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ALGEBRAIC OPERATORS AND MOMENTS ON ALGEBRAIC SETS

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Abstract. Boundedness and essential normality of algebraic Hilbert space operators with applications to complex moment problems on *real* algebraic sets is investigated here.

In solving the complex moment problem positive definiteness is insufficient. Usually some additional assumptions on the sequence in question are required, cf., for instance, [17], [2], [18], [5] and [15]. Inspired by the paper [12] (which goes back to [11]) we consider here the complex moment problem on real algebraic sets. It turns out that among additional assumptions needed in this case is the natural requirement the underlying sequence to satisfy an algebraic condition (cf. Proposition 1, ii)) which comes from the equation describing the algebraic set. The results of Sections 5, 6 and 7, when confronted with discussion contained in Section 8, show that an appropriate choice of positive definiteness is essential.

As Proposition 2 explains there is a link between the complex moment problem on real algebraic sets and essential normality of algebraic operators. This is the reason why studying algebraic operators forms a substantial part of the paper. The most important question in this matter is when algebraic operators may have enough bounded or quasianalytic vectors. Bounded vectors for (*a priori* unbounded) algebraic operators force their boundedness. This leads to the moment problem on compact subsets of algebraic sets. Quasianalytic vectors of algebraic operators contribute to the moment problem on unbounded algebraic sets.

The essential feature of our paper is that we do *not* restrict ourselves to the complex dimension one. However, in the one dimensional case our considerations cover algebraic sets described by polynomials having a dominating coefficient; among them there are the standard algebraic curves (like lemniscates and so on) or, more generally, equipotential ones. All this may impact Approximation Theory like the very classical cases of the real line and the unit circle do (cf. [7], [8] and [9]) as well as may have some connection with Stochastic Processes cf., for instance, [6].

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Algebraic operators

1. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,-\rangle$. Let $\mathbf{B}(\mathcal{H})$ stand for the ring of all bounded linear operators on $\mathcal{H}$ and let $\sigma_{\mathrm{ap}}(A)$ denote the approximate point spectrum of $A\in\mathbf{B}(\mathcal{H})$. For a given dense linear subspace $\mathcal{D}$ of $\mathcal{H}$ we denote by $\mathbf{L}(\mathcal{D})$ the set of all linear operators A in $\mathcal{H}$ such that domain $\mathcal{D}(A)=\mathcal{D}$ and $A\mathcal{D}\subset\mathcal{D}$, and by $\mathbf{L}^{\#}(\mathcal{D})$ the set of all those $A\in\mathbf{L}(\mathcal{D})$ for which there exists an operator $A^{\#}\in\mathbf{L}(\mathcal{D})$ such that $\langle Af,g\rangle=\langle f,A^{\#}g\rangle$, $f,g\in\mathcal{D}$.

Notice: $A^{\#}=A^*|_{\mathcal{D}}$ where A^* is the Hilbert space adjoint of A.

Define $\mathcal{A}'$ for $\mathcal{A}\subset\mathbf{L}^{\#}(\mathcal{D})$ as the set $\{T\in\mathbf{L}^{\#}(\mathcal{D});\,TA=AT,\ A\in\mathcal{A}\}$. We say that two operators A and B from $\mathbf{L}^{\#}(\mathcal{D})$ *doubly commute* if $A\in\{B,B^{\#}\}'$. $N\in\mathbf{L}^{\#}(\mathcal{D})$ is said to be *formally normal* if $\|Nf\|=\|N^{\#}f\|$, $f\in\mathcal{D}$, or equivalently, if $N^{\#}N=NN^{\#}$.

Given $A\in\mathbf{L}(\mathcal{D})$, denote by $\mathcal{B}_a(A)$, $a\geq 0$, the set of all $f\in\mathcal{D}$ for which there is a non-negative number c_f such that

$$\|A^n f\|\leq c_f\, a^n\,,\qquad n\geq 0\ .$$

Put $\mathcal{B}(A)=\bigcup_{a\geq 0}\mathcal{B}_a(A)$. In fact $\mathcal{B}_a(A)$ as well as $\mathcal{B}(A)$ form linear subspaces of $\mathcal{D}$, which are invariant for A.

Denote by $\mathcal{Q}(A)$ the set of all $f\in\mathcal{D}$ such that

$$\sum_{n=1}^{\infty}\|A^n f\|^{-1/n}=+\infty\ .$$

It is clear that $\mathcal{B}(A)\subset\mathcal{Q}(A)$.

Lemma 1. *Suppose $A\in\mathbf{L}(\mathcal{D})$ and $\mathcal{X}$ is a subset of $\mathcal{D}$ such that $A(\mathcal{X})\subset\mathcal{X}$. If there are an integer $d\geq 1$ and a non-negative number a such that*

$$\|A^d f\|\leq a\max\Big\{\|A^s f\|;\ 0\leq s<d\Big\}\,,\qquad f\in\mathcal{X}\ , \tag{1}$$

then $\operatorname{lin}\mathcal{X}\subset\mathcal{B}_{a'}(A)$ where $a'=\max\{1,a\}$.

If, in addition, A satisfies the following inequality

$$\|Af\|^2\leq\|f\|\,\|A^2 f\|\,,\qquad f\in\mathcal{D}\ , \tag{2}$$

then $\|Af\|\leq a'\,\|f\|$, $f\in\operatorname{lin}\mathcal{X}$.

Proof: Set

$$b_f=\max\Big\{\|A^s f\|;\ 0\leq s<d\Big\}\,,\qquad f\in\mathcal{X}\ .$$

Suppose A satisfies (1). We show that

$$b_{Af} \le \max\{1, a\}\, b_f\,, \qquad f \in \mathcal{X}\ . \tag{3}$$

Indeed, using (1), we have, $f \in \mathcal{X}$,

$$\begin{aligned} b_{Af} = \max\Big\{\|A^s f\|;\ 1 \le s < d+1\Big\} &\le \max\{b_f,\ \|A^d f\|\} \\ &\le \max\{b_f, a b_f\} = \max\{1, a\}\, b_f\ . \end{aligned}$$

Since $A(\mathcal{X}) \subset \mathcal{X}$, applying the induction argument to (1) and (3), we get

$$\|A^{d+s} f\| \le (\max\{1, a\})^{s+1}\, b_f\,, \qquad s \ge 0, \quad f \in \mathcal{X}\ ,$$

and, finally,

$$\|A^n f\| \le (\max\{1, a\})^n\, b_f\,, \qquad n \ge 0, \quad f \in \mathcal{X}\ .$$

This is precisely $\mathcal{X} \subset \mathcal{B}_{a'}(A)$ and, consequently, $\operatorname{lin} \mathcal{X} \subset \mathcal{B}_{a'}(A)$.

Assume (2). Then (cf. Proposition 1 of [19]), for $f \in \mathcal{D}$, we have

$$\|Af\| \le \|f\|^{n/(n+1)}\, \|A^{n+1} f\|^{1/(n+1)}\,, \qquad n = 1, 2, \dots\ . \tag{4}$$

If, moreover, A satisfies (1), then, due to $\operatorname{lin} \mathcal{X} \subset \mathcal{B}_{a'}(A)$ and inequality (4), we get that $\|Af\| \le a'\, \|f\|$, $f \in \operatorname{lin} \mathcal{X}$. For reader's convenience we include the proof of (4) here. First we show that

$$\|A^n f\| \le \|f\|^{1/(n+1)}\, \|A^{n+1} f\|^{n/(n+1)}\,, \qquad n = 1, 2, \dots\ . \tag{5}$$

If $n = 1$ this is precisely (2). Assume (5) for $n-1$. Then, by (2),

$$\|A^n f\|^2 = \|A A^{n-1} f\|^2 \le \|A^{n-1} f\|\, \|A^2 A^{n-1} f\| \le \|f\|^{1/n}\, \|A^n f\|^{(n-1)/n}\, \|A^{n+1} f\|\ .$$

Dividing both sides by $\|A^n f\|^{(n-1)/n}$, we get (5) for n. Now pass to the proof of (4). If $n = 1$ this is again (2). Assume (4) for $n-1$. Then, by (5),

$$\begin{aligned} \|Af\| \le \|f\|^{(n-1)/n}\, \|A^n f\|^{1/n} &\le \|f\|^{(n-1)/n} \Big\{\|f\|^{1/(n+1)}\, \|A^{n+1} f\|^{n/(n+1)}\Big\}^{1/n} \\ &= \|f\|^{n/(n+1)}\, \|A^{n+1} f\|^{1/(n+1)}\ . \end{aligned}$$

This completes the proof of (4). ∎

Remark 1. It is worthwhile to notice that there is a large class of unbounded operators satisfying (2), namely the class of hyponormal operators (cf. [19, Proposition 3]). Even more, one can extend the class of operators for which the second conclusion of Lemma 1 holds true replacing condition (2) by inequalities of Kato–Protter type (cf. again [19]).

