PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 1 – 1994

SKEW POLYNOMIAL RINGS SATISFYING R-BND PROPERTY

Refaat M. Salem

Abstract: In this paper we show that, a prime right noetherian ring A satisfies $r-(T)<\infty$ iff $A[x,\sigma]$ satisfies $r-Bnd$.

Introduction

In [6] Robson has found a relation between the Krull dimension (in the sense of Rentschler, [5]) and the upper of the number of generators of right ideals in polynomial rings over simple right noetherian rings. Also Stafford in [8] studied the relation between a ring A and its polynomial ring $A[x]$ if one of them satisfies $r-Bnd$ property. Here we extend Robson's result to the Ore extension over simple right noetherian rings and study the relation between the properties $T(A)<\infty$ and the $r-Bnd$ of the skew polynomial ring $A[x,\sigma]$.

1 – Definition and basic concepts

All rings here with identity and all modules are unitary. A right A-module M is said to be torsion right A-module if for each $m \in M$ there exists an element $s \in S$ (the m-set of regular elements of A) such that $ms=0$. A ring A is said to be $T(A)=n$ if every finitely generated torsion right A-module can be generated by n elements. The ring A (the module M_A) is said to be $r-Bnd\,n$, if every right ideal of A (submodule of M_A) can be generated by n elements. A right A-module M is said to be completely faithful, if every nonzero subfactor module is faithful, it is clear that each module over simple rings is completely faithful. The ring Krull dimension of A_A denoted by $K(A_A)$ is deviation of the lattice of submodule of A_A [6]. Consider the Ore extension ring of A, that is the ring

Received: June 22, 1991; *Revised*: December 24, 1992.

$R = A[x, \sigma, \delta]$, where σ is an automorphism of A and δ is a σ-derivation of A, where addition is componentwise and multiplication is given as

$$(ab)\,x = x\,\sigma(ab) + \delta(ab) \quad \text{and} \quad \delta(ab) = \sigma(a)\,\delta(b) + \delta(a)\,b\ .$$

The ideal I of A is called σ-ideal if $\sigma(I) \subseteq I$, it is well known that if A is a right noetherian ring, then $\sigma(I) = I$. The ideal I is called σ-prime ideal of A, if whenever J, K are σ-ideals of A such that $JK \subseteq I$, then $J \subseteq I$ or $K \subseteq I$. The ring A is called a σ-prime ring if (0) is a σ-prime ideal of A. The ideal I is called a σ-maximal ideal of A, if there is no proper σ-ideal J such that $I \subset J \subset A$.

2 – Preliminary remarks and results

1. Let $R = A[x, \sigma, \delta]$, where A is any ring, σ is an automorphism of A and δ is a σ-derivation of A. Let S be a multiplicative set of regular elements of A, such that $\sigma(S) \subseteq S$ and S satisfies the right Ore condition. Let $Q = AS^{-1}$, then

- **i)** σ and δ can be extended in a unique manner to an automorphism σ of Q and to a σ-derivation δ of Q.
- **ii)** S is the multiplicative set of regular elements of R, R satisfies the right Ore condition for S and $RS^{-1} = Q[x, \sigma, \delta]$ ([2], Theorem 7.1.2).

2. If A is a simple right Artinian ring, then $R = A[x, \sigma, \delta]$ is a principal right ring, where σ is an automorphism of A and δ is a σ-derivation ([2], Corollary 6.2.2).

3. Suppose $n < \infty$ and A is a ring with $K(A) > n$. If M is a completely faithful noetherian right A-module, such that $K(M) = n$, then M can be generated by $n+1$ elements [7].

4. If A is a simple right noetherian ring with $K(A) = n$, then any right ideal of A can be generated by $n+1$ elements [7].

5.

