e_2 = e_2^2$ and $u_1, u_2 \in U(R)$. If $R\big(1 - e_2(1 - e_1)\big)R \neq R$, then there exists a maximal ideal M of R such that $R\big(1 - e_2(1 - e_1)\big)R \subseteq M$. Clearly,

$J(R) \subseteq M$. Hence, $\bar{x} = \overline{e_1} + \overline{u_1} = \overline{e_2} + \overline{u_2}$ in R/M. By Theorem 3.1, R/M is a division ring. This implies that $\overline{e_i}$ are $\bar{0}$ or $\bar{1}$. If $\overline{e_1} \neq \overline{e_2}$, then $1_{R/M}$ is the sum of two units, a contradiction. Therefore we get $e_1 - e_2 \in M$. This implies that $e_2(1-e_1) = (e_1 - e_2)(e_1 - 1) \in M$, and so $1 = e_2(1-e_1) + \big(1 - e_2(1-e_1)\big) \in M$, a contradiction. As a result, $R\big(1 - e_2(1-e_1)\big)R = R$. As $e_2(1-e_1) \in R$ is an idempotent, we get $e_2(1-e_1) = 0$, and so $e_2 = e_2e_1$. Likewise, $e_1 = e_1e_2$. Consequently, $e_1 = e_2$, and then $u_1 = u_2$. Therefore R is uniquely clean. □

Let $S(R)$ be the nonempty set of all ideals of a ring R generated by central idempotents. By Zorn's lemma, $S(R)$ contains maximal elements. As usually, we say that R/P is a Pierce stalk if P is a maximal element of the set $S(R)$, and that P is a Pierce ideal. Let $\mathrm{Pier}(R)$ be the set of all Pierce ideals of R.

Proposition 3.1. *Every uniquely π-clean ring is the subdirect product of rings R_i, where each $R_i/J(R_i)$ is torsion.*

Proof. Let R be a uniquely π-clean ring. [**9**, Remark 11.2] says that the intersection of all Pierce ideals of R is zero, i.e., $\bigcap\{\, P \mid P \in \mathrm{Pier}(R)\} = 0$. Let $\varphi_P : R \to R/P$ be the natural epimorphism. Then $\bigcap_{P\in \mathrm{Pier}(R)} \ker \varphi_P = \bigcap_{P \in \mathrm{Pier}(R)} P = 0$. Hence, R is the subdirect product of all R/P, where $P \in \mathrm{Pier}(R)$. In view of Lemma 2.1, R is an Abelian exchange ring. Let $P \in \mathrm{Pier}(R)$. Then R/P is an exchange ring. As R is indecomposable, we see that R/P is a local ring. By an argument in [**6**], R/P is uniquely π-clean, and so $R/P/J(R/P)$ is potent from Lemma 2.3, as needed. □

Lemma 3.3. *Let R be an Abelian exchange ring. Then $J^*(R) = J(R)$.*

Proof. Let M be a maximal ideal of R. If $J(R) \nsubseteq M$, then $J(R) + M = R$. Write $x + y = 1$ with $x \in J(R)$, $y \in M$. Then $y = 1 - x \in U(R)$, an absurd. Hence, $J(R) \subseteq M$. This implies that $J(R) \subseteq J^*(R)$. Let $x \in J^*(R)$, and let $r \in R$. If $R(1-xr)R \neq R$, then we can find a maximal ideal M of R such that $R(1-xr)R \subseteq M$, and so $1 - xr \in M$. It follows that $1 = xr + (1-xr) \in M$, which is imposable. Therefore $R(1-xr)R = R$. In light of Lemma 3.2, $1 - xr \in U(R)$, and then $x \in J(R)$. This completes the proof. □

Theorem 3.2. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *R is an exchange ring;*
(2) *$R/J^*(R)$ is potent and every idempotent uniquely lifts modulo $J^*(R)$.*

Proof. Suppose R is uniquely π-clean. Then R is an Abelian exchange ring by Lemma 2.1. In view of Lemma 3.3, $J^*(R) = J(R)$. It follows from Lemma 2.3 that $R/J^*(R)$ is potent. Let $e - e^2 \in J(R)$. Then we can find an idempotent $f \in R$ such that $e - f \in J(R)$. Since $(e-f)^2\big(1 - (e-f)\big) = 0$, we get $e = f$, as desired.