We have to point out that any unbounded Hilbert space operator A with invariant domain $\mathcal{D}(A)$ such that $A^d = 0$, for some $d \geq 2$ (cf. [14]), satisfies (1) with $\mathcal{X} = \mathcal{D}(A)$. However it fails to satisfy (2). On the other hand, any unbounded symmetric operator with invariant domain satisfies (2) and does not (1) with $\mathcal{X} = \mathcal{D}(A)$.

2. In the sequel we adapt the usual multiindex notation understanding by $\mathbb{N}$ the set $\{0, 1, ...\}$. Given a (not necessarily commutative) ring $\mathbf{R}$, denote by $\mathbf{R}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\iota]$, as usually, the set of all polynomials in $\kappa + \iota$ indeterminates with coefficients in $\mathbf{R}$. Given $p \in \mathbf{R}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\iota]$, we *always* denote by P_{ij} (and $P_i = P_{i0}$ if $i = 0$) the coefficients of p and by p^{H} the homogenous part of p of the highest degree. Occasionally we have to regard a polynomial $p \in \mathbf{R}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$ as a member of $\mathbf{R}[Z_2, ..., Z_\kappa, \overline{Z}_2, ..., \overline{Z}_\kappa][Z_1, \overline{Z}_1]$; then, to avoid any confusion with p^{H}, we denote by p_{H} the homogenous (with respect to Z_1 and $\overline{Z}_1$) part of p of the highest degree (again with respect to Z_1 and $\overline{Z}_1$).

Caution: distinguish p_{H} from p^{H} when $\kappa > 1$.

If $\mathbf{R}$ is a ring of operators, then for $\mathbf{A} = (A_1, ..., A_\kappa) \subset \mathbf{L}(\mathcal{D})$ (resp. $\mathbf{A} = (A_1, ..., A_\kappa) \subset \mathbf{L}^{\#}(D)$) and $p \in \mathbf{R}[Z_1, ..., Z_\kappa]$ (resp. $p \in \mathbf{R}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$) we put

$$p(\mathbf{A}) = \sum_{i \in \mathbb{N}^\kappa} P_i \, \mathbf{A}^i \quad (\text{resp. } p(\mathbf{A}, \mathbf{A}^{\#}) = \sum_{i,j \in \mathbb{N}^\kappa} P_{ij} \, A^i \, A^{\#j}) \ ,$$

where $\mathbf{A}^{\#} = (A_1^{\#}, ..., A_\kappa^{\#})$.

The question we deal with now is when $p(A, A^{\#}) = 0$ for some nonconstant $p \in \mathbf{B}(\mathcal{H})[Z, \overline{Z}]$ implies that A is bounded. Consider first the case of $\mathbf{B}(\mathcal{H})[Z]$. As an immediate consequence of our Lemma 1 we get

Theorem 1. *Let $A \in \mathbf{L}(\mathcal{D})$ satisfy (2) and let $p \in \mathbf{B}(\mathcal{H})[Z]$ be a polynomial of degree $d \geq 1$ such that $p(A) = 0$. If $0 \notin \sigma_{\mathrm{ap}}(P_d)$, then A is a bounded and*

$$\|A\| = \max\Big\{1, \|P_d^{-1}\| \Big(\sum_{j=0}^{d-1} \|P_j\|\Big)\Big\} \ .$$

Proof: Since 0 does not belong to $\sigma_{\mathrm{ap}}(A)$, the following inequality holds

$$\|f\| \leq \|P_d^{-1}\| \, \|P_d f\| \, , \qquad f \in \mathcal{D} \ .$$

On the other hand, $p(A)=0$ implies

$$\|P_d^{-1}\|^{-1}\,\|A^d f\| \le \|P_d A^d f\| = \Big\|\sum_{j=0}^{d-1} P_j A^j f\Big\|$$
$$\le \max\Big\{\|A^s f\|;\ 0\le s<d\Big\}\Big(\sum_{j=0}^{d-1}\|P_j\|\Big)\ .$$

Thus the operator A satisfies (1) with $a=\|P_d^{-1}\|(\sum_{j=0}^{d-1}\|P_j\|)$. Applying Lemma 1 we get the conclusion. ∎

Now consider the case $\mathbf{B}(\mathcal{H})[Z,\overline{Z}]$. We say that P_{kl} is a *dominating coefficient* of a polynomial $p\in\mathbf{B}(\mathcal{H})[Z,\overline{Z}]$ of degree $d\ge 1$, if $k+l=d$, $0\notin\sigma_{\mathrm{ap}}(P_{kl})$ and

$$\|P_{kl}^{-1}\|^{-1} > \sum_{\substack{i+j=d\\ i\ne k,\ j\ne l}} \|P_{ij}\|\ .$$

Extending the class of admissible polynomials we restrict the class of operators to formally normal ones so as to get

Theorem 2. *Let $p\in\mathbf{B}(\mathcal{H})[Z,\overline{Z}]$ be of degree $d\ge 1$. Suppose $N\in\mathbf{L}^\#(\mathcal{D})$ is a formally normal operator such that $p(N,N^\#)=0$. Let $\mathcal{X}$ be an arbitrary subset of $\mathcal{D}$ such that $N(\mathcal{X})\subset\mathcal{X}$. If there is $\alpha>0$ such that*

$$\|p^{\mathrm{H}}(N,N^*)f\| \ge \alpha^{-1}\|N^d f\|\,,\qquad f\in\mathcal{X}\ , \tag{6}$$

then

$$\|Nf\| \le \max\Big\{1,\alpha\Big(\sum_{i+j<d}\|P_{ij}\|\Big)\Big\}\|f\|\,,\qquad f\in\operatorname{lin}\mathcal{X}\ . \tag{7}$$

In the case $\mathcal{X}=\mathcal{D}$, the closure N^- of N is a bounded normal operator. If p has a dominating coefficient, then (6) is satisfied with $\mathcal{X}=\mathcal{D}$ and $\alpha=(\|P_{kl}^{-1}\|^{-1}-\sum_{\substack{i+j=d\\ i\ne k,\ j\ne l}}\|P_{ij}\|)^{-1}$.

Proof: Assume (6). Since N satisfies (2), condition (7) follows directly from Lemma 1 as well as from the inequalities

$$\alpha^{-1}\|N^d f\| \le \|p^{\mathrm{H}}(N,N^*)f\| = \Big\|\sum_{i+j<d} P_{ij}\,N^i\,N^{*j}f\Big\| \le \sum_{i+j<d}\|P_{ij}\|\,\|N^i\,N^{*j}f\| =$$
$$= \sum_{i+j<d}\|P_{ij}\|\,\|N^{i+j}f\| \le \max\Big\{\|N^s f\|;\ 0\le s<d\Big\}\Big(\sum_{i+j<d}\|P_{ij}\|\Big),\quad f\in\mathcal{X}\ .$$

Since for $\mathcal{X} = \mathcal{D}$, the operator N^- is formally normal and bounded, it must necessarily be normal.