- **i)** Let A be a ring with Krull dimension and $c \in S$ (the m-set of regular elements), then $K\{(A/cA)_A\} < K(A_A)$ ([4], Lemma 6.3.9, pp. 131.)
- **ii)** If M_A is finitely generated, then $K(M) \leq K(A_A)$ ([4], Lemma 6.2.5, pp. 131).
- **iii)** If N_A is a submodule of M_A, $k(M) = \sup\{K(N), K(M/N)\}$ ([4], Lemma 6.2.4, pp. 180).

iv) Let M_A have Krull dimension and also be the sum of submodules each of which has Krull dimension $\leq \alpha$, then $K(M) \leq \alpha$ ([4], Lemma 6.2.14, pp. 184).

6. Let I be a nonzero σ-ideal of the σ-prime right noetherian ring A, then $I \cap S \neq \varphi$, where S is the m-set of regular elements ([1], Proposition I.12, pp. I.14).

7. The following are equivalent for a ring A with an automorphism σ of A and a σ-derivation δ of A

i) A is right noetherian.

ii) $A[x, \sigma, \delta]$ is right noetherian ([3], Theorem 2.2.15).

3 – The main results

Lemma 1. *Let A be a prime right Goldie ring and $R = A[x, r, \delta]$, where σ is an automorphism of A and δ is a σ-derivation. If I is a right idea of $R = A[x, \sigma, \delta]$, then I contains an element g such that $I/J = I/g$ and $A[x, \sigma, \delta]$ is a torsion right A-module.*

Proof: Since, A is a prime right Goldie ring, then A has a right quotient ring Q which is simple Artinian. By Remark 1 ii) the automorphism σ of A and the σ-derivation δ of A can be extended in a unique manner to an automorphism σ of Q and a σ-derivation δ of Q. Consider the Ore extension ring $Q[x, \sigma, \delta]$, then by Remark 1 ii) $Q[x, \sigma, \delta] = As^{-1}[x, \sigma, \delta] = A[x, \sigma, \delta]S^{-1}$. Using Remark 2 $Q[x, \sigma, \delta]$ is a principal right ideal ring. Let I be a right ideal of $R = A[x, \sigma, \delta]$, then $I_S = IQ[x, \sigma, \delta] = \{ks^{-1} \mid k \in I,\ s \in S\}$. Since I_S is an ideal in $Q[x, \sigma, \delta]$, then $I_S = hA[x, \sigma, \delta]_S$, where $h \in I_S$. Since, S satisfies the right Ore condition, then $I_S = gs^{-1}A[x, \sigma, \delta]_S = gA[x, \sigma, \delta]_S$, where $g \in I <_r A[x, \sigma, \delta]$. Let $J = gA[x, \sigma, \delta]$ and consider the right A-module $M = I/J$, this is a torsion right A-module. Since $I \subseteq I_S = gA[x, \sigma, \delta]_S = \{ks^{-1} \mid k \in I,\ s \in S\}$, then each $i \in I$ can be written as $i = gf$, where $f \in A[x, \sigma, \delta]_S$. Thus $i = gms_1^{-1}$ where $m \in A[x, \sigma, \delta]$ and $s_1 \in S$. Accordingly, $is_1 = gm \in J = gA[x, \sigma, \delta]$ and $m = I/J$ is a torsion right A-module. ∎

Lemma 2. *Let A be a prime right noetherian ring which satisfies $r-(T) = n$, then the Ore extension ring $R = A[x, \sigma, \delta]$ of A satisfies $r - Bnd(n+1)$.*

Proof: Let I be a nonzero right ideal of R. Since, A is right noetherian, then R is right noetherian by Remark 7 and $I = \sum_{i=1}^{k} g_i R$ say $k > n+1$. Using Lemma 1 there exists a nonzero element $g \in I$ such that $I/J = I/gR$ is a torsion right A-module. Therefore, for each $g_i \in I$, there exists $r_i \in S$ (the m-set of regular elements) such that $g_i r_i \in J$. Now, if we define the R-homomorphism

$$\Phi\colon \sum_{i=1}^{k}{}^{\oplus}(g_iR/g_ir_iR) \to \sum_{i=1}^{k} g_iR/\sum_{i=1}^{k} g_ir_i$$