Conversely, assume that (1) and (2) hold. Let $e \in R$ be an idempotent, and let $r \in R$. Then $\overline{er(1-e)} \in R/J^*(R)$ is potent. This implies that $\overline{er(1-e)} = \bar{0}$, and so $er(1-e) \in J^*(R)$. Since $e - e$, $e - \big(e + er(1-e)\big) \in J^*(R)$, by the uniqueness, we get $e = e + er(1-e)$, and so $er = ere$. Likewise, $re = ere$; hence that $er = re$.

Thus, R is Abelian. In light of Lemma 3.3, $J^*(R) = J(R)$. Therefore we complete the proof, in terms of Lemma 2.3. $\square$

Corollary 3.2. *Let R be a ring which has finitely many maximal ideals. Then R is uniquely π-clean if and only if*

(1) *R is an exchange ring;*
(2) *$R/J^*(R)$ is the direct sum of finitely many torsion rings and every idempotent uniquely lifts modulo $J^*(R)$.*

Proof. $\Rightarrow$: Clearly, R is an exchange ring. Let M be a maximal ideal of R. As in the proof of Lemma 3.3, we see that $J(R) \subseteq M$. This shows that $M \in J\text{-}\operatorname{spec}(R)$. Therefore R/M is torsion by Theorem 3.1. Since R has finitely many maximal ideals $M_1, \ldots, M_n$, we see that $R/J^*(R) \cong R/M_1 \oplus \cdots \oplus R/M_n$. Therefore $R/J^*(R)$ is the direct sum of finitely many torsion rings, as desired.

$\Leftarrow$: As every torsion ring is potent, we see that $R/J^*(R)$ is potent. Therefore we complete the proof, by Theorem 3.2. $\square$

Theorem 3.3. *Let R be a ring. Then R is uniquely π-clean if and only if*

(1) *For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $a^m - e \in J^*(R)$.*
(2) *$J^*(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N}\}$.*

Proof. One direction is obvious by Lemma 3.3 and Theorem 2.4.

Conversely, assume that (1) and (2) hold. Let $x \in N(R)$. Then $x^m - 1 \in U(R)$ for all $m \in \mathbb{N}$. By hypothesis, we have $x \in J^*(R)$, and so $N(R) \subseteq J^*(R)$. Let $e \in R$ be an idempotent, and let $r \in R$. Then $e + er(1-e) + 0 = e + er(1-e)$ with $0, er(1-e) \in J^*(R)$. By the uniqueness, we get $er = ere$. Similarly, we have $re = ere$. That is, $er = re$. We infer that R is Abelian. For any $a \in R$, there exist an $m \in \mathbb{N}$ and a unique $e \in R$ such that $w := a^m - e \in J^*(R)$. Then $a^m = (1-e) + (2e - 1 + w)$. But $2e - 1 + w = (1-2e)\big((1-2e)w - 1\big) \in U(R)$, by (2). If there exists an idempotent $f \in R$ such that $a^m - f \in U(R)$, then $e - f = (a^m - f) - (a^m - e) = (a^m - f)\big(1 - (a^m - f)^{-1}(a^m - e)\big) \in U(R)$. It follows from $(e + f - 1)(e - f)^2 = 0$ that $f = 1 - e$, and we are through. $\square$

Let $P(R)$ be the intersection of all prime ideals of R, i.e., $P(R)$ is the prime radical of R. As is well known, $P(R)$ is the intersection of all minimal prime ideals of R.

