PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 103(117) (2018), 103–112 DOI: https://doi.org/10.2298/PIM1817103K

A NOTE ON RECURRENT BIVECTORS IN 4-DIMENSIONAL LORENTZ MANIFOLDS

Bahar Kırık

Dedicated to the memory of Professor Mileva Prvanović

Abstract. We study recurrence properties of the second order skew-symmetric tensor fields, which are referred to as bivectors, on a 4-dimensional manifold admitting a Lorentz metric. Considering the known classification scheme for these tensor fields, recurrent bivectors which can be scaled to be parallel are first determined and these results are associated with the holonomy theory. This examination then identifies proper recurrence of such bivectors on the manifold. The link between these bivectors and the holonomy group is investigated and some theorems are proved.

1. Introduction

Let M be a smooth, connected manifold admitting a smooth metric g of Lorentz signature $(+,+,+,-)$. Then, (M,g) is called a *space-time* and these are important in Einstein's general theory of relativity. It will be assumed that (M,g) is not flat and throughout the following, ∇ will denote the Levi-Civita connection of (M,g). The tangent space to M at $m \in M$ is denoted by T_mM and the inner product $g(u,v)$ arising from $g(m)$ will be written as $u.v$ for $u,v \in T_mM$. A nonzero member u of T_mM is called *spacelike*, *timelike* or *null* if $u.u > 0$, $u.u < 0$ or $u.u = 0$, respectively. For this signature, we can choose a pseudo orthonormal basis of mutually orthogonal vectors x,y,z,t for T_mM such that

$$x.x = y.y = z.z = -t.t = 1.$$

Alternatively, an associated *null* basis l,n,x,y can be chosen with $\sqrt{2}l = z+t$, $\sqrt{2}n = z-t$ so that l and n are null vectors satisfying $l.n = 1$. Also, one can build up a *complex null* basis $l,n,s,\bar{s}$ where l and n are as above, $\sqrt{2}s = x+iy$, $\bar{s}$ is the complex conjugate of s and so s and $\bar{s}$ are complex and null. The Riemann curvature tensor arising from ∇ is denoted by *Riem* with components $R^a{}_{bcd}$ and one has the type (0, 4) curvature tensor with components $R_{abcd} = g_{ae}R^e{}_{bcd}$. A spanning set will be denoted by the symbol $\langle\ \rangle$.

2010 *Mathematics Subject Classification:* 53C29; 53C50.
Key words and phrases: bivector, recurrent tensor, Lorentz signature, holonomy.

Looking in the literature, there has been much interest in the geometry of 4-dimensional manifolds which admit different metric signatures. More precisely, for a 4-dimensional manifold, these signatures can only be positive definite or Lorentz or $(+,+,-,-)$ which is referred to as neutral signature (see, e.g., [**4, 7, 9, 22**]). Recurrent tensor fields are one of the geometric objects which attract considerable attention and these tensor fields have been studied by many authors in various contexts (for example, see [**2, 3, 6, 10, 12, 16, 17, 21**]). Besides, recurrent and parallel vector fields have a significant place in holonomy theory (see Section 2.3) and in this perspective, these vector fields were investigated in, e.g., [**3, 5, 6, 10, 12**]. On the other hand, a second order symmetric tensor can be classified by finding all possible Jordan canonical forms and Segre types. For each signature, these classifications are known and the schemes can be found in, e.g., [**5, 8, 18**]. Some studies on the recurrence structure for second order symmetric tensor fields and some applications of recurrent tensors to the Ricci tensor and the curvature tensor can be found, for instance, in [**2, 6, 10, 12, 16, 17, 21**]. This paper studies the features of recurrence for second order skew-symmetric tensors called bivectors. It will be based on the classification of these tensor fields on a 4-dimensional Lorentz manifold and these analyses will be combined with holonomy theory. Also, some brief remarks will be given for signature $(+,+,+,+)$.

2. Some preliminaries

In this section, some basic concepts about bivectors and their classification on 4-dimensional Lorentz manifolds (including brief remarks for other metric signatures), recurrent tensors and holonomy structure are given.

2.1. Bivectors and their classification. The 6-dimensional vector space of all 2-forms at $m \in M$, which will be denoted by $\Lambda_m M$, is a Lie algebra under matrix commutation denoted by $[\ ]$. A member F of $\Lambda_m M$ is referred to as *bivector* and if we denote the components of F by F_{ab} then $F_{ab} = -F_{ba}$ and so the rank of F must be even. If this rank is 2, F is called a *simple bivector* and if it is 4, then F is called a *nonsimple bivector*. If F is simple, then it can be written as $F^{ab} = u^a v^b - v^a u^b$ where $u, v \in T_m M$. The 2-space spanned by u and v is uniquely determined by F and it is called the *blade* of F. In this case, F or its blade will be denoted by $u \wedge v$. A simple bivector is called *spacelike* (respectively, *timelike* or *null*) if its blade is a *spacelike* (each nonzero member of it is spacelike) (respectively, *timelike* (it contains exactly two distinct null directions) or *null* (it contains exactly one null direction)) 2-space at m.

