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KOROVKIN TYPE THEOREM FOR FUNCTIONS OF TWO VARIABLES VIA LACUNARY EQUISTATISTICAL CONVERGENCE

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ABSTRACT. Aktuğlu and Gezer [**1**] introduced the concepts of lacunary equistatistical convergence, lacunary statistical pointwise convergence and lacunary statistical uniform convergence for sequences of functions. Recently, Kaya and Gönül [**11**] proved some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence by using test functions 1, $\frac{x}{1+x}$, $\frac{y}{1+y}$, $(\frac{x}{1+x})^2+(\frac{y}{1+y})^2$. We apply the notion of lacunary equistatistical convergence to prove a Korovkin type approximation theorem for functions of two variables by using test functions 1, $\frac{x}{1-x}$, $\frac{y}{1-y}$, $(\frac{x}{1-x})^2+(\frac{y}{1-y})^2$.

1. Introduction and preliminaries

The following concept of statistical convergence for sequences of real numbers was introduced by Fast [**6**]. Let $K \subseteq \mathbb{N}$ and $K_n = \{j : j \leqslant n,\ j \in K\}$. Then the *natural density* of K is defined by $\delta(K) := \lim_{n\to\infty} |K_n|/n$ if the limit exists, where $|K_n|$ denotes the *cardinality* of the set K_n.

A sequence $x = (x_j)$ of real numbers is said to be *statistically convergent* to the number L if, for every $\epsilon > 0$, the set $\{j : j \in \mathbb{N},\ |x_j - L| \geqslant \epsilon\}$ has natural density zero, that is, if, for each $\epsilon > 0$, we have

$$\lim_n \frac{1}{n}\big|\{j : j \leqslant n,\ |x_j - L| \geqslant \epsilon\}\big| = 0.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.

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Fridy and Orhan [**7**] defined the notion of lacunary statistical convergence as follows. Let θ be a lacunary sequence; the number sequence x is S_θ-convergent to L provided that for every $\epsilon > 0$),

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geqslant \epsilon\}| = 0.$$

In this case we write S_θ-limit $x = L$ or $x_k \to L(S_\theta)$.

The concept of equistatistical convergence was introduced by Balcerzak et al. [**2**] and was subsequently applied for deriving approximation theorems in [**1**, **8**–**10**, **19**]. In [**1**], Aktuglu and Gezer [**1**] generalized the idea of statistical convergence to lacunary equistatistical convergence. Recently, Kaya and Gönül [**11**] established some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence. Korovkin type approximation theorems for various kinds of statistical convergence are studied in [**3**–**5**, **14**–**18**]. In this paper, we prove such type of theorem via lacunary equistatistical convergence by using the test functions $1, \frac{x}{1-x}$ and $(\frac{x}{1-x})^2$.

Let $C[a,b]$ be the linear space of all real-valued continuous functions f on $[a,b]$. We know that $C[a,b]$ is a Banach space with the norm given by

$$\|f\|_{C[a,b]} := \sup_{x \in [a,b]} |f(x)| \qquad (f \in C[a,b]).$$

Let f and f_n $(n \in \mathbb{N})$ be real-valued functions defined on a subset X of the set $\mathbb{N}$ of positive integers.

Definition 1.1. A sequence (f_k) of real-valued functions is said to be *lacunary equi-statistically convergent* to f on X if, for every $\epsilon > 0$, the sequence $(S_r(\epsilon, x))_{r \in \mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if, for every $\epsilon > 0$, we have $\lim_{r \to \infty} \|S_r(\epsilon, x)\|_{C(X)} = 0$, where

$$S_r(\epsilon, x) := \frac{1}{h_r} \big| \{k : k \in I_r,\ |f_k(x) - f(x)| \geqslant \epsilon\} \big|$$

and $\mathcal{C}(X)$ denotes the space of all continuous functions on X. In this case, we write

$$f_k \rightrightarrows f \ \ (\theta\text{-equistat}).$$

Definition 1.2. A sequence (f_k) is said to be *lacunary statistically pointwise convergent* to f on X if, for every $\epsilon > 0$ and for each $x \in X$, we have

$$\lim_r \frac{1}{h_r} \big| \{k : k \in I_r,\ |f_k(x) - f(x)| \geqslant \epsilon\} \big| = 0.$$

In this case, we write $f_r \to f$ $(\theta\text{-stat})$.