Corollary 3.3. *Let R be a uniquely π-clean in which every prime ideal is maximal. Then*

$$P(R) = \{x \in R \mid x^m - 1 \in U(R) \text{ for all } m \in \mathbb{N}\}.$$

Proof. As every maximal ideal is prime, we deduce that $J^*(R) = P(R)$, and therefore we complete the proof by Theorem 3.3. $\square$

4. Certain Classes

In this section we investigate certain classes of uniquely π-clean rings. We now recall the concept of ideal-extensions. Let R be a ring with an identity and S be a ring (not necessary unitary), and let S be an R-R-bimodule in which $(s_1s_2)r = s_1(s_2r)$, $r(s_1s_2) = (rs_1)s_2$ and $(s_1r)s_2 = s_1(rs_2)$ for all $s_1, s_2 \in S$, $r \in R$. The ideal extension $I(R;S)$ is defined to be the additive Abelian group $R \oplus S$ with multiplication $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2 + r_1s_2 + s_1r_2)$ (see [**10**]). We start this section by examining when an ideal extension is uniquely π-clean.

Theorem 4.1. *The ideal-extension $I(R;S)$ is uniquely π-clean and S is idempotent-free if and only if*

(1) *R is uniquely π-clean;*
(2) *If $e = e^2 \in R$, then $es = se$ for all $s \in S$;*
(3) *If $s \in S$, then there exists an $s' \in S$ such that $ss' = s's$ and $s+s'+ss' = 0$.*

Proof. Assume that (1)–(3) hold. Let $e \in S$ be an idempotent. Then $(-e) + s' + (-e)s' = 0$ for some $s' \in S$. Hence, $(1 - e)(1 + s') = 1$, and so $e = 0$. That is, S is idempotent-free. Let $(a, s) \in I(R;S)$. Then $a \in R$ is uniquely π-clean. Thus, we have some $n \in \mathbb{N}$ such that $a^n \in R$ is uniquely clean. Write $a^n = e + u$, $e = e^2 \in R$, $u \in U(R)$. Hence, $(a, s)^n = (a^n, x) = (e, 0) + (u, x)$ for some $x \in S$. Clearly, $(e, 0)^2 = (e, 0)$. As $x \in S$, we see that $u^{-1}x \in S$, and so we have some $t \in S$ such that $u^{-1}x + t + u^{-1}xt = 0$ and $u^{-1}xt = tu^{-1}x$. This implies that $1 + u^{-1}x = (1+t)^{-1} \in U(R)$. One easily checks that $(u, x)^{-1} = (u^{-1}, -(1+u^{-1}x)^{-1}u^{-1}xu^{-1})$; hence, $(u, x) \in U\big(I(R;S)\big)$. Write $(a, s)^n = (f, y) + (v, w), (f, y)^2 = (f, y)$ and $(v, w) \in U\big(I(R;S)\big)$. Then $f = f^2 \in R$, $y = 0$ and $v \in U(R)$. Clearly, $a^n = f + v$. Further, $x = y + w = w$. This implies that $f = e$, $v = u$, and so $(f, y) = (e, 0)$, $(v, w) = (u, x)$. As a result, $(a, s) \in I(R;S)$ is uniquely π-clean, and so $I(R;S)$ is uniquely π-clean.

Assume that $I(R;S)$ is uniquely π-clean and S is idempotent-free. Then R is uniquely π-clean. Let $e = e^2 \in R$ and $s \in S$. In view of Lemma 2.1, $(e, 0) = (e, 0)^2 \in I(R;S)$ is central. Hence, $(e, 0)(0, s) = (0, s)(e, 0)$, and so $es = se$. For any $s \in S$, there exists some $n \in \mathbb{N}$ such that $(1, s)^n \in I(R;S)$ is uniquely clean. Write $(1, s)^n = (1, x) = (f, y)+(u, v)$ where $x \in S, (f, y) \in I(R;S)$ is an idempotent and $(u, v) \in I(R;S)$ is a unit. Clearly, $f = 0$, and so $y = 0$. This implies that $x = y + v = v$; hence, $(1, x) \in I(R;S)$ is a unit. Further, $(1, s) \in I(R;S)$ is a unit. Write $(1, s)^{-1} = (1, s')$ for a $s' \in S$. Then $ss' = s's$ and $s + s' + ss' = 0$, hence the result. □

Corollary 4.1. *Let R be uniquely π-clean. Then $S = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ is uniquely π-clean.*

Proof. Let $T = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then $S \cong I(R;T)$. Then the result follows by Theorem 4.1. □

A ring R is called potently J-clean if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in J(R)$. We shall show that such rings form a subclass of uniquely