When g has Lorentz signature, the classification of bivectors is known from general relativity theory and in a null basis l, n, x, y at $m \in M$, the canonical forms and corresponding Segre types are given as follows (for details we refer to, e.g., [**19, 5, 8**]).

$$F_{ab} = \alpha(x_a y_b - y_a x_b) \ \ \text{(spacelike, Segre type } \{(11)z\bar{z}\}) \tag{2.1}$$

$$F_{ab} = \alpha(l_a n_b - n_a l_b) \ \ \text{(timelike, Segre type } \{11(11)\}) \tag{2.2}$$

$$F_{ab} = \alpha(l_a x_b - x_a l_b) \ \ \text{(null, Segre type } \{(31)\}) \tag{2.3}$$

$$F_{ab} = \alpha(l_a n_b - n_a l_b) + \beta(x_a y_b - y_a x_b) \quad \text{(nonsimple, Segre type}\{11z\bar{z}\}) \tag{2.4}$$

where $\alpha, \beta \in \mathbb{R}$ and for the nonsimple case $\alpha \neq 0 \neq \beta$.

Moreover, if the metric is positive definite, all simple bivectors are spacelike and the Segre type is $\{(11)z\bar{z}\}$. If F is nonsimple, in an orthonormal basis x, y, z, w at m, it can be characterized by $F = \alpha(x \wedge y) + \beta(z \wedge w)$ ($\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0 \neq \beta$) (see, e.g., [**14**]). In this case, F has a pair of complex eigenvectors (eigenvalues) $x \pm iy$ ($\pm i\alpha$) and $z \pm iw$ ($\pm i\beta$) and its Segre type is written $\{z\bar{z}w\bar{w}\}$ (or $\{(zz)(\bar{z}\bar{z})\}$ if $\alpha = \pm\beta$).

When g has neutral signature, such a classification has been done in [**8**] but it will not be needed here.

2.2. Recurrent tensor fields. A global, smooth tensor field T on M is called a *recurrent tensor* if $\nabla T = T \otimes \lambda$ for some 1-form λ which is necessarily smooth on M. It is first useful to remark that since a recurrent tensor need not be nowhere-zero on the manifold M, it will be assumed that T is nowhere-zero on the nonempty, connected, open subset U on which it is studied (for details of this section see [**12**, p. 263]). Geometrically, any recurrent tensor has the property that given any $m, m' \in U$ and curve $m \to m'$ in U the value of T at m' is proportional to the parallel transport of $T(m)$ along c at m' and the proportionality ratio depends on c and λ. If the 1-form λ vanishes on U, T is called *parallel* (or *covariantly constant*). Let T be recurrent on U and suppose that there exists a nowhere-zero function φ on U such that φT is parallel on U. Then λ is the gradient $\nabla(-\log|\varphi|)$. Conversely, if λ is a gradient, that is, $\lambda = \nabla\psi$ on U then $e^{-\psi}T$ is parallel on U. This allows us to give a definition of proper recurrence for bivectors in the following way. The Ricci identity for a nowhere-zero recurrent bivector F is given by as follows:

$$(\nabla_d\nabla_c - \nabla_c\nabla_d)F_{ab} = F_{ae}R^e{}_{bcd} + F_{eb}R^e{}_{acd} = F_{ab}(\nabla_d\lambda_c - \nabla_c\lambda_d). \tag{2.5}$$

It can be seen from (2.5) that if $F_{ae}R^e{}_{bcd} + F_{eb}R^e{}_{acd}$ vanishes on U, then λ is a gradient on some neighbourhood N of m and hence the bivector F can be scaled to be parallel on N. According to this, a bivector F will be called *properly recurrent on* U if the subset $\mathcal{V} \equiv \{m \in U : (F_{ae}R^e{}_{bcd} + F_{eb}R^e{}_{acd})(m) \neq 0\}$ is open and dense in the subspace topology on U. A similar definition can be given for a nowhere-zero recurrent vector field on U (see [**12**]). Also, it can be shown that the Segre type of F, including degeneracies, is the same at each point of U and the eigenvalues of F can be regarded as constant functions on U (see [**13**]). Another useful remark is that if a bivector F is recurrent and nowhere-zero on U and if $F^{ab}F_{ab} \neq 0$ at $m \in U$, then $F^{ab}F_{ab}$ is nowhere-zero on U and a contraction of the recurrence condition, $\nabla_c F_{ab} = \lambda_c F_{ab}$, with F^{ab} shows that λ is the gradient of $\frac{1}{2}(\log|F_{ab}F^{ab}|)$ and so F can be scaled to be parallel on U. Hence, only bivectors satisfying $F^{ab}F_{ab} = 0$ on U may be properly recurrent. So, if the metric is positive definite, any recurrent bivector can be scaled to be parallel. As a last remark, it can be seen that if a bivector F is recurrent, the dual bivector $\overset{*}{F}$ is also recurrent (see [**10**]).