Definition 1.3. A sequence (f_r) is said to be *lacunary statistically uniformly convergent* to f on X if (for every $\epsilon > 0$), we have

$$\lim_r \frac{1}{h_r} \big| \{k : k \in I_r,\ \|f_k - f\|_{C(X)} \geqslant \epsilon\} \big| = 0.$$

In this case, we write $f_r \rightrightarrows f$ $(\theta\text{-stat})$.

DEFINITION 1.4. (see [**10**]). A sequence (f_r) of real-valued functions is said to be *equistatistically convergent* to f on X if, for every $\epsilon > 0$, the sequence $(P_{n,\epsilon}(x))_{r\in\mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if (for every $\epsilon > 0$) we have $\lim_{n\to\infty} \|P_{n,\epsilon}(x)\|_{C(X)} = 0$, where

$$P_{n,\epsilon}(x) = \frac{1}{n}\big|\{k : k \leqslant n.\ |f_k(x) - f(x)| \geqslant \epsilon\}\big| = 0.$$

In this case, we write $f_k \rightsquigarrow f$ (equistat).

The following implications of the above definitions and concepts are trivial.

$$f_k \rightrightarrows f\ \ (\theta\text{-stat}) \Rightarrow f_k \rightsquigarrow f\ \ (\theta\text{-equistat}) \Rightarrow f_k \to f\ \ (\theta\text{-stat}).$$

Furthermore, in general, the reverse implications do not hold true.

2. Main Results

Let $I = [0, A]$, $J = [0, B]$, $A, B \in (0, 1)$ and $K = I \times J$. We denote by $C(K)$ the space of all continuous real valued functions on K. This space is equipped with the norm $\|f\|_{C(K)} := \sup_{(x,y)\in K} |f(x, y)|$, $\quad f \in C(K)$. Let $H_\omega(K)$ denote the space of all real valued functions f on K such that

$$|f(s,t) - f(x,y)| \leqslant \omega\left(f; \sqrt{\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 + \left(\frac{t}{1-t} - \frac{y}{1-y}\right)^2}\right),$$

where ω is the modulus of continuity, i.e.

$$\omega(f;\delta) = \sup_{(s,t),(x,y)\in K} \left\{\big|f(s,t) - f(x,y)\big| : \sqrt{(s-x)^2 + (t-y)^2} \leqslant \delta\right\}.$$

It is to be noted that any function $f \in H_\omega(K)$ is continuous and bounded on K.

In [**1**], Aktuğlu and Gezer proved the Korovkin theorem for lacunary eqistatistiacal convergence by using the test functions $1, x$ and x^2; while Kaya and Gönül [**11**] used the test functions 1, $\frac{x}{1+x}$, $(\frac{y}{1+y})$, $(\frac{x}{1+x})^2 + (\frac{y}{1+y})^2$. Recently, Srivastava et al [**19**] defined and studied the λ-eqistatistiacal convergence of positive linear operators by using the notion of λ-statistical convergence [**15**]. In this paper, we apply the notion of lacunary equistatistical convergence to prove a Korovkin type approximation theorem for functions of two variables by using test functions 1, $\frac{x}{1-x}$, $(\frac{y}{1-y})$, $(\frac{x}{1-x})^2 + (\frac{y}{1-y})^2$.

Let T be a linear operator which maps $C[a, b]$ into itself. We say that T is *positive* if, for every non-negative $f \in C[a, b]$, we have $T(f, x) \geqslant 0$ $(x \in [a, b])$.