π-clean rings. A ring R is an exchange ring if and only if $R/J(R)$ is an exchange ring, and every idempotent lifts modulo $J(R)$. We have

LEMMA 4.1. *Every potently J-clean ring is an exchange ring.*

PROOF. Let R be a potently J-clean ring. Then $R/J(R)$ is potent, and so it is an exchange ring. Let $\overline{e} \in R/J(R)$ be an idempotent. Then we have a potent $p \in R$ such that $w := e - p \in J(R)$. Write $p = p^n$ for some $n \geqslant 2$. Then $p^{n-1} \in R$ is an idempotent. Moreover, $e = p + w$, and so $e^{n-1} = p^{n-1} + v$ for some $v \in J(R)$. But $e - e^{n-1} \in J(R)$. Hence, $e - p^{n-1} = \big(e - e^{n-1}\big) + \big(e^{n-1} - p^{n-1}\big) \in J(R)$. So idempotents can be lifted modulo $J(R)$. Therefore R is an exchange ring. $\square$

THEOREM 4.2. *Every Abelian potently J-clean ring is uniquely π-clean.*

PROOF. Let R be an Abelian potently J-clean ring. Then R is an exchange ring by Lemma 4.1. Thus, every idempotent in R lifts modulo $J(R)$. For any $a \in R$, there exists a potent $p \in R$ such that $a - p \in J(R)$. This implies that $\bar{a} \in R/J(R)$ is potent, and so $R/J(R)$ is potent. According to Lemma 2.3, R is uniquely π-clean. $\square$

COROLLARY 4.2. *Let R be Abelian. If for any sequence of elements $\{a_i\} \subseteq R$ there exists a $k \in \mathbb{N}$ and $n_1, \cdots, n_k \geqslant 2$ such that $(a_1 - a_1^{n_1}) \cdots (a_k - a_k^{n_k}) = 0$, then R is uniquely π-clean.*

PROOF. For any $a \in R$, we have a $k \in \mathbb{N}$ and $n_1, \cdots, n_k \geqslant 2$ such that $(a - a^{n_1}) \cdots (a - a^{n_k}) = 0$. This implies that $a^k = a^{k+1} f(a)$ for some $f(t) \in \mathbb{Z}[t]$. By Herstein's theorem, R is periodic. Therefore every element in R is the sum of a potent element and a nilpotent element.

Clearly, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings R_i.

Case 1. There exists a subring S_i of R_i which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$ where D_i is a division ring.

Case 2. $R_i \cong M_{m_i}(D_i)$ for a division ring D_i. Clearly, the hypothesis is inherited by all subrings, all homomorphic images and all corners of R, we claim that, for any sequence of elements $\{a_i\} \subseteq M_2(D_i)$ there exists $s \in \mathbb{N}$ and $m_1, \ldots, m_s \geqslant 2$ such that $(a_1 - a_1^{m_1}) \cdots (a_s - a_s^{m_s}) = 0$. Choose $a_i = e_{12}$ if i is odd and $a_i = e_{21}$ if i is even. Then $(a_1 - a_1^{m_1})(a_2 - a_2^{m_2}) \cdots (a_s - a_s^{m_s}) = a_1 a_2 \cdots a_s \neq 0$, a contradiction. This forces $m_i = 1$ for all i. We infer that each R_i is reduced, and then so is $R/J(R)$. If $a \in N(R)$, we have some $n \in \mathbb{N}$ such that $a^n = 0$, and thus $\bar{a}^n = 0$ is $R/J(R)$. Hence, $\bar{a} \in J\big(R/J(R)\big) = 0$. This implies that $a \in J(R)$, and so $N(R) \subseteq J(R)$. Therefore R is potently J-clean, hence the result by Theorem 4.2. $\square$

5. Uniquely π-nil Clean Rings

In this section, we explore uniquely π-nil-clean rings, and completely determine when a ring is uniquely π-nil-clean ring.