2.3. Holonomy structure. The holonomy group of (M, g) (more precisely of ∇) denoted by Φ is a Lie group and so it has a Lie algebra ϕ (for details, see [**15**]).

TABLE 1. Holonomy algebras for $(+,+,+,-)$

Type	Basis	Parallel vector fields	Recurrent vector fields	Type	Basis	Parallel vector fields	Recurrent vector fields
R_2	$l\wedge n$	$\langle x,y\rangle$	$l,\, n$	R_9	$l\wedge n,\, l\wedge x,\, l\wedge y$	—	l
R_3	$l\wedge x$	$\langle l,y\rangle$	—	R_{10}	$l\wedge n,\, l\wedge x,\, n\wedge x$	$\langle y\rangle$	—
R_4	$x\wedge y$	$\langle l,n\rangle$	—	R_{11}	$l\wedge x,\, l\wedge y,\, x\wedge y$	$\langle l\rangle$	—
R_5	$l\wedge n+\omega(x\wedge y)$	—	—	R_{12}	$l\wedge x,\, l\wedge y,\, l\wedge n+\omega(x\wedge y)$	—	l
R_6	$l\wedge n,\, l\wedge x$	$\langle y\rangle$	l	R_{13}	$x\wedge y,\, y\wedge z,\, x\wedge z$	$\langle t\rangle$	—
R_7	$l\wedge n,\, x\wedge y$	—	$l,\, n$	R_{14}	$l\wedge n,\, l\wedge x,\, l\wedge y,\, x\wedge y$	—	l
R_8	$l\wedge x,\, l\wedge y$	$\langle l\rangle$	—	R_{15}	$o(1,3)$	—	—

When g has Lorentz signature, then ϕ is a subalgebra of the orthogonal algebra of g, that is, $o(1,3)$. The possible holonomy algebras in bivector representation are known (see, e.g., [**5, 9, 10, 20**]) and using the labellings R_1 (flat), $R_2,\ldots,R_{15}$ given in [**20**], these algebras are presented in Table 1. Here, $0\neq\omega\in\mathbb{R}$. Additionally, if $0\neq k\in T_mM$ is an eigenvector of each member of ϕ, then there exists a local recurrent vector field which is smooth on some neighbourhood of the point m and whose value at m is k. Further, if each eigenvalue for k is zero for all $F\in\phi$, then this vector field can be chosen to be parallel. Thus, recurrent and parallel vector fields are shown in Table 1.

3. Recurrence structures of real bivectors

In this section, the recurrence structures of real bivectors are studied on 4-dimensional Lorentz manifolds. As discussed in Section 2.2, if a recurrent bivector satisfies $F^{ab}F_{ab}\neq 0$, then F can be scaled to be parallel. So, let us first investigate the solutions of $\nabla F=0$. Suppose that F is spacelike and parallel. Then, taking the covariant derivative of (2.1) and considering $\nabla F=0$, a contraction of the resulting equation by x^ay^b shows that $\nabla\alpha=0$, i.e., α is constant on U. Thus, we get

$$(\nabla_c x_a)y_b+x_a\nabla_c y_b-(\nabla_c y_a)x_b-y_a\nabla_c x_b=0. \tag{3.1}$$

By contracting (3.1) with x^al^b and x^an^b, we obtain, $l^b\nabla_c y_b=n^b\nabla_c y_b=0$ (or $y^b\nabla_c l_b=y^b\nabla_c n_b=0$ since $x^ax_a=1$, $l^ax_a=l^ay_a=n^ay_a=0$). Applying these to the derivative $\nabla_c l_b=e_cx_b+f_cy_b+h_cl_b$ (for some 1-forms e,f,h on U) give $e_c=f_c=0$ and so, one has $\nabla_c l_b=h_cl_b$. Hence, l is recurrent. Performing similar contractions and considering the basic conditions, the other tetrad derivatives are found $\nabla_c n_b=-h_cn_b$ (and so n is recurrent), $\nabla_c x_b=q_cy_b$ and $\nabla_c y_b=-q_cx_b$ for some 1-form q on U. Moreover, from the derivatives of x and y, one has $\nabla_c(x_b\pm iy_b)=\mp iq_c(x_b\pm iy_b)$, in other words $x\pm iy$ are complex, recurrent null vector fields. From Table 1, the possible holonomy types are R_2, R_4 (here l and n may be chosen parallel) or R_7. The case when F is timelike [that is, $F=\alpha(l\wedge n)$ from (2.2)] and parallel is completely analogous. In this case, $\overset{*}{F}=\alpha(x\wedge y)$ and $\nabla F=0$, then $\nabla\overset{*}{F}=0$ and hence one gets the expressed holonomy types as above. Conversely, for each of these holonomy types the bivectors $x\wedge y$ and $l\wedge n$ are easily checked to be parallel.