Suppose now that p has a dominating coefficient P_{kl}. Then the fact that $0 \notin \sigma_{\mathrm{ap}}(P_{kl})$ implies

$$\begin{aligned}
\|p^{\mathrm{H}}(N,N^*)f\| &= \Big\| P_{kl}N^kN^{*l}f + \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} P_{ij}N^iN^{*j}f \Big\| \\
&\geq \|P_{kl}N^kN^{*l}f\| - \Big\| \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} P_{ij}N^iN^{*j}f \Big\| \\
&\geq \|P_{kl}^{-1}\|^{-1}\, \|N^kN^{*l}f\| - \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} \|P_{ij}\|\, \|N^iN^{*j}f\| \\
&= \Big(\|P_{kl}^{-1}\|^{-1} - \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} \|P_{ij}\| \Big) \|N^d f\| \ .
\end{aligned}$$

Thus N satisfies (6) with $\alpha = (\|P_{kl}^{-1}\|^{-1} - \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} \|P_{ij}\|)^{-1}$ and $\mathcal{X} = \mathcal{D}$. ∎

Remark 2. If p and N satisfy all the assumptions of Theorem 2 and, in addition, $p \in \mathbb{C}[Z,\overline{Z}]$ has a dominating coefficient, then the conclusion (7) can be strengthened as follows

$$\|N\| \leq \max\{a, a^{1/d}\}, \quad \text{where } \ a = \Big(|P_{kl}| - \sum_{\substack{i+j=d\\ i\neq k,\ j\neq l}} |P_{ij}| \Big)^{-1} \Big(\sum_{i+j<d} |P_{ij}| \Big) \ .$$

This inequality differs from (7) only when $a < 1$. If so, because we have already proved that N^- is a bounded normal operator, $\|N\|$ = diameter of the support of the spectral measure of $N^- \leq$ diameter$\{z;\ p(z,\overline{z}) = 0\} \leq a^{1/d}$.

3. One of the main goals of this paper is the question of *essential normality* of algebraic operators or their systems. From this point of view, the question of boundedness of these operators becomes an intermediate (though interesting for itself) step toward this goal. Essential normality of algebraic operators which are, in fact, unbounded is a much more difficult question. The following theorem (being in flavor of the paper [13] of Nelson) is a attempt in this direction. First we state a version of Lemma 1.

Lemma 2. *Let $A \in \mathbf{L}^{\#}(\mathcal{D})$ doubly commute with a formally normal operator $M \in \mathbf{L}^{\#}(\mathcal{D})$. Suppose $\mathcal{X}$ is a subset of $\mathcal{D}$ such that $A(\mathcal{X}) \subset \mathcal{X}$ and $M(\mathcal{X}) \subset \mathcal{X}$. If there are an integer $d \geq 1$ and a non-negative number a such that*

$$\|A^d f\| \leq a \max\Big\{ \|A^iM^jf\|;\ 0 \leq i < d,\ j = 0,1 \Big\}, \qquad f \in \mathcal{X}\ , \tag{8}$$

then $\mathcal{X} \cap \mathcal{Q}(M) \subset \mathcal{Q}(A)$.

Proof: Applying induction to (8) and employing invariance of $\mathcal{X}$ we get

$$\|A^{d+s}f\| \leq b^{s+1} \max\Big\{\|A^i M^j f\|;\ 0 \leq i < d,\ 0 \leq j \leq s+1\Big\}, \quad f \in \mathcal{X},\ \ s \geq 0\ ,$$

where $b = \max\{1, a\}$. Since M is formally normal and doubly commutes with A, this implies

$$\begin{aligned}\|A^{d+s}f\| &\leq b^{s+1} \max\Big\{\langle A^{\#i}A^i f, M^{\#j}M^j f\rangle^{\frac{1}{2}};\ 0 \leq i < d,\ 0 \leq j \leq s+1\Big\} \\ &\leq b^{s+1} \max\Big\{\|A^{\#i}A^i f\|^{\frac{1}{2}};\ 0 \leq i < d\Big\} \max\Big\{\|M^{2j}f\|^{\frac{1}{2}};\ 0 \leq j \leq s+d\Big\}\ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f \in \mathcal{X},\ \ s \geq 0\ .\end{aligned}$$

Finally, we come to

$$\|A^n f\| \leq a_f^n \max\Big\{\|M^{2j}f\|^{\frac{1}{2}};\ 0 \leq j \leq n\Big\}, \quad f \in \mathcal{X},\ \ n \geq d\ , \tag{9}$$

with a suitable a_f. Take $f \in \mathcal{X} \cap \mathcal{Q}(M)$ and set $s_n = \|M^{2n}f\|^2$, $n \geq 0$. Since M is formally normal, $\{s_n\}$ is a Stieltjes moment sequence. Since

$$\sum_{n=1}^{\infty} \|M^n f\|^{-1/n} = +\infty\ ,$$

we get, by (2)$\Leftrightarrow$(3) of [15, pg. 32], that

$$\sum_{n=1}^{\infty} s_n^{-1/4n} = +\infty\ .$$

Applying the equivalence of Appendix with $w = \frac{1}{4}$ we get

$$\sum_{n=1}^{\infty} \max\Big\{s_j;\ 0 \leq j \leq n\Big\}^{-1/4n} = +\infty\ .$$

This and the inequality (9) implies that $f \in \mathcal{Q}(A)$. ∎

Theorem 3. *Let* $p \in \mathbf{L}^{\#}(\mathcal{D})[Z, \overline{Z}]$ *be of degree* $d \geq 1$. *Let* $M, N \in \mathbf{L}^{\#}(\mathcal{D})$ *be doubly commuting formally normal operators such that* $p(N, N^{\#}) = 0$. *Let* $\mathcal{X}$ *be a subset of* $\mathcal{D}$ *invariant for both* M *and* N. *Suppose there is* $\beta > 0$ *such that*

$$\|P_{kl}f\| \leq \beta \max\{\|f\|, \|Mf\|\}\,, \quad f \in \mathcal{D},\ \ k+l < d\ . \tag{10}$$

If either **i)** N *satisfies (6) with* $\alpha > 0$ *or* **ii)** p^{H} *has a dominating coefficient*(1), *then* $\mathcal{X} \cap \mathcal{Q}(M) \subset \mathcal{Q}(N)$. *In the case* $\mathcal{X} = \operatorname{lin} \mathcal{Q}(M) = \mathcal{D}$, N^{-} *is normal.*

(1) In this case there is no need to require the coefficients of p^{H} to have invariant domains.

Proof: Suppose N satisfies (6). Then (10) implies (cf. the proof of Theorem 2) that

$$\|N^d f\| \le \alpha \sum_{i+j<d} \|P_{ij} N^i N^{*j} f\| \le \alpha\,\beta \sum_{i+j<d} \max\Big\{\|N^{i+j} f\|, \|N^{i+j} M f\|\Big\}, \quad f \in \mathcal{X} .$$

Thus N satisfies the assumptions of Lemma 2 with $A = N$. This gives the conclusion for the case i). Under the assumption ii) we show the inequality (6) in the same way as in the proof of Theorem 2. If $\mathcal{X} = \operatorname{lin} \mathcal{Q}(M) = \mathcal{D}$, then $\operatorname{lin} \mathcal{Q}(N) = \mathcal{D}$ and, consequently, by Theorem 1 of [15], N^- is normal. ∎

The moment problem

4. For a given polynomial $p \in \mathbb{C}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$ of degree $d \ge 1$ and for a given multisequence $c\colon \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ we define the set

$$\mathcal{Z}(p) = \Big\{z \in \mathbb{C}^\kappa;\ p(z, \overline{z}) = 0\Big\}$$

and the multisequence $c_p\colon \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ by

$$c_p(k,l) = \sum_{i,j} P_{ij}\, c(i+k, j+l) .$$

Recall that c is said to be

1) *positive definite* if for any function $\xi\colon \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ of finite support

$$\sum_{i,j} \sum_{k,l} c(i+l, j+k)\, \xi(i,j)\, \overline{\xi(k,l)} \ge 0 ;$$

2) a *complex moment multisequence* if there exists a positive Borel measure μ on $\mathbb{C}^\kappa$ such that

$$c(i,j) = \int_{\mathbb{C}^\kappa} z^i\, \overline{z}^j\, \mu(dz)\,, \quad i,j \in \mathbb{N}^\kappa . \tag{11}$$

The point is to characterize (in terms of positive definiteness) those c's which are complex moment multisequences with representing measures concentrated on the set $\mathcal{Z}(p)$.