We prove the following result:

THEOREM 2.1. *Let $\theta = (k_r)$ be a lacunary sequence, and let (L_r) be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Then for all $f \in H_\omega(K)$*

$$L_r(f; x, y) \rightsquigarrow f(x, y)\ (\theta\text{-equistat}) \tag{2.1}$$

if and only if

$$L_r(f; x, y) \rightsquigarrow g_i(x, y)\ (\theta\text{-equistat})\ (i = 0, 1, 2, 3), \tag{2.2}$$

with $g_0(x) = 1$, $g_1(x) = \frac{x}{1-x}$, $g_2(x) = \frac{y}{1-y}$ and $g_3(x) = \left(\frac{x}{1-x}\right)^2 + \left(\frac{y}{1-y}\right)^2$.

Proof. Since each of the functions f_i belongs to $H_\omega(K)$, conditions (2.2) follow immediately. Let $g \in H_\omega(K)$ and $(x, y) \in K$ be fixed. Then for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(s,t) - f(x,y)| < \varepsilon$ holds for all $(s,t) \in K$ satisfying $\left|\frac{s}{1-s} - \frac{x}{1-x}\right| < \delta_1$, $\left|\frac{t}{1-t} - \frac{y}{1-y}\right| < \delta_2$. Let

$$K(\delta_1, \delta_2) := \Big\{(s,t) \in K : \Big|\frac{s}{1-s} - \frac{x}{1-x}\Big| < \delta_1,\ \Big|\frac{t}{1-t} - \frac{y}{1-y}\Big| < \delta_2\Big\}.$$

Hence

$$\begin{aligned} |f(s,t) - f(x,y)| &= |f(s,t) - f(x,y)|_{\chi_{K(\delta_1,\delta_2)}(s,t)} \\ &\quad + |f(s,t) - f(x,y)|_{\chi_{K \setminus K(\delta_1,\delta_2)}(s,t)} \\ &\leqslant \varepsilon + 2N_{\chi_{K \setminus K(\delta_1,\delta_2)}(s,t)}, \end{aligned} \tag{2.3}$$

where χ_D denotes the characteristic function of the set D and $N = \|f\|_{C_B(K)}$ Further we get

$$\chi_{K \setminus K(\delta_1,\delta_2)}(s,t) \leqslant \frac{1}{\delta_1^2}\Big(\frac{s}{1-s} - \frac{x}{1-x}\Big)^2 + \frac{1}{\delta_2^2}\Big(\frac{t}{1-t} - \frac{y}{1-y}\Big)^2. \tag{2.4}$$

Combining (2.3) and (2.4) and choosing $\delta := \min\{\delta_1, \delta_2\}$, we get

$$|f(s,t) - f(x,y)| \leqslant \varepsilon + \frac{2N}{\delta^2}\Big\{\Big(\frac{s}{1-s} - \frac{x}{1-x}\Big)^2 + \Big(\frac{t}{1-t} - \frac{y}{1-y}\Big)^2\Big\}.$$

After using the linearity and positivity of operators $\{L_r\}$, we get

$$\begin{aligned} |L_r(f;x,y) - f(x,y)| \leqslant \varepsilon + M\{&|L_r(g_0;x,y) - g_0(x,y)| + |L_r(g_1;x,y) - g_1(x,y)| \\ &+ |L_r(g_2;x,y) - g_2(x,y)| + |L_r(g_3;x,y) - g_3(x,y)|\}, \end{aligned}$$

which implies that

$$|L_r(f;x,y) - f(x,y)| \leqslant \epsilon + B\sum_{i=0}^{3} |L_r(g_i;x,y) - g_i(x,y)|, \tag{2.5}$$

where $M := \varepsilon + N + \frac{4N}{\delta^2}$. Now for a given $\rho > 0$, choose $\epsilon > 0$ such that $\epsilon < \rho$. Then, for each $i = 0, 1, 2, 3$, set $\psi_\rho(x,y) := |\{k \in \mathbb{N} : |L_k(f;x,y) - f(x,y)| \geqslant \rho\}|$ and $\psi_{i,\rho}(x,y) := |\{k \in \mathbb{N} : |L_k(g_i;x,y) - g_i(x,y)| \geqslant \frac{\rho-\epsilon}{4K}\}|$ for ($i = 0, 1, 2, 3$), it follows from (2.5) that $\psi_\rho(x,y) \subseteq \bigcup_{i=0}^{3} \psi_{i,\rho}(x,y)$. Hence

$$\frac{\|\psi_\rho(x,y)\|_{C_B(K)}}{h_r} \leqslant \sum_{i=o}^{3}\Big(\frac{\|\psi_{i,\rho}(x,y)\|_{C_B(K)}}{h_r}\Big). \tag{2.6}$$