Now, let $F = l \wedge x'$ (null) be parallel with $l.x' = 0$ and so we can choose a tetrad l, n, x, y and then $x' = ax + by$ $(a, b \in \mathbb{R})$. Then the covariant derivative F contracted with l^a gives $l^a(\nabla_c x'_a) = -x'^a(\nabla_c l_a) = 0$ whilst a contraction with x'^a shows that l is recurrent with recurrence 1-form $\nabla[-\frac{1}{2}\log(x'.x')]$ which is a gradient and so l can be scaled to be parallel. This means that if F is null, M admits a nowhere-zero, parallel, null eigenvector l of F (cf. [**10**]). Then, from Table 1, the only possible holonomy types of M are R_3, R_4, R_8 or R_{11}. On the other hand, from (2.5) the condition for the commutator of F,

$$F_{ae}R^e{}_{bcd} + F_{eb}R^e{}_{acd} = F_{ae}R^e{}_{bcd} - F_{be}R^e{}_{acd} = 0 \tag{3.2}$$

must be satisfied for $F = l \wedge x'$. This condition is equivalent to $[F, G] = 0$ where G is any bivector in the range space of the curvature map f defined by $f : \Lambda_m M \to \Lambda_m M$ given by $G^{ab} \to R^{ab}{}_{cd}G^{cd}$ and which holds immediately for the holonomy type R_3. In fact, Table 1 shows that l and y are parallel vector fields for holonomy type R_3. Then the bivector $l \wedge y$ is parallel and so R_3 admits a parallel null bivector. For holonomy type R_4 with algebra $G = x \wedge y$, we have $[F, G] \neq 0$ and hence, condition (3.2) is not satisfied. For holonomy type R_8, an exponentiation from the algebra gives $\nabla_b x_a = l_a r_b$ for some 1-form r and then $l \wedge x$ is parallel. For holonomy type R_{11}, one has

$$\begin{aligned} R_{abcd} = A_1 G_{ab}G_{cd} + A_2 H_{ab}H_{cd} + A_3 J_{ab}J_{cd} + A_4(G_{ab}H_{cd} + H_{ab}G_{cd}) \\ + A_5(G_{ab}J_{cd} + J_{ab}G_{cd}) + A_6(H_{ab}J_{cd} + J_{ab}H_{cd}) \end{aligned} \tag{3.3}$$

where A_i $(i = 1, ..., 6)$ are smooth functions, $G = l \wedge x$, $H = l \wedge y$ and $J = x \wedge y$. Using (3.3) in the commutator of $F = l \wedge x'$ and contracting the resulting equation with $n^a y^b$ and $n^b x^a$ shows that $A_3 = A_5 = A_6 = 0$ and the range space of the curvature map is spanned by the bivectors G and H. However, the Ambrose–Singer theorem [**1**] says that if one fixes $m \in M$ and for any $m' \in M$ computes the range space, rg$f$, of the curvature map and parallel transports rg$f$ to m along some curve $c : m' \to m$ and does this for all such m' and c, the collection of bivectors obtained at m spans ϕ. Therefore, since l is recurrent, the Ambrose-Singer theorem gives the contradiction that ϕ contains only null members whose blade contains l. Hence, R_3 and R_8 are the only possible holonomy types.

Next, let F be nonsimple and parallel. Then, $\nabla F = \nabla \overset{*}{F} = 0$ are satisfied for $F = \alpha(l \wedge n) + \beta(x \wedge y)$ and $\overset{*}{F} = \alpha(x \wedge y) - \beta(l \wedge n)$ $(\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0 \neq \beta)$ in some basis l, n, x, y. Under these conditions, we get $F_{ab}F^{ab}$ and $F_{ab}\overset{*}{F}{}^{ab}$ are constant. This gives $\alpha^2 - \beta^2$ and $\alpha\beta$ constant. Now, $(\alpha^2 + \beta^2)^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2$ and so $\alpha^2 + \beta^2$ is constant. Hence, α and β are constant and so $\nabla(x \wedge y) = \nabla(l \wedge n) = 0$. Therefore, l and n are recurrent whose recurrence 1-forms differ only in sign, $x \pm iy$ are complex recurrent vector fields and the possible holonomy types are R_2, R_4 or R_7 from an earlier result.