Proposition 1. *Let $p \in \mathbb{C}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$ and let $c\colon \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ be a complex moment multisequence with representing measure μ. Then c is positive definitive and the following conditions are equivalent*

i) *The closed support of μ is contained in $\mathcal{Z}(p)$,*

ii) $c_p = 0$.

Proof: Positive definiteness of c is a matter of direct verification. If the closed support of μ is contained in $\mathcal{Z}(p)$, then, by (11), we have

$$c_p(k,l) = \sum_{i,j} P_{ij}\, c(i+k, j+l) = \int_{\mathcal{Z}(p)} p(z,\overline{z})\, z^k\, \overline{z}^l\, \mu(dz) = 0\,, \qquad k,l \in \mathbb{N}^\kappa\ ,$$

and, consequently, $c_p = 0$. Conversely, if $c_p = 0$, then (11) implies once more

$$\int_{\mathbb{C}^\kappa} p(z,\overline{z})\, z^k\, \overline{z}^l\, \mu(dz) = 0\,, \qquad k,l \in \mathbb{N}^\kappa\ .$$

Thus

$$\int_{\mathbb{C}^\kappa} |p(z,\overline{z})|^2\, \mu(dz) = 0\ ,$$

which implies i). ∎

The next proposition makes transparent relationship between algebraic operators and the complex moment problem on algebraic sets.

Proposition 2. *Suppose $p \in \mathbb{C}[Z_1,...,Z_\kappa,\overline{Z}_1,...,\overline{Z}_\kappa]$. Then the following conditions are equivalent*

1) *Each positive definite multisequence $c\colon\ \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ satisfying $c_p = 0$ is a moment multisequence on $\mathcal{Z}(p)$;*

2) *Each doubly commuting κ-tuple $\mathbf{N}$ of formally normal operators in $\mathbf{L}^{\#}(\mathcal{D})$ such that $\mathcal{D} = \operatorname{lin}\{\mathbf{N}^{\#k}\mathbf{N}^l f;\ k,l \in \mathbb{N}^\kappa\}$ for some $f \in \mathcal{D}$, which satisfies $p(\mathbf{N},\mathbf{N}^{\#}) = 0$ extends to a κ-tuple of normal operators whose spectral measures commute.*

If, in addition, $\mathcal{Z}(p)$ is bounded, then either of the conditions 1) and 2) is equivalent to the following one

3) *Each doubly commuting κ-tuple $\mathbf{N}$ of formally normal operators in $\mathbf{L}^{\#}(\mathcal{D})$, which satisfies $p(\mathbf{N},\mathbf{N}^{\#}) = 0$ is composed of bounded operators.*

Proof:

1)⇒2) Suppose $\mathbf{N} = (N_1,...,N_\kappa)$ is a doubly commuting κ-tuple of formally normal operators such that $\mathcal{D} = \operatorname{lin}\{\mathbf{N}^{\#k}\mathbf{N}^l f;\ k,l \in \mathbb{N}^\kappa\}$ for some $f \in \mathcal{D}$. Suppose $\mathbf{N}$ satisfies $p(\mathbf{N},\mathbf{N}^{\#}) = 0$.

Set $c_{mn} = \langle \mathbf{N}^m f, \mathbf{N}^n f\rangle$. Then c is positive definite and $c_p = 0$. Thus c satisfies (11) with μ concentrated on $\mathcal{Z}(p)$. This implies that $\mathbf{N}$ is unitarily equivalent to the restriction of a κ-tuple $(M_{z_1},...,M_{z_\kappa})$ to $\mathbb{C}[Z_1,...,Z_\kappa,\overline{Z}_1,...,\overline{Z}_\kappa] \subset \mathcal{L}^2(\mu)$, where M_{z_j} is the operator of multiplication by the independent variable z_j on $\mathcal{L}^2(\mu)$. Thus $(M_{z_1},...,M_{z_\kappa})$ is the required extension.

2)⇒1) Take c to be positive definite with $c_p = 0$. Then there is (cf. [17] or [5]) a Hilbert space $\mathcal{H}$, $f \in \mathcal{H}$ and a doubly commuting κ-tuple $\mathbf{N} = (N_1,...,N_\kappa)$ of formally normal operators in $\mathbf{L}^{\#}(\mathcal{D})$ such that $\mathcal{D} = \operatorname{lin}\{\mathbf{N}^{\#i}\mathbf{N}^j f;\ i,j \in \mathbb{N}^\kappa\}$ ($\mathcal{D}^- = \mathcal{H}$) and

$$c(i,j) = \langle \mathbf{N}^i f, \mathbf{N}^j f\rangle \ .$$

We show that $c_p = 0$ implies $p(\mathbf{N},\mathbf{N}^{\#}) = 0$. Take $g = \mathbf{N}^{\#i}\mathbf{N}^j f$ and $h = \mathbf{N}^{\#k}\mathbf{N}^l f$. Then, since $\mathbf{N}$ is doubly commuting, we have

$$\begin{aligned}\langle p(\mathbf{N},\mathbf{N}^{\#})g,h\rangle = \sum_{s,t} P_{st}\langle \mathbf{N}^{j+k+s} f, \mathbf{N}^{i+l+t} f\rangle &= \sum_{s,t} P_{st}\, c(s+j+k, t+i+l) \\ &= c_p(j+k, i+l) = 0 \ .\end{aligned}$$

Since $\mathcal{D} = \operatorname{lin}\{\mathbf{N}^{\#i}\mathbf{N}^j f;\ i,j \in \mathbb{N}^\kappa\}$, this forces $p(\mathbf{N},\mathbf{N}^{\#}) = 0$. Since $\mathbf{N}$ extends to a κ-tuple of normal operators with the joint spectral measure E, the spectral theorem gives (11) with $\mu(\cdot) = \langle E(\cdot)f, f\rangle$. Now Proposition 1 implies 1).

Suppose now $\mathcal{Z}(p)$ is bounded. Since 3)⇒2) trivially, pass to the proof of 1)⇒3). Take a doubly commuting κ-tuple $\mathbf{N} = (N_1,...,N_\kappa)$ of formally normal operators such that $p(\mathbf{N},\mathbf{N}^{\#}) = 0$. Take an arbitrary $f \in \mathcal{D}$ and set $c_{mn} = \langle \mathbf{N}^m f, \mathbf{N}^n f\rangle$. Then c is positive definite and $c_p = 0$. Thus c satisfies (11) with $\mu = \mu_f$ concentrated on $\mathcal{Z}(p)$.

Define $\mathcal{D}_f = \operatorname{lin}\{\mathbf{N}^j f;\ j \in \mathbb{N}^\kappa\}$ and $\mathbf{N}_f = \mathbf{N}|_{\mathcal{D}_f}$. Then one can show that $\mathbf{N}_f$ is unitarily equivalent to the restriction of $(M_{z_1},...,M_{z_\kappa})$ to $\mathbb{C}[Z_1,...,Z_\kappa] \subset \mathcal{L}^2(\mu)$. Since $\mathcal{Z}(p)$ is bounded, these operators are bounded with bound independent of f. Consequently $\mathbf{N}$ comprises bounded operators too. ∎

Note. Condition 2) without cyclicity of $\mathbf{N}$ looks as follows

4) *Each doubly commuting κ-tuple $\mathbf{N}$ of formally normal operators in $\mathbf{L}^{\#}(\mathcal{D})$, which satisfies $p(\mathbf{N},\mathbf{N}^{\#}) = 0$ extends to a κ-tuple of normal operators whose spectral measures commute.*

When the set $\mathcal{Z}(p)$ is bounded, condition 4) is equivalent to 2). However, in view of the examples of [4] it is very unlikely that, dropping boundedness of $\mathcal{Z}(p)$, the implication 2)⇒4) may be still preserved.