Now using hypothesis (2.2) and Definition 1.1, the right-hand side of (2.6) tends to zero as $r \to \infty$. Therefore, we have $\lim_{r\to\infty} \frac{1}{h_r}\|\psi_\rho(x,y)\|_{C_B(K)} = 0$ for every $\rho > 0$, i.e.. (2.1) holds. $\square$

Example 2.1. Consider the following Meyer-König and Zeller [**13**] (of two variables) operators:

$$B_{m,n}(f;x,y) := (1-x)^{m+1}(1-y)^{n+1}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\Big(\frac{j}{j+m+1}, \frac{k}{k+n+1}\Big)\binom{m+j}{j}\binom{n+k}{k} x^j y^k,$$

where $f \in H_\omega(K)$, and $K = [0, A] \times [0, B]$, $A, B \in (0, 1)$.

Since, for $x \in [0, A]$, $A \in (0, 1)$, we have $1/(1-x)^{n+1} = \sum_{k=0}^{\infty}\binom{n+k}{k}x^k$, it is easy to see that $B_{m,n}(g_0; x, y) = f_0(x, y)$. Also, we obtain

$$\begin{aligned}
B_{m,n}(g_1; x, y) &= (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\binom{m+j}{j}\binom{n+k}{k}x^j y^k \\
&= (1-x)^{m+1}(1-y)^{n+1}x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k \\
&= (1-x)^{m+1}(1-y)^{n+1}x\frac{1}{(1-x)^{m+2}}\frac{1}{(1-y)^{n+1}} = \frac{x}{(1-x)},
\end{aligned}$$

and similarly $B_{m,n}(g_2; x, y) = \frac{y}{(1-y)}$.

Finally, we get

$$\begin{aligned}
&B_{m,n}(g_3; x, y) \\
&\quad = (1-x)^{m+1}(1-y)^{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Big\{\Big(\frac{j}{m+1}\Big)^2 + \Big(\frac{k}{n+1}\Big)^2\Big\}\binom{m+j}{j}\binom{n+k}{k}x^j y^k \\
&\quad = (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{j}{m+1}\frac{(m+j)!}{m!(j-1)!}\binom{n+k}{k}x^{j-1}y^k \\
&\qquad + (1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{k}{n+1}\binom{m+j}{j}\frac{(n+k)!}{n!(k-1)!}x^j y^{k-1} \\
&\quad = (1-x)^{m+1}(1-y)^{n+1}\frac{x}{m+1}\Big\{x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(m+j+1)!}{(m+1)!(j-1)!}\binom{n+k}{k}x^{j-1}y^k \\
&\qquad\qquad + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+j+1}{j}\binom{n+k}{k}x^j y^k\Big\} \\
&\qquad + (1-x)^{m+1}(1-y)^{n+1}\frac{y}{n+1}\Big\{y\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k+1)!}{(n+1)!(k-1)!}\binom{m+j}{j}x^j y^{k-1} \\
&\qquad\qquad + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k+1}{k}\binom{m+j}{j}x^j y^k\Big\} \\
&\quad = \frac{m+2}{m+1}\Big(\frac{x}{1-x}\Big)^2 + \frac{1}{m+1}\frac{x}{1-x} + \frac{n+2}{n+1}\Big(\frac{y}{1-y}\Big)^2 + \frac{1}{n+1}\frac{y}{1-y} \\
&\qquad\qquad \to \Big(\frac{x}{1-x}\Big)^2 + \Big(\frac{y}{1-y}\Big)^2.
\end{aligned}$$

Therefore, we have $B_n(f_i; x, y) \rightsquigarrow g_i(x, y)$ (θ-equistat) ($i = 0, 1, 2, 3$). Hence by Theorem 2.1, we have $B_n(f; x, y) \rightsquigarrow g(x, y)$ (θ-equistat).