Now, suppose that F is nowhere-zero and *properly recurrent* on U, that is, $\nabla_c F_{ab} = \lambda_c F_{ab}$ for some recurrence 1-form λ on U. If F is spacelike or timelike, then $F^{ab}F_{ab} \neq 0$. So, the proper recurrence for such bivectors is not possible here. If F is nonsimple, using (2.4), one gets $F^{ab}F_{ab} = 2(\beta^2 - \alpha^2)$. One can deduce from here that if $\alpha \neq \pm\beta$ then $F^{ab}F_{ab} \neq 0$. This yields that λ is a gradient and hence F can be scaled to be parallel. Thus, for the proper recurrence of nonsimple bivectors, we only need to check the case when $\alpha = \pm\beta$ (that is, the case $F^{ab}F_{ab} = 0$). With $\alpha = \beta$ (the case $\alpha = -\beta$ will be similar) we have $F = G + \overset{*}{G}$, $\overset{*}{F} = \overset{*}{G} - G$ with G, $\overset{*}{G}$ simple and nonnull, $G = \alpha(l \wedge n)$, $\overset{*}{G} = \alpha(x \wedge y)$. Then $\nabla_c F_{ab} = \lambda_c F_{ab}$, $\nabla_c \overset{*}{F}_{ab} = \lambda_c \overset{*}{F}_{ab}$ and so $\nabla_c G_{ab} = \lambda_c G_{ab}$, $\nabla_c \overset{*}{G}_{ab} = \lambda_c \overset{*}{G}_{ab}$ and since $G^{ab}G_{ab} \neq 0$, λ is a gradient. As a result, a nonsimple bivector cannot be properly recurrent and so the only possibility for proper recurrence is the null bivector case.

In that case, let us suppose that F is null with $F = l \wedge x$ in some tetrad l, n, x, y. Then if F is recurrent, i.e., $\nabla_c(l_a x_b - x_a l_b) = (l_a x_b - x_a l_b)\lambda_c$. Contractions with l^a and x^a show that l is recurrent with a recurrence 1-form equal to λ. So, if λ is not a gradient, the null bivector F is properly recurrent and this yields a properly recurrent null vector l. From Table 1, the potential holonomy types are R_2, R_6, R_7, R_9, R_{12} or R_{14}. For holonomy type R_2, $l \wedge x$ is clearly properly recurrent. For holonomy types R_6 and R_9 exponentiation from the algebra gives in each case $\nabla_b x_a = l_a r_b$ for some 1-form r and so $l \wedge x$ is properly recurrent for these types also.

Now consider types R_7, R_{12} and R_{14}. For types R_7 and R_{14}, the argument given earlier using the Ambrose-Singer theorem and the recurrence of l shows that $x \wedge y$ must appear in the range of the curvature map at some $m \in M$ and hence in some open neighbourhood U of m. Now (3.2) adapted to the case when $F = l \wedge x'$ is properly recurrent shows that a necessary condition for proper recurrence of F is that $[F, T]$ is proportional to F for each T in the range of f. This fails for the choice $T = x \wedge y$. A similar argument rules out for the R_{12} case because of the bivector $l \wedge n + \omega(x \wedge y)$ (which, as above, must be in the range of f at some $m \in M$).

All these results proved above are put together in the following theorem.

Theorem 3.1. *Let M be a smooth, connected, 4-dimensional manifold admitting a Lorentz metric and F be a nowhere-zero, recurrent bivector on some open subset U of M. Then, the following conditions hold.*

(i) *If $\nabla F = 0$ with F nonnull (simple or nonsimple), then the holonomy type is either R_2, R_4 or R_7.*
(ii) *If $\nabla F = 0$ with F is null, then the holonomy type is either R_3 or R_8.*
(iii) *If F is properly recurrent on U, then F must be null and the holonomy type is either R_2, R_6 or R_9.*

4. Complex bivectors

In this section, complex bivectors on 4-dimensional Lorentz manifolds will be considered and the above idea about the recurrence structure will be expanded to these bivectors. Let the set of complex bivectors at the point m be denoted by CB(m) which is a 6-dimensional complex vector space. Define 3-dimensional subspaces of CB(m) by $\overset{+}{Z}_m \equiv \{G \in \mathrm{CB}(m) : \overset{*}{G} = -iG\}$ and $\overset{-}{Z}_m \equiv \{G \in \mathrm{CB}(m) : \overset{*}{G} = iG\}$ where $*$ is the Hodge duality operator defined for members of this complex vector space. Thus, one has $\mathrm{CB}(m) = \overset{+}{Z}_m \oplus \overset{-}{Z}_m$. A member of $\overset{+}{Z}_m$ is called a *self-dual bivector* and it can be written in the form $\overset{+}{F} \equiv F + i\overset{*}{F}$ for a real bivector F. Furthermore, a member of $\overset{-}{Z}_m$ is called an *anti self-dual bivector* and can be written in the form $\overset{-}{F} \equiv F - i\overset{*}{F}$ for a real bivector F. It is clear that $\overset{+}{F}$ and $\overset{-}{F}$ are conjugates. According to these, $\overset{+}{F} = l \wedge n + i(x \wedge y)$ and $\overset{-}{F} = l \wedge n - i(x \wedge y)$ are examples of self-dual and anti self-dual bivectors, respectively.