as

$$\Phi(g_1a_1 + H_1, ..., g_ka_k + H_k) = (g_1a_1 + ... + g_ka_k) + H \ ,$$

where $H_i = g_ir_iR$ and $H = \sum_{i=1}^{k} g_ir_iR$, then it is easily verified that Φ is a well defined onto R-homomorphism. Also, since $g_ir_i \in J$ for each $i = 1, ..., k$, then $g_ir_iR \subseteq J$ and $\sum_{i=1}^{k} g_ir_iR \subseteq J$, thus

$$\varphi\colon \sum_{i=1}^{k} g_iR/\sum_{i=1}^{k} g_ir_iR \to \sum_{i=1}^{k} g_iR/gR = I/J \ ,$$

is onto. Consequently,

$$\tau\colon \sum_{i=1}^{k}{}^{\oplus} g_iR/g_ir_iR \to I/J \ ,$$

where $\tau = \Phi \circ \varphi$ is also onto. Moreover, if we define R-homomorphism

$$\Theta\colon \sum_{i=1}^{k}{}^{\oplus} R/r_iR \to \sum_{i=1}^{k}(g_iR/g_ir_iR)$$

as

$$\Theta(a_1 + H'_1, ..., a_kH'_k) = (g_1a_1 + H_1, ..., g_ka_k + H_k) \ ,$$

where $H'_i{=}r_iR$, then it is easily verified Θ is a well defined onto R-homomorphism.

Summarizing I/J is the homomorphic image of $\sum_{i=1}^{k\,\oplus} R/r_iR$. Since, A satisfies $r - T(A) = n$ and $\bigoplus_{i=1}^{k} A/r_iA$ is finitely generated torsion right A-module, hence $\bigoplus_{i=1}^{k} A/r_iA$ can be generated by n elements as A-module. Therefore, $\bigoplus_{i=1}^{k} R/r_iR$ can be generated by n elements as R-module. Since I/J is its homomorphic image, then it is generated as an R-module by n elements. Hence, I is generated by $n+1$ elements and the lemma is proved. ∎

The following result shows how can the right (left) Krull dimension [6] play an important role in determining the upper bound of the number of generators of the right (left) ideals in Ore extension rings.

Proposition 3. *Let A be a simple right noetherian ring and $K(A_A) = n$, then both A and $R = A[x, \sigma, \delta]$ satisfies $r - Bnd(n+1)$.*

Proof: Since A is a simple right noetherian ring and $K(A_A) = n$, then by Remark 4 A satisfies $r - Bnd(n+1)$. Also, using Remark 7 R is right noetherian.

Then by the same argument used in Lemma 1 one can easily check that any nonzero right ideal I of R contains a nonzero element g and the right R-submodule $J = gR$ such that I/J is a torsion right A-module. Let $I = \sum_{i=1}^{k} a_i R$, where $K > n+1$, since I/J is a torsion right A-module, then for each a_i there exists $r_i \in S$ such that $a_i r_i \in J$. Also, as in Lemma 2 it can be easily verified that I/J is the homomorphic image of $\sum_{i=1}^{k \oplus}(A/r_iA)[x,\sigma,\delta]$. Since, each r_i is regular and $K(A) = n$, then by Remark 5 i) $K(A/r_iA) < n$ for each $i = 1, ..., k$. Consider the right A-module $M = \sum_{i=1}^{k} A/r_iA$, since M is finitely generated A-module, then $K(M) \leq n$ by Remark 5 ii) and since, M is the sum of submodules each of Krull dimension $< n$, then by Remark 5 iv) $K(M) < n$. Since, M is the homomorphic image of $\bigoplus_{i=1}^{k} A/r_iA$, we get that $K(\bigoplus_{i=1}^{k} A/r_iA) \leq K(M) < n$ by Remark 5 iii). Since A is simple and $K(\bigoplus_{i=1}^{k} A/r_iA) < n$, then by Remark 3 