5. Now we are ready to apply our operator-theoretical results to the complex moment problem on algebraic sets proposing a few of the ways of doing this. Start with $\kappa = 1$.

Theorem 4. *Suppose $p \in \mathbb{C}[Z,\overline{Z}]$ has a dominating coefficient. Then for any $c\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ the following conditions are equivalent*

1) *c is positive definite and $c_p = 0$;*

2) *c is a complex moment bisequence on $\mathcal{Z}(p)$.*

Proof: The only implication we have to prove is 1)⇒2) (the reverse implication is in Proposition 1). Since $\mathcal{Z}(p)$ is bounded, the implication 1)⇒2) follows from Theorem 2 and Proposition 2. ∎

Remark 3. It should be noticed that Theorem 4 can be deduced from Theorem 6 of [3]. More precisely, Cassier's result, when applied to algebraic curves, allows us to replace in Theorem 4 the assumption "*$p \in \mathbb{C}[Z,\overline{Z}]$ has a dominating coefficient*" by "$|p^{\mathrm{H}}(z,\overline{z})| > 0$ *for* $z \neq 0$". This can be done by reducing the whole story to the two parameter (real) moment problem on the compact set

$$\Big\{(x,y)\in\mathbb{R}^2;\ -(\operatorname{Re}p(x+iy,x-iy))^2-(\operatorname{Im}p(x+iy,x-iy))^2\ge 0\Big\}\ .$$

It is important to emphasize the fact that the Hilbert space approach of our paper, as opposed to Cassier's, is constructive. *A rebours*, though Cassier's paper does not involve any Operator Theory, we can use his result, via implication 1)⇒3) of Proposition 2, to verify boundedness of some algebraic formally normal operators. Namely we get a version of Theorem 2.

Corollary 1. *Suppose $p \in \mathbb{C}[Z,\overline{Z}]$ and $|p^{\mathrm{H}}(z,\overline{z})| > 0$ for $z \neq 0$. Then any formally normal operator $N \in \mathbf{L}^{\#}(\mathcal{D})$, satisfying $p(N,N^{\#}) = 0$ must necessarily be bounded.* ∎

We wish to point out that some classical curves fall in Theorem 4. Among them there are the equipotential ones $\mathcal{Z}(p\,\overline{p}-1)$, where $p\in\mathbb{C}[Z]$. Then $c_{p\overline{p}-1}=0$ reads as

$$\sum_{k,l} P_k\,\overline{P}_l\,c(k+i,l+j) = c(i,j)\,, \qquad i,j\in\mathbb{N}\ ,$$

(the case of Bernoulli's lemniscate, $p(Z) = Z^2-1$, has been considered in [12] where different kind of characterization appears). Also some other standard curves like Cassini's oval, Pascal's helix and the four-leafed rose have dominating coefficients.

Pass to the case of $\kappa = 2$.

Theorem 5. *Suppose $p \in \mathbb{C}[Z_1,Z_2,\overline{Z}_1,\overline{Z}_2]$ is given by*

$$p(z_1,z_2,\overline{z}_1,\overline{z}_2) = |q(z_2,\overline{z}_2)|^2 + |r(z_1,z_2,\overline{z}_1,\overline{z}_2)|^2\,, \qquad z_1,z_2\in\mathbb{C}\ ,$$

where $q \in \mathbb{C}[Z_2, \overline{Z}_2]$, $r \in \mathbb{C}[Z_1, Z_2, \overline{Z}_1, \overline{Z}_2]$ *and* $r_{\mathrm{H}} \in \mathbb{C}[Z_1, \overline{Z}_1]$. *Let* q *as well as* r_{H} *have dominating coefficients. Then for any* $c\colon \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{C}$ *the following conditions are equivalent:*

1) c *is a positive definite and* $c_p = 0$,

2) c *is a complex moment 4-sequence on* $\mathcal{Z}(p)$.

Proof: As before the only implication we have to prove is 1)⇒2). Take a doubly commuting pair $\mathbf{N} = (N_1, N_2)$ of formally normal operators such that $p(\mathbf{N}, \mathbf{N}^\#) = 0$. This means that

$$q(N_2, N_2^\#)^\# \, q(N_2, N_2^\#) + r(N_1, N_2, N_1^\#, N_2^\#)^\# \, r(N_1, N_2, N_1^\#, N_2^\#) = 0 \ ,$$

so $q(N_2, N_2^\#) = 0$ and $r(N_1, N_2, N_1^\#, N_2^\#) = 0$. The first of these equalities combined with Theorem 2 gives us that N_2^- is a bounded operator. Thus $s = r(Z, N_2^-, \overline{Z}, N_2^*)$ belongs to $\mathbf{B}(\mathcal{H})[Z, \overline{Z}]$, it has a dominating coefficient and $s(N_1, N_1^\#) = 0$. Applying again Theorem 2 we get that N_1^- is a bounded operator. Since $\mathcal{Z}(p)$ is bounded, the conclusion 2) follows from Proposition 2. ∎

Notice that the Cartesian product of two circles is well suited to Theorem 5.

Remark 4. Making again (cf. Remark 3) use of a nonconstructive approach of [3] we can get a stronger version of Theorem 5 replacing "*q has a dominating coefficient*" by "*$q \in \mathbb{C}[Z_2, \overline{Z}_2]$ and $|q^{\mathrm{H}}(z, \overline{z})| > 0$ for $z \neq 0$*". To prove this use Corollary 1 instead of Theorem 2 in showing that N_2^- is bounded.

6. Another application of our operator methods to moment problems is as follows.

Theorem 6. *Let* $p \in \mathbb{C}[Z_1, \dots, Z_\kappa, \overline{Z}_1, \dots, \overline{Z}_\kappa]$ *be such that* p_{H} *is of the form* $qZ_1^i \overline{Z}_1^j$ *with* $i+j \geq 1$, *where* $q \in \mathbb{C}[Z_2, \dots, Z_\kappa, \overline{Z}_2, \dots, \overline{Z}_\kappa]$. *Suppose a multisequence* $c\colon \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ *is positive definite and* $c_p = 0$. *Suppose, moreover,* c *satisfies the following conditions*

$$c_{q\overline{q}}(n\delta_1, n\delta_1) \geq \gamma \, c(n\delta_1, n\delta_1) \quad \text{for some } \gamma > 0 \ \text{ and sufficiently large } n \in \mathbb{N} \ , \tag{12}$$

$$\liminf c(n\delta_j, n\delta_j)^{1/n} < +\infty \qquad \text{for } \ j = 2, \dots, \kappa \ , \tag{13}$$

where $\delta_j = (0, \dots, 1, \dots, 0) \in \mathbb{N}^\kappa$ *with* 1 *on the* j*-th position. Then* c *is a complex moment multisequence on a compact subset of* $\mathcal{Z}(p)$.

Proof: Since c is positive definite, there is an κ-tuple $\mathbf{N} = (N_1, \dots, N_\kappa)$ of doubly commuting formally normal operators in $\mathbf{L}^\#(\mathcal{D})$ such that $\mathcal{D} = \mathrm{lin}\{\mathbf{N}^{\#i}\mathbf{N}^j f;$

$i, j \in \mathbb{N}^\kappa\}$ for some $f \in \mathcal{D}$ and

$$c(i,j) = \langle \mathbf{N}^i f, \mathbf{N}^j f\rangle \ . \tag{14}$$

Using Remark 2 of [17], we infer from (13) that each N_l, $l = 2, ..., \kappa$, is bounded. Like in the proof of Proposition 2 we show that $c_p = 0$ implies $p(\mathbf{N}, \mathbf{N}^\#) = 0$.