On the other hand, using the complex null basis $l, n, s, \bar{s}$ expressed in Section 1, one has a basis V, W, Z for $\overset{+}{Z}_m$ which are defined, respectively, by $V = l \wedge \bar{s}$, $W = n \wedge s$ and $Z = l \wedge n + \bar{s} \wedge s$. Then, a basis for $\overset{-}{Z}_m$ is $\bar{V}$, $\bar{W}$, $\bar{Z}$, that is, the conjugates of V, W, Z (for details, see [**5**, pp. 177–178]). In this case, the following conditions between these bivectors are satisfied:

$$W_{ab}V^{ab} = \bar{W}_{ab}\bar{V}^{ab} = 2, \quad Z_{ab}Z^{ab} = \bar{Z}_{ab}\bar{Z}^{ab} = -4$$

and the other such contractions between any two of them are all zero.

Like the classification of real bivectors given in Section 2.1, one can classify the complex bivectors as follows (some details can be found in [**11**]). If $\overset{+}{F}$ is a *simple* member of $\overset{+}{Z}_m$, then its blade is *totally null*, that is, it is spanned by a pair of orthogonal, complex null vectors (the same applies to $\overset{-}{Z}_m$). Therefore, any simple member $\overset{+}{F} \in \overset{+}{Z}_m$ can be written as $\overset{+}{F} = p \wedge q$ for complex vectors p and q which satisfy $p.p = q.q = p.q = 0$. It can be shown that up to a (complex) scaling, the blade of a complex totally null bivector has a unique real null direction (see [**11**]). Let k be the unique (up to scaling) real null vector satisfying

$$\overset{+}{F}_{ab}k^b = 0. \tag{4.1}$$

Then, for the real bivectors F and $\overset{*}{F}$, the conditions $F_{ab}k^b = 0$ and $\overset{*}{F}_{ab}k^b = 0$ hold. This shows that F, $\overset{*}{F}$ are *real null* bivectors and also k is the (real) unique, up to scaling, common eigenvector of these bivectors and it is null. In this case, the self-dual bivector $\overset{+}{F}$ equipped with these conditions will be called *null*, otherwise it will be called *nonnull*. In the latter case, F and $\overset{*}{F}$ are both nonnull. According to these classifications, $\overset{+}{F} = l \wedge n + i(x \wedge y)$ and $\overset{+}{G} = l \wedge x - i(l \wedge y)$ are examples of nonnull and null self-dual bivectors, respectively.

With the inspiration from the real case, we shall define the recurrence structure for self-dual bivectors (similarly, it can be done for anti self-dual bivectors). A self-dual bivector is called *complex recurrent* if the condition

$$\nabla_c \overset{+}{F}_{ab} = \overset{+}{F}_{ab}P_c \tag{4.2}$$

is satisfied for some complex 1-form P. Then we have from (4.2)

$$\nabla_c F_{ab} = F_{ab}\xi_c - \overset{*}{F}_{ab}\mu_c, \quad \nabla_c \overset{*}{F}_{ab} = F_{ab}\mu_c + \overset{*}{F}_{ab}\xi_c \tag{4.3}$$

where $P = \xi + i\mu$ with ξ, μ being real 1-forms. It can be seen from (4.3) that if the imaginary part of the complex 1-form P is zero then the real bivectors F and $\overset{*}{F}$ are recurrent and the converse is also true. Thus, we have the following lemma.

LEMMA 4.1. *Let a self-dual (or anti self-dual) bivector be complex recurrent. A necessary and sufficient condition for its real and imaginary parts to be recurrent is that the recurrence* 1*-form of the self-dual (or anti self-dual) bivector is real.*

On the other hand, for a self-dual bivector, one can calculate that

$$\overset{+}{F}_{ab}\overset{+}{F}^{ab} = 2F_{ab}F^{ab} + 2iF_{ab}\overset{*}{F}^{ab} \tag{4.4}$$

since $F_{ab}F^{ab} = -\overset{*}{F}_{ab}\overset{*}{F}^{ab}$ for Lorentz signature. It is useful to note that a real bivector F is simple if and only if $\overset{*}{F}$ is simple if and only if $\overset{*}{F}_{ab}F^{ab} = 0$ (see [**5**, pp. 174–175]). Therefore, we obtain from (4.4) that F (and $\overset{*}{F}$) is simple if and only if $\overset{+}{F}_{ab}\overset{+}{F}^{ab} = 2F_{ab}F^{ab}$, that is, $\overset{+}{F}_{ab}\overset{+}{F}^{ab}$ is real. In addition to these, we can conclude that F and $\overset{*}{F}$ are (real) null bivectors if and only if $F_{ab}F^{ab} = F_{ab}\overset{*}{F}^{ab} = 0 \Leftrightarrow \overset{+}{F}_{ab}\overset{+}{F}^{ab} = 0$, that is, $\overset{+}{F}$ is null. Besides this, $\overset{+}{F}$ is nonnull if and only if $\overset{+}{F}_{ab}\overset{+}{F}^{ab} \neq 0$.