LEMMA 5.1. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is uniquely π-nil-clean.*
(2) *R is an Abelian periodic ring.*

Proof. (1) $\Rightarrow$ (2). Let $e \in R$ be an idempotent and $r \in R$. Choose $a = e + er(1-e)$. Then we can find some $m \in \mathbb{N}$ such that $a^m \in R$ is uniquely nil clean. As $a = a^m = e + er(1-e) = \big(e + er(1-e)\big) + 0$, by the uniqueness, we get $er(1-e) = 0$, and so $er = ere$. Likewise, $re = ere$, and so $er = re$. Therefore R is Abelian. Let $a \in R$. Then there exists some $n \in \mathbb{N}$ such that $a^n = f + u$, where $f = f^2 \in R$ and $u \in N(R)$. Hence, $a^{2n} = f + v$ for a $v \in N(R)$ and $uv = vu$. This shows that $a^n - a^{2n} = u - v \in N(R)$. Thus, we have a $k \in \mathbb{N}$ such that $a^{nk} = a^{nk+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In light of Herstein's theorem, R is periodic.

(2) $\Rightarrow$ (1) Let $a \in R$. Since R is periodic, there exists some $m \in \mathbb{N}$ such that $a^m \in R$ is an idempotent. Write $a^m = e + w$ where $e = e^2 \in R$ and $w \in N(R)$. Then $a^m - e = w \in N(R)$. As R is Abelian, we see that $\big(a^m - e\big)^3 = a^m - e$. Thus, $\big(a^m - e\big)\big(1 - (a^m - e)^2\big) = 1$, and so $a^m = e$, as required. $\square$

As every finite ring is periodic, it follows from Lemma 5.1 that every finite commutative ring is uniquely π-nil-clean, e.g., $\mathbb{Z}_n[\alpha] = \{a + b\alpha \mid a, b \in \mathbb{Z}_n,\ \alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ i^2 = -1\}$.

The above observation leads us to the following result alluded to earlier.

Theorem 5.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely π-nil-clean.*
(2) *R is uniquely π-clean and $J(R)$ is nil.*
(3) *For any $a \in R$, there exist some $m \in \mathbb{N}$ and a unique idempotent $e \in R$ such that $a^m - e \in P(R)$.*

Proof. (1) $\Rightarrow$ (3). By virtue of Lemma 5.1, R is an Abelian periodic ring. In view of [**2**, Theorem 2], $N(R)$ forms an ideal of R, and so $N(R) = P(R)$. For any $a \in R$, there exists some $m \in \mathbb{N}$ such that a^m is uniquely nil clean. Write $a^m = e + w$ with $e = e^2$ and $w \in N(R)$. Therefore $a^m - e \in P(R)$, as required.

(3) $\Rightarrow$ (2). Let $e \in R$ be an idempotent, and let $r \in R$. Then we have an idempotent $f \in R$ such that $er(1-e) = f + w$ for a $w \in P(R)$. Hence, $1 - f = 1 - er(1-e) + w = \big(1 - er(1-e)\big)\big(1 + (1 + er(1-e))w\big) \in U(R)$. We infer that $f = 0$, and so $er(1-e) = w \in P(R)$. But we have a unique expression $e + er(1-e) = e + er(1-e) + 0$ where $er(1-e), 0 \in P(R)$. By the uniqueness, we get $e = e + er(1-e)$, and so $er = ere$. Similarly, $re = ere$. Therefore $er = re$, i.e., R is Abelian.

Let $x \in J(R)$. Write $x = h + v$ with $h = h^2 \in R, v \in P(R)$. Then $h = x - v \in J(R)$; hence that $h = 0$. It follows that $J(R) = P(R)$. Accordingly, for any $a \in R$, there exist some $m \in \mathbb{N}$ and a unique idempotent $e \in R$ such that $a^m - e \in J(R)$.

If $x \in N(R)$, then we have an idempotent $g \in R$ and a $u \in P(R)$ such that $x = g + u$, and so $g = x - u$. As R is Abelian, we see that $xu = ux$, and then $g \in N(R)$. This shows that $g = 0$. Consequently, $x = u \in P(R) \subseteq J(R)$. We infer that $N(R) \subseteq J(R)$. In light of Corollary 2.4, R is uniquely π-clean, as desired.