Suppose (12) is satisfied for $n \geq n_0$. Then we have, with $d = i + j$,

$$\begin{aligned} \left\| q(N_2, ..., N_\kappa, N_2^*, ..., N_\kappa^*)\, N_1^i N_1^{*j} N_1^n f \right\|^2 &= c_{q\overline{q}}\Big((d+n)\delta_1, (d+n)\delta_1\Big) \geq \\ &\geq \gamma\, c\Big((d+n)\delta_1, (d+n)\delta_1\Big) = \gamma \|N_1^{d+n} f\|^2 , \quad n \geq n_0 \ . \end{aligned} \tag{15}$$

If $q(N_2, ..., N_\kappa, N_2^*, ..., N_\kappa^*) = 0$, then, by (15), $N_1^n f = 0$ for $n \geq d + n_0$. Thus, by (14), $\liminf c(n\delta_1, n\delta_1)^{1/n} = 0$.

If $q(N_2, ..., N_\kappa, N_2^*, ..., N_\kappa^*) \neq 0$, then we can proceed as follows. Set $r = p(Z, N_2^-, ..., N_\kappa^-, \overline{Z}, N_2^*, ..., N_\kappa^*)$. Then $r \in \mathbf{B}(\mathcal{H})[Z, \overline{Z}]$, $r(N_1, N_1^\#) = 0$ and $r^{\mathrm{H}} = q(N_2^-, ..., N_\kappa^-, N_2^*, ..., N_\kappa^*) Z^i \overline{Z}^j$. Since (15) implies (6) with $N = N_1$, $p = r$ and $\mathcal{X} = \{N_1^n f : n \geq n_0\}$, we can apply Theorem 2 to get N_1 is bounded on $\operatorname{lin} \mathcal{X}$. Consequently, since $N_1^{n_0} f \in \mathcal{X}$, we have

$$\liminf c(n\delta_1, n\delta_1)^{1/n} = \liminf \|N_1^n f\|^{2/n} < +\infty \ .$$

Exploiting once more Remark 2 of [17] we infer that the operator N_1^- is bounded. Now the joint spectral measure E of the family $(N_1^-, ..., N_\kappa^-)$ gives (11) with $\mu(\cdot) = \langle E(\cdot) f, f\rangle$ having compact support. Then Proposition 1 implies the conclusion. ∎

Corollary 2. *Let $p \in \mathbb{C}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$ be such that either*
i) *$p_{\mathrm{H}} \in \mathbb{C}[Z_1, \overline{Z}_1]$ and p_{H} has a dominating coefficient or*
ii) *p_{H} is of the form $q Z_1^i \overline{Z}_1^j$ with $i + j \geq 1$, where $q \in \mathbb{C}[Z_2, ..., Z_\kappa, \overline{Z}_2, ..., \overline{Z}_\kappa]$ and*

$$q\,\overline{q} \geq \gamma \quad \textit{for some } \gamma > 0 \ . \tag{16}$$

Then for any multisequence $c : \mathbb{N}^\kappa \times \mathbb{N}^\kappa \to \mathbb{C}$ the following conditions are equivalent

1) *c is positive definite, $c_p = 0$ and c satisfies the condition (13),*

2) *c is a complex moment multisequence on a compact subset of $\mathcal{Z}(p)$.*

Proof: The only implication which deserves proof is 1)⇒2). Let $\mathcal{D}$, f and $\mathbf{N}$ be as in the proof of Theorem 6. Like there, $N_2^-, ..., N_\kappa^-$ are commuting bounded normal operators.

Consider the case i). Since, according to our assumptions, the polynomial r (being the same as in the proof of Theorem 6) has a dominating coefficient, it follows from Theorem 2 that N_1 is bounded. Consequently, the joint spectral measure E of the family $(N_1^-, ..., N_\kappa^-)$ leads us to the conclusion 2).

Consider the case ii). Let F be the joint spectral measure of $(N_2^-, ..., N_\kappa^-)$. Then, by (16), we have

$$\begin{aligned} c_{q\overline{q}}(n\delta_1, n\delta_1) &= \left\| q(N_2, ..., N_\kappa, N_2^*, ..., N_\kappa^*)\, N_1^n f \right\|^2 \\ &= \int_{\mathbb{C}^{\kappa-1}} |q(z,\overline{z})|^2 \, \langle F(dz) N_1^n f, N_1^n f\rangle \\ &\geq \gamma \int_{\mathbb{C}^{\kappa-1}} \langle F(dz) N_1^n f, N_1^n f\rangle = \gamma \|N_1^n f\|^2 = \gamma\, c(n\delta_1, n\delta_1)\,, \qquad n \geq 0\,. \end{aligned}$$

Thus c satisfies condition (12). Applying Theorem 6 we get 2). ∎

Notice that Theorem 6 as well as Corollary 2 accept unbounded $\mathcal{Z}(p)$ as well. To present some further examples let

$$p(z_1, z_2, \overline{z}_1, \overline{z}_2) = |z_1|^4 - |s(z_2)|^2 \, (z_1^2 + \overline{z}_1^2) \;,$$

where $s \in \mathbb{C}[Z_2]$. Then the section $\{z_1 \in \mathbb{C};\, p(z_1, z_2, \overline{z}_1, \overline{z}_2) = 0\}$ is bounded for any z_2, while the projections of the set $\mathcal{Z}(p)$ on $\mathbb{C} \times \{0\}$ as well as on $\{0\} \times \mathbb{C}$ are unbounded provided $\deg s > 0$. This example also stresses another difference in our approach and Cassier's (he has to assume that $\mathcal{Z}(p)$ itself is bounded while we, allowing $\mathcal{Z}(p)$ unbounded, get the resulting measure to have the bounded support).

7. So far we have dealt with the situation where representing measures are compactly supported. The noncompact case seems to be much more involved at least from the operator theoretic point of view. First we wish to contribute somewhat to this case considering parabolic curves.

Proposition 3. *Suppose that a polynomial $p \in \mathbb{C}[Z, \overline{Z}]$ is of the form $p = Z - \overline{Z} + q(Z + \overline{Z})$ where $q \in \mathbb{C}[Z]$. Then for any $c : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ the following conditions are equivalent:*

1) *c is positive definite and $c_p = 0$;*

2) *c is a complex moment bisequence on $\mathcal{Z}(p)$.*

Proof: While the implication 2)⇒1) is trivial, the reverse follows from Proposition 2. Indeed, let $N \in \mathbf{L}^{\#}(\mathcal{D})$ be a formally normal operator which satisfies $p(N, N^{\#}) = 0$. Thus $\operatorname{Im} N = r(\operatorname{Re} N)$ with $r = \frac{1}{2} iq(2Z)$. Since the operator $\operatorname{Re} N$ is symmetric, there is a selfadjoint extension T of $\operatorname{Re} N$ in $\mathcal{K} \supset \mathcal{H}$. Thus the normal operator $T + ir(T)^-$ extends N. The conclusion 2) is a consequence of Proposition 2. ∎

The next result is more advanced.

Theorem 7. *Let $p \in \mathbb{C}[Z_1, Z_2, \overline{Z}_1, \overline{Z}_2]$ be such that p_{H} is of the form $qZ_1^i\overline{Z}_1^j$ with $i + j \geq 1$ and $q \in \mathbb{C}[Z_2, \overline{Z}_2]$. Let $c\colon \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{C}$ be a positive definite 4-sequence such that $c_p = 0$. Suppose, moreover, that c satisfies the following two conditions*

$$c_{q\overline{q}}(n, n) \geq \gamma\, c(n, n) \quad \text{for some } \gamma > 0 \text{ and all } n \in \mathbb{N}^2 \tag{17}$$

and

$$\sum_{n=1}^{\infty} c(wn\delta_2, wn\delta_2)^{-1/2n} = +\infty\,, \qquad w = \max\Big\{1, \deg p_{kl};\ k{+}l{<}i{+}j\Big\}\ , \tag{18}$$

where $p_{kl} \in \mathbb{C}[Z_2, \overline{Z}_2]$ are the coefficients of $p \in \mathbb{C}[Z_2, \overline{Z}_2][Z_1, \overline{Z}_1]$. Then c is a complex moment 4-sequence on $\mathcal{Z}(p)$.