Now suppose that $\overset{+}{F}$ is complex recurrent with recurrence 1-form $P = \xi + i\mu$. Let the real and imaginary parts of P be the gradients of some functions. In other words, assume that there exists some nowhere zero functions η and θ on U such that $\xi = \nabla\eta$ and $\mu = \nabla\theta$. Then, using (4.3), it can obtained that the real and imaginary parts of $e^{-(\eta+i\theta)}\overset{+}{F} = e^{-\eta}[(\cos\theta)F + (\sin\theta)\overset{*}{F}] + ie^{-\eta}[(\cos\theta)\overset{*}{F} - (\sin\theta)F]$ are both parallel. This means that a complex recurrent bivector with a (complex) recurrence 1-form P whose real and imaginary parts are gradients can be scaled to be a complex parallel bivector whose real and imaginary parts are parallel.

Firstly, assume that $\overset{+}{F}$ is nonnull and complex recurrent. Then, $\overset{+}{F}_{ab}\overset{+}{F}^{ab} \neq 0$ and (4.2) gives that $\nabla_c(\overset{+}{F}_{ab}\overset{+}{F}^{ab}) = 2P_c(\overset{+}{F}_{ab}\overset{+}{F}^{ab})$. So, P is a complex gradient and, according to the previous argument, $\overset{+}{F}$ can be scaled to a complex parallel bivector. Thus, only complex self-dual bivectors satisfying $\overset{+}{F}_{ab}\overset{+}{F}^{ab} = 0$ (so $\overset{+}{F}$ is null) may be properly (complex) recurrent.

Let us now suppose that $\overset{+}{F}$ is null and complex recurrent. Taking the covariant derivative of (4.1) and using (4.2), we get $\overset{+}{F}_{ab}\nabla_c k^b = 0$. Then, we obtain

$$\overset{*}{F}_{ab}(\nabla_c k^b)q^c = F_{ab}(\nabla_c k^b)q^c = 0$$

for all $q \in T_mM$. Since k is the unique null direction of the real bivectors F and $\overset{*}{F}$ (and it is the unique null direction of $\overset{+}{F}$ at the same time) and $\overset{+}{F}$ is complex recurrent, the parallel propagation preserves k being null and due to the discreteness of the real null eigenvector, k must be recurrent. Therefore, if a self-dual null bivector is complex recurrent, then there exists a (real) recurrent null vector field.

Conversely, let k be a recurrent null vector field satisfying equation (4.1). Then, $\nabla_b k_a = k_a r_b$ for some (real) 1-form r. In this case, the covariant derivative of (4.1) gives $(\nabla_c \overset{+}{F}_{ab})k^b = 0$ (and so $(\nabla_c F_{ab})k^b = (\nabla_c \overset{*}{F}_{ab})k^b = 0$). Therefore, $(\nabla_c \overset{+}{F}_{ab})k^b q^c = 0$ for all q. So, $(\nabla_c \overset{+}{F}_{ab})q^c$ is a self-dual bivector and it can be written in a linear combination of the basis V, W, Z for $\overset{+}{Z}_m$ given earlier with $k = l$. A contraction with k^b then shows that $(\nabla_c \overset{+}{F}_{ab})q^c$ is a multiple of the bivector $\overset{+}{F}$ for all q. This shows that $\overset{+}{F}$ is complex recurrent. Similar results can be done for $\overset{-}{F}$. Hence, we have the following theorem.

Theorem 4.1. *Let M be a smooth, connected, 4-dimensional manifold admitting a Lorentz metric and $\overset{+}{F}$ (or $\overset{-}{F}$) be a nowhere-zero, self-dual (or anti self-dual), complex null bivector on some open subset U of M. Then, $\overset{+}{F}$ (or $\overset{-}{F}$) is complex recurrent if and only if there exists a real, null, recurrent vector field on U.*