(2) $\Rightarrow$ (1). In view of Lemma 2.1, R is Abelian. In view of Lemma 2.3, $R/J(R)$ is potent. Let $a \in R$. Then $\bar{a} = \overline{a^m} (m \geqslant 2)$, and so $a - a^m \in J(R)$. As $J(R)$ is nil, every idempotent lifts modulo $J(R)$. Hence, we can find some $n \in \mathbb{N}$ such that

$(a-a^m)^n = 0$, and so $a^n = a^{n+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In terms of Herstein's theorem, R is periodic. This completes the proof, by Lemma 5.1. □

Corollary 5.1. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is uniquely π-nil-clean.*
(2) *$R/J(R)$ is potent, R is Abelian and $J(R)$ is nil.*
(3) *For any $a \in R$, there exists some $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in P(R)$.*

Proof. (1) $\Leftrightarrow$ (2) is proved by Theorem 5.1 and Lemma 2.3.

(1) $\Rightarrow$ (3) This is obvious, in view of Lemma 5.1 and Theorem 5.1.

(3) $\Rightarrow$ (1). For any $a \in R$, there exist some $m \in \mathbb{N}$ and a central idempotent $e \in R$ such that $a^m - e \in P(R)$. Write $a^m - f \in P(R)$ for an idempotent $f \in R$. Then $e - f = (a^m - f) - (a^m - e) \in P(R)$. As $(e-f)^3 = e - f$, we conclude that $e = f$, and we are through by Theorem 5.1. □

Let $n \geqslant 2$ be a fixed integer. Following Yaqub [**8**], a ring R is said to be generalized n-like provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$.

Proposition 5.1. *Every generalized n-like ring is uniquely π-nil-clean.*

Proof. Let $a \in R$. Then $a^{2n} - 2a^{n+1} + a^2 = 0$, and so $(a - a^n)^2 = 0$. Thus, $a - a^n \in N(R)$. Hence, $a^m = a^{m+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. Accordingly, R is periodic by Herstein's theorem.

Let $e, f \in R$. Since R is a generalized n-like ring, we have

$$\big((1-e)f\big)^n e = \big((1-e)fe\big)^n - (1-e)fe + (1-e)fe = 0;$$
$$\big((1-e)f\big)^n = (1-e)f + (1-e)f - (1-e)f = (1-e)f.$$

Reiterating in the last, we get $(1-e)f = \big((1-e)f\big)^{2n}$, and so $(1-e)fe = 0$. Hence, $fe = efe$. Likewise, $ef = efe$. Therefore $ef = fe$. We infer that R is Abelian. Therefore we conclude that R is uniquely π-nil-clean, in terms of Lemma 5.1. □

Let $R = \Big\{ \left(\begin{smallmatrix} x & y & z \\ 0 & x^2 & 0 \\ 0 & 0 & x \end{smallmatrix}\right) \mid x, y, z \in GF(4) \Big\}$. It is easy to check that for each $a \in R$, $a^7 = a$ or $a^7 = a^2 = 0$. Therefore R is a generalized 7-like ring. By Proposition 5.1, R is uniquely π-clean which is a noncommutative periodic ring.

An element $a \in R$ is uniquely weakly nil-clean provided that a or $-a$ is uniquely nil-clean. A ring R is uniquely weakly nil-clean ring provided that every element in R is uniquely weakly nil-clean [**5**].

Lemma 5.2. *Every uniquely weakly nil-clean ring is uniquely π-nil-clean.*

Proof. Let R be a uniquely weakly nil-clean ring. In view of [**5**, Theorem 12], R is Abelian. Let $a \in R$. Then there exists an idempotent $e \in R$ such that $a - e \in N(R)$ or $-a - e \in N(R)$. If $a - e \in N(R)$, then $a - a^2 \in N(R)$. If $-a - e \in N(R)$, then $a + a^2 \in N(R)$. In any case, $a^n = a^{n+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In view of Herstein's theorem, R is periodic. Therefore R is uniquely π-nil-clean, in terms of Lemma 5.1. □

[**5**, Theorem 12] says that a ring R is a uniquely weakly nil-clean ring if and only if R is Abelian, $J(R)$ is nil and $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings. We have

THEOREM 5.2. *A ring R is a uniquely weakly nil-clean ring if and only if*

(1) *R is uniquely π-nil-clean;*
(2) *$R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings.*

PROOF. $\Rightarrow$: In view of Lemma 5.2, proving (1). Further, proving (2) in terms of [**4**, Theorem 18].