Proof: Let $\mathcal{D}$, f and $\mathbf{N} = (N_1, N_2)$ be as in the proof of Theorem 6 ($\kappa = 2$). Define $r \in \mathcal{L}^{\#}(\mathcal{D})[Z, \overline{Z}]$ by $r = p(Z, N_2, \overline{Z}, N_2^{\#})$. Then $r(N_1, N_1^{\#}) = 0$. Set $\mathcal{X} = \{\mathbf{N}^m f;\ m \in \mathbb{N}^2\}$, $M = N_2^w$ and $d = i + j$. Condition (18) implies that $f \in \mathcal{Q}(M) \subset \mathcal{Q}(N_2)$. Theorem 1 of [15] implies, in turn, that $\operatorname{lin} \mathcal{Q}(N_2) = \mathcal{D}$ and N_2^- is normal. Let F be a spectral measure of N_2^-. Take $h \in \mathcal{D}$ and put $\mu(\cdot) = \langle F(\cdot)h, h\rangle$. Since the coefficients R_{kl} of r are equal to $p_{kl}(N_2, N_2^{\#})$ we can find constants β_1, β_2 and β_3 such that (Δ being the closed unit disc at zero)

$$\begin{aligned}\|R_{kl}h\|^2 &= \int_{\mathbb{C}} |p_{kl}(z,\overline{z})|^2\,\mu(dz) = \int_{\Delta} |p_{kl}(z,\overline{z})|^2\,\mu(dz) + \int_{\mathbb{C}\setminus\Delta} |p_{kl}(z,\overline{z})|^2\,\mu(dz)\\ &\leq \beta_1\|h\|^2 + \beta_2 \int_{\mathbb{C}} |z|^{2w}\,\mu(dz) \leq \beta_3 \max\{\|h\|^2, \|Mh\|^2\}\,, \qquad k + l < d\ .\end{aligned}$$

This means that assumption (10) of Theorem 3 is satisfied with $p = r$. It follows from (17) that for any $g = N_1^k N_2^l f \in \mathcal{X}$ we have

$$\begin{aligned}\Big\|q(N_2, N_2^{\#})N_1^i N_1^{\#j} g\Big\|^2 &= \Big\langle q\overline{q}(N_2, N_2^{\#})N_1^{i+j+k}N_2^l f, N_1^{i+j+k}N_2^l f\Big\rangle =\\ &= c_{q\overline{q}}\Big((i+j+k)\delta_1 + l\delta_2, (i+j+k)\delta_1 + l\delta_2\Big)\\ &\geq \gamma\, c\Big((i+j+k)\delta_1 + l\delta_2, (i+j+k)\delta_1 + l\delta_2\Big) = \gamma\|N_1^{i+j}g\|^2\ .\end{aligned}$$

If $q(N_2,N_2^{\#}) = 0$, then the above inequality leads to $f \in \mathcal{B}(N_1)$. If $q(N_2,N_2^{\#}) \neq 0$, then $r^{\mathrm{H}} = q(N_2, N_2^{\#})Z^i\overline{Z}^j$ and the same inequality implies that N_1 and r satisfy assumption i) of Theorem 3 with $N = N_1$. Consequently, $f \in \mathcal{Q}(N_1)$. In any of these two cases $f \in \mathcal{Q}(N_1) \cap \mathcal{Q}(N_2)$. Using Theorem 2 of [15], we get that the spectral measures of N_1^- and N_2^- commute. The joint spectral measure of the pair (N_1^-, N_2^-) does the job. ∎

Corollary 3. *Let $p \in \mathbb{C}[Z_1, Z_2, \overline{Z}_1, \overline{Z}_2]$ be such that either* **i)** $p_{\mathrm{H}} \in \mathbb{C}[Z_1, \overline{Z}_1]$ *and p_{H} has a dominating coefficient or* **ii)** p_{H} *is of the form $qZ_1^i\overline{Z}_1^j$ with $i+j \geq 1$, where $q \in \mathbb{C}[Z_2, \overline{Z}_2]$ satisfies (16). Let $c\colon \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{C}$ be a positive definite 4-sequence such that $c_p = 0$. Suppose, moreover, c satisfies (18). Then c is a complex moment 4-sequence on $\mathcal{Z}(p)$.*

Proof: Let $\mathcal{D}$, f, $\mathbf{N} = (N_1, N_2)$ and r be as in the proof of Theorem 7. Like in that proof condition (18) implies that N_2^- is normal. Let F be the spectral measure of N_2^-. Consider the case i). Then the polynomial r satisfies (10) with $M = N_2^w$ and consequently, by Theorem 3, $f \in \mathcal{Q}(N_1) \cap \mathcal{Q}(N_2)$. This leads to the conclusion.

Assume ii). Then, like in the proof of Corollary 2, we have

$$c_{q\overline{q}}(k,l,k,l) = \left\|q(N_2, N_2^{\#})N_1^k N_2^l f\right\|^2 = \int_{\mathbb{C}} |q(z,\overline{z})z^l|^2 \left\langle F(dz)N_1^k f, N_1^k f\right\rangle$$
$$\geq \gamma \int_{\mathbb{C}} |z^l|^2 \left\langle F(dz)N_1^k f, N_1^k f\right\rangle = \gamma \|N_1^k N_2^l f\|^2 = \gamma\, c(k,l,k,l)\ .$$

In this case the conclusion follows from Theorem 7. ∎

Remark 5. Notice that, in general, the noncompact case requires more than positive definiteness of c and $c_p = 0$. Indeed, the polynomial $p = Z_1\overline{Z}_1$ satisfies all the assumptions of Corollary 3 though there are positive definite 4-sequences c satisfying $c_p = 0$, which are not moment 4-sequences on $\mathcal{Z}(p) = \{0\} \times \mathbb{C}$ (to see this notice that an arbitrary bisequence can be identified with a 4-sequence c satisfying $c_p = 0$, and invoke [1, p. 193, Th. 3.5]). This is because such sequences do not satisfy (18).

8. Considering the complex moment problem it is tempting to assume, instead of positive definiteness of a bisequence c (in the sense of this paper), that all the matrices

$$C_n = (c(i,j))_{i,j=0}^n$$

are positive definite. When $\mathcal{Z}(p)$ is the unit circle centered at 0, a bisequence c satisfying $c_p = 0$ must necessarily be of Toeplitz type (cf. [11]) and positive definiteness of c and this of all the matrices C_n (as well as that on the group of $\mathbf{Z}$) coincide. However for other sets this is *not* longer true (for the Bernoulli lemniscate see [12]). Even if one wants to assume a little bit more, namely that there is a Hilbert space operator S such that

$$c(i,j) = \langle S^i f, S^j f\rangle \ , \tag{19}$$

with some vector f, the situation does not improve too much. Indeed, consider the matrix

$$S = \begin{pmatrix} \alpha & 0 \\ 1 & -\alpha \end{pmatrix},$$

where α satisfies the equality $\alpha^2 = 1 - z$ with $|z| = 1$ and $f = (1,0)$. Then the bisequence c defined by (19) (notice that here S is even bounded) satisfies the condition ii) of Proposition 1 for the Bernoulli lemniscate, that is

$$c(m+2,n+2) = c(m+2,n) + c(m,n+2) \ , \qquad m,n \geq 0 \ .$$

Then so defined c is *not* a complex moment bisequence, because S, being cyclic, is not subnormal (cf. Proposition 3 of [16]); the latter follows from the fact that $\|S^*g\| > \|Sg\|$ for $g = (0,1)$.