According to the above theorem, we proved that the existence of a real, null, recurrent vector field is equivalent to the existence of a complex recurrent, self-dual, null bivector. However, this equivalence cannot hold for the real null bivector case as we showed in Section 3. More precisely, we proved that a real, recurrent null bivector implies a real, recurrent null vector field but the converse is not true. For instance, for holonomy types R_7, R_{12} and R_{14}, l is a recurrent (null) vector field but using the Ambrose-Singer theorem we showed the nonexistence of a real null recurrent bivector for these holonomy types. In addition to these, for holonomy types R_2, R_4 and R_7, $\overset{+}{F} = l \wedge n + i(x \wedge y)$ is a nonnull, self-dual bivector and it is (complex) parallel because of the fact that for each of these holonomy types, $\nabla_c l_a = r_c l_a$, $\nabla_c n_a = -r_c n_a$, $\nabla_c x_a = q_c y_a$ and $\nabla_c y_a = -q_c x_a$ for some 1-forms r and q and so $\nabla(l \wedge n) = \nabla(x \wedge y) = 0$. Moreover, for these holonomy types, there is a real, null recurrent vector field (in fact, there are two; l and n which are parallel for type R_4 and which are properly recurrent for types R_2 and R_7) and so there exists a complex recurrent null bivector from Theorem 4.1. Hence, the existence of a nonnull complex recurrent bivector implies the existence of a complex recurrent null bivector. Besides, for holonomy type R_2, the self-dual bivector $\overset{+}{F} = l \wedge x - i(l \wedge y)$ is null and complex recurrent with recurrence 1-form real and equal to that of l. Also, for holonomy type R_6, by remembering the exponentiation from the algebra, one has $\nabla_b x_a = l_a q_b$ for some 1-form q and then $\overset{+}{F}$ is a (null) complex recurrent bivector with a real recurrence 1-form. For holonomy types R_{11}, R_{12} and R_{14}, the bivector $l \wedge x - i(l \wedge y)$ is complex, null and complex recurrent, but no real null recurrent bivector exists for these holonomy types.

Acknowledgements. The author would like to express her sincere thanks to Professor Graham S. Hall for his valuable comments and suggestions. She also wishes to thank the organizers of 19th Geometrical Seminar for their wonderful hospitality and financial support.

References

1. W. Ambrose, I. Singer, *A theorem on holonomy*, Trans. Am. Math. Soc. **75** (1953), 428.
2. D. K. Datta, *Some theorems on symmetric recurrent tensors of the second order*, Tensor (N.S.) **15** (1964), 61–65.
3. ______, *Recurrent tensors and holonomy group*, Bull. Aust. Math. Soc. **10** (1974), 71–77.
4. R. Ghanam, G. Thompson, *The holonomy Lie algebras of neutral metrics in dimension four*, J. Math. Phys. **42**(5) (2001), 2266–2284.
5. G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific, 2004.
6. ______, *Recurrence conditions in space-time*, J. Phys A. **10**(1) (1977), 29–42.
7. ______, *Four-dimensional Ricci-flat manifolds which admit a metric*, Filomat **29**(3) (2015), 563–571.
8. ______, *Some general, algebraic remarks on tensor classification, the group $O(2,2)$ and sectional curvature in* 4*-dimensional manifolds of neutral signature*, Colloq. Math. **150** (2017), 63–86.
9. ______, *The geometry of 4-dimensional Ricci flat manifolds which admit a metric*, J. Geom. Phys. **89** (2015), 50–59.
10. ______, *Covariantly constant tensors and holonomy structure in general relativity*, J. Math. Phys. **32**(1) (1991), 181–187.
11. ______, *Some comparisons of the Weyl conformal tensor and its classification in* 4*-dimensional Lorentz, neutral or positive definite signatures*, Int. J. Mod. Phys. D **26**(5) (2017), 1741004.
12. G. S. Hall, B. Kırık, *Recurrence structures in* 4*-dimensional manifolds with metric of signature* $(+,+,-,-)$, J. Geom. Phys. **98** (2015), 262–274.
13. G. S. Hall, A. D. Rendall, *Local and global algebraic structures in general relativity*, Int. J. Theor. Phys. **28** (1989), 365–375.
14. G. S. Hall, Z. Wang, *Projective structure in* 4*-dimensional manifolds with positive definite metrics*, J. Geom. Phys. **62** (2012), 449–463.
15. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, (vol 1), Interscience, New York, 1963.
16. E. M. Patterson, *On symmetric recurrent tensors of the second order*, Quart. J. Math. Oxford **22** (1951), 151–158.
17. ______, *Some theorems on Ricci recurrent spaces*, J. London Math. Soc. **27**(3) (1952), 287–295.
18. A. Z. Petrov, *Einstein Spaces*, Pergamon, 1969.
19. R. K. Sachs, *Gravitational waves in general relativity. VI. the outgoing radiation condition*, Proc. Roy. Soc. London, Series A **264** (1961), 309–338.
20. J. F. Schell, *Classification of four-dimensional Riemannian spaces*, J. Math. Phys. **2** (1961), 202–206.
21. A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. **52** (1950), 36–64.
22. Z. Wang, G. S. Hall, *Projective structure in* 4*-dimensional manifolds with metric of signature* $(+,+,-,-)$, J. Geom. Phys. **66** (2013), 37–49.

Marmara University, Faculty of Arts and Sciences
Department of Mathematics, Istanbul, Turkey
`bahar.kirik@marmara.edu.tr`
`baharkirik@gmail.com`