3. Rate of Lacunary Equistatistical Convergence

In this section we study the rate of lacunary equistatistical convergence of a sequence of positive linear operators as given in [**11**].

Definition 3.1. Let (a_n) be a positive non-increasing sequence. A sequence (f_r) is said to be *lacunary equistatistically convergent to a function* f *with the rate* β $(0<\beta<1)$ if for every $\epsilon>0$,

$$\lim_{r\to\infty}\frac{\Lambda_{r,\epsilon}(x,y)}{r^{-\beta}}=0$$

uniformly with respect to $(x,y)\in K$ or equivalently, for every $\epsilon>0$,

$$\lim_{r\to\infty}\frac{\|\Lambda_{r,\epsilon}(x,y)\|_{C_B(X)}}{r^{-\beta}}=0,$$

where

$$\Lambda_r(x,\epsilon):=\frac{1}{h_r}|\{k\in I_r:|f_k(x,y)-f(x,y)|\geqslant\varepsilon\}|=0.$$

In this case, we write $f_r-f=o(r^{-\beta})$ (θ-equistat) on K.

We have the following basic lemma.

Lemma 3.1. *Let (f_r) and (g_r) be sequences of functions belonging to $H_\omega(K)$. Assume that $f_r-f=o(r^{-\beta_1})$ (θ-equistat) on X and $g_r-g=o(r^{-\beta_2})$ (θ-equistat). Let $\beta=\min\{\beta_1,\beta_2\}$. Then the following statement holds:*

(i) $(f_r+g_r)-(f+g)=o(r^{-\beta})$ *(θ-equistat),*
(ii) $(f_r-f)(g_r-g)=o(r^{-\beta_1})$ *(θ-equistat),*
(iii) $\mu(f_r-f)=o(r^{-\beta_1})$ *(θ-equistat) for any real number μ,*
(iv) $\sqrt{|f_r-f|}=o(r^{-\beta_1})$ *(θ-equistat).*

We recall that the modulus of continuity of a function $f\in H_\omega(K)$ is defined by

$$\omega(f;\delta)=\sup_{s,x\in K}\{|f(s)-f(x)|:|s-x|\leqslant\delta\}\ \ (\delta>0).$$

Now we prove the following result.

Theorem 3.1. *Let $\{L_r\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Assume that the following conditions hold:*

(a) $L_r(g_0;x,y)-g_0=o(r^{-\beta_1})$ *(θ-equistat) on K,*
(b) $\omega(f;\delta_{r,x},\delta_{r,y})=o(r^{-\beta_2})$ *(θ-equistat) on K,*

where $\delta_{r,x}=\sqrt{L_r\big(\big(\big(\frac{s}{1-s}-\frac{x}{1-x}\big)^2,x\big)\big)}$ and $\delta_{r,y}=\sqrt{L_r\big(\big(\big(\frac{t}{1-t}-\frac{y}{1-y}\big)^2,y\big)\big)}$. Then for all $f\in H_\omega(K)$, we have $L_r(f;x,y)-f(x,y)=o(r^{-\beta})$ (θ-equistat) on K, where $\beta=\min\{\beta_1,\beta_2\}$.

Proof. Let $f\in H_\omega(K)$ and $(x,y)\in K$. Then it is well known that,

$$|L_r(f;x,y)-f(x,y)|\leqslant M|L_r(g_0;x,y)-g_0(x,y)| + \big(L_r(g_0;x,y)+\sqrt{L_r(g_0;x,y)}\big)\omega(f;\delta_{r,x},\delta_{r,y}),$$

where $M = \|f\|_{H_\omega(K)}$. This yields that

$$\begin{aligned}|L_r(f;x,y) - f(x,y)| \leqslant M|(L_r(g_0;x,y) - g_0(x,y)| + 2\omega(f;\delta_{r,x},\delta_{r,y}) \\ + \omega(f;\delta_{r,x},\delta_{r,y})|(L_r(g_0;x,y) - g_0(x,y)|.\end{aligned}$$

Now using the conditions (a), (b) and Lemma 3.1 in the above inequality, we get $L_r(f) - f = o(r^{-\beta})$ (θ-equistat) on K. □

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