Consider now a polynomial $q \in \mathbb{C}[Z]$ and again a bisequence c satisfying (19) with $S \in \mathbf{B}(\mathcal{H})$ and some $f \in \mathcal{H}$. Suppose that

$$c_{q\overline{q}-1} = 0 \ . \tag{20}$$

Then, by (19),

$$\Big\langle q(S)S^m f, q(S)S^n f\Big\rangle = c_{q\overline{q}}(m,n) = c_1(m,n) = \langle S^m f, S^n f\rangle \ .$$

Consequently, $V = q(S)$ is an isometry on $\operatorname{clo\,lin}\{S^n f;\ n \geq 0\}$. Suppose, moreover, that

$$\mathcal{H} = \operatorname{clo\,lin}\Big\{q(S)^n f;\ n \geq 0\Big\} \ . \tag{21}$$

Because V is a cyclic subnormal operator and $VS = SV$, we can use Theorem 3 of [20] to get subnormality of S. This implies that c is a complex moment bisequence and Proposition 1 localizes the representing measure on $\mathcal{Z}(p)$. Thus we come to the following conclusion:

(*) *A bisequence c given by (19) with $S \in \mathbf{B}(\mathcal{H})$ and satisfying (20) and (21) is a complex moment bisequence on $\mathcal{Z}(p)$.*

Notice that the conditions (19) and (21) can be expressed explicitly in terms of the bisequence c itself. Also the assumption $S \in \mathbf{B}(\mathcal{H})$ in (*) can not be removed.

Come back to the Bernoulli lemniscate. Since in this case $q = Z^2 - 1$ and, consequently, $\operatorname{clo}\operatorname{lin}\{q(S)^n f;\, n \geq 0\} = \operatorname{clo}\operatorname{lin}\{S^{2n} f;\, n \geq 0\}$, then, assuming that $c(0,0) = 1$ and $G_n = \det C_n > 0$ for any $n > 0$, one can show that the condition (21) is equivalent to the following one

$$\{\alpha_n\}_{n=0}^{\infty} \in \ell^2 \ \text{ and } \ \sum_n \alpha_n G_{n,m} = 0 \ \text{ for all } m \geq 0 \quad \Rightarrow \quad \{\alpha_n\}_{n=0}^{\infty} = 0\ ,$$

where $G_{-1} = 1$ and

$$G_{n,m} = (G_n G_{n-1})^{-\frac{1}{2}} \det \begin{pmatrix} c(0,0) & c(0,1) & \ldots & c(0,n-1) & c(0,2m) \\ c(1,0) & c(1,1) & \ldots & c(1,n-1) & c(1,2m) \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ c(n,0) & c(n,1) & \ldots & c(n,n-1) & c(n,2m) \end{pmatrix}.$$

A natural question one may ask for is whether there are complex moment sequences (viz. measures) on the Bernoulli lemniscate for which (21) is satisfied. The latter is equivalent to

$$\mathcal{H}^2(\mu) = \operatorname{clo}\operatorname{lin}\{Z^{2n}:\ n \geq 0\}\ , \tag{22}$$

where $\mathcal{H}^2(\mu)$ is the closure of $\mathbb{C}[Z]$ in $\mathcal{L}^2(\mu)$ for μ being a representing measure of the underlying complex moment bisequence. Here we wish to give a partial answer to this question. Let μ be a positive finite Borel measure on the Bernoulli lemniscate, which is zero on the open lower half plane. Assume at least one of the points $2^{\frac{1}{2}}$ and $-2^{\frac{1}{2}}$ is not an atom of μ. Let $\varphi(z) = z^2 - 1$ be understood as a mapping of the Bernoulli lemniscate onto the circle $\mathbb{T}$ and let m be the Lebesgue measure on $\mathbb{T}$. Then

(**) *The condition (22) is fulfilled provided*

$$\int_{\mathbb{T}} \log(d\mu \circ \varphi^{-1}/dm)\, dm = -\infty\ .$$

Indeed, it follows from the Szegö theorem [10] that the constant function 1 is in the $\mathcal{L}^2(\mu\circ\varphi^{-1})$-closure of $\operatorname{lin}\{Z^n;\, n \geq 1\}$. This implies 1 is in the $\mathcal{L}^2(\mu)$-closure of $\operatorname{lin}\{(Z^2-1)^n;\, n \geq 1\}$. Consequently, the $\mathcal{L}^2(\mu)$-closure of $\operatorname{lin}\{(Z^2-1)^n;\, n \geq 1\}$ is the same as the $\mathcal{L}^2(\mu)$-closure of $\operatorname{lin}\{Z^{2n};\, n \geq 0\}$. Fix a (measurable) branch of the square root on $\mathbb{C}$ such that $(z^2)^{\frac{1}{2}} = z$ for z in the upper half plane containing only one of the real halfaxes (depending on which of the points $2^{\frac{1}{2}}$ and $-2^{\frac{1}{2}}$ we take into account). Since $(Z+1)^{\frac{1}{2}} \in \mathcal{L}^2(\mu \circ \varphi^{-1}) =$ the $\mathcal{L}^2(\mu \circ \varphi^{-1})$-closure of $\operatorname{lin}\{Z^n;\, n \geq 1\}$ and at least one of the points $2^{\frac{1}{2}}$ and $-2^{\frac{1}{2}}$ is not an atom of μ, $Z \in \mathcal{L}^2(\mu)$-closure of $\operatorname{lin}\{(Z^2-1)^n;\, n \geq 1\} = \mathcal{L}^2(\mu)$-closure of $\operatorname{lin}\{Z^{2n};\, n \geq 0\}$. This, in turn, implies that the condition (22) is fulfilled.

Notice that if both $2^{\frac{1}{2}}$ and $-2^{\frac{1}{2}}$ are atoms of μ, condition (22) may fail though $\int_{\mathbb{T}} \log(d\mu \circ \varphi^{-1}/dm)dm = -\infty$.

Considerations of this section may be regarded as being complementary to those of [12].

Appendix

The following fact, of rather technical nature, has been isolated so as to make it more convenient for further use.

Let $\{s_n\}_{n=0}^{\infty}$ be a Stieltjes moment sequence and let $w \in (0, +\infty)$. Then

$$\sum_{n=1}^{\infty} s_n^{-w/n} = +\infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \max\{s_j;\ 0 \le j \le n\}^{-w/n} = +\infty\ .$$

Indeed, supposing $s_n > 0$ for all n, we can decompose the sequence (using the representing measure) as $s_n = s_{0,n} + s_{1,n}$, where $\{s_{0,n}\}$ is the decreasing Stieltjes moment sequence while $\{s_{1,n}\}$ is the increasing one. The only nontrivial case happens when both $s_{0,n}$ and $s_{1,n}$ never vanish. Then

$$\begin{aligned}\max\{s_j;\ 0 \le j \le n\}^{w/n} &\le (s_{0,0} + s_{1,n})^{w/n} \le \left((s_{0,0}\, s_{1,0}^{-1} + 1)\, s_{1,n}\right)^{w/n} \\ &\le (s_{0,0}\, s_{1,0}^{-1} + 1)^{w/n}\, s_n^{w/n}, \qquad n \ge 1\ .\end{aligned}$$

This proves the implication "$\Rightarrow$". The reverse implication is obvious. ∎

Note added on May 15, 1991. The paper was completed around the end of 1989 and since then it has been widely circulating. In the meantime the paper [21] by Schmüdgen was submitted (on May 14, 1990) and already published. He proved a moment theorem (which in fact generalizes that of Cassier, cf. *Remark* 3) making a substantial use of the *Positivstellensatz* of the theory of semi-algebraic sets. Schmüdgen's result impacts our Theorems 4, 5 and Corollary 1. Moreover, when combined with our Proposition 2, it leads to the following:

Suppose $p \in \mathbb{C}[Z_1, ..., Z_\kappa, \overline{Z}_1, ..., \overline{Z}_\kappa]$. If $\mathcal{Z}(p)$ is bounded, then a doubly commuting κ-tuple $\mathbf{N}$ of formally normal operators in $\mathbf{L}^{\#}(\mathcal{D})$, which satisfies $p(\mathbf{N}, \mathbf{N}^{\#}) = 0$ is composed of bounded operators.

The remaining parts of our paper are independent of [2]. Besides its simplicity, the operator theoretic approach we develop is applicable to the unbounded case as well.

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