$\Leftarrow$: As R is uniquely π-nil-clean, in view of Corollary 5.1, R is Abelian and $J(R)$ is nil, then by (2) and in light of [**5**, Theorem 18], R is a uniquely weakly nil-clean ring. □

COROLLARY 5.2. *A ring R is a uniquely weakly nil-clean ring if and only if for any $a \in R$, there exists a central idempotent $e \in R$ such that $a - e \in P(R)$ or $a + e \in P(R)$.*

PROOF. $\Rightarrow$: In view of Corollary 5.1, R is uniquely π-nil-clean. For any $a \in R$, by hypothesis, we see that $\bar{a}$ or $-\bar{a}$ is an idempotent in $R/J(R)$. By virtue of [**1**, Theorem 1.12], $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings.

$\Leftarrow$: Let $a \in R$. By (2), there exists a central idempotent $e \in R$ such that $a - e \in P(R)$ or $a + e \in P(R)$. Hence, $a^2 - e = (a - e)(a + e) \in P(R)$. Thus, R is uniquely π-nil-clean, by Theorem 5.1. Let $x \in J(R)$. Then there exists a central idempotent $f \in R$ such that $x - f$ or $x + f$ is in $P(R)$. If $x - f \in P(R)$, then $f \in J(R)$, and so $f = 0$. This implies that $x \in P(R)$. If $x + f \in J(R)$, similarly, $x \in P(R)$. Hence, $J(R) \subseteq P(R)$. We infer that $J(R) = P(R)$. Thus, $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of two such rings, by [**1**, Theorem 1.12]. In light of Theorem 5.2, the result follows. □

A ring R is uniquely nil-clean provided that every element in R is uniquely nil-clean. [**5**, Corollary 13] says that R is a uniquely nil-clean ring if and only if R is a uniquely weakly nil-clean ring and $2 \in J(R)$. Further, we derive

COROLLARY 5.3. *A ring R is a uniquely nil-clean ring if and only if*

(1) *R is uniquely π-nil-clean;*
(2) *$R/J(R)$ is a Boolean ring.*

PROOF. $\Rightarrow$: Clearly, R is uniquely π-nil-clean. In view of [**3**, Theorem 4.5], $R/J(R)$ is Boolean.

$\Leftarrow$: By virtue of Theorem 5.2, R is a uniquely weakly nil-clean ring. As $\overline{2}^2 = \overline{2}$, we see that $2 \in J(R)$. Therefore R is a uniquely nil-clean ring, in terms of [**5**, Corollary 13]. □

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References

1. M. S. Ahn, D. D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mt. J. Math. **36** (2006), 783–798.
2. A. Badawi, *On Abelian π–regular rings*, Commun. Algebra **25** (1997), 1009–1021.
3. H. Chen, *On uniquely clean rings*, Commun. Algebra **39** (2011), 189–198.
4. H. Chen, *Rings Related Stable Range Conditions*, Series in Algebra **11**, World Scientific, Hackensack, NJ, 2011.
5. A. Cimpean, P. Danchev, *Weakly nil-clean index and uniquely weakly nil-clean rings*, Internat. Electronic J. Algebra **21** (2017), 180–197.
6. T. K. Lee, Y. Zhou, *From Boolean rings to clean rings*, Contemp. Math. **609** (2014), 223–232.
7. D. Lu, W. Yu, *On prime spectrums of commutative rings*, Commun. Algebra **34** (2006), 2667–2672.
8. H. Tominaga, A. Yaqub, *On generalized n-like rings and related rings*, Math. J. Okayama Univ. **23** (1981), 199–202.
9. A. A. Tuganbaev, *Rings Close to Regular*, Kluwer, Dordrecht, Boston, London, 2002.
10. W. K. Nicholson, Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit*, Glasg. Math. J. **46** (2004), 227–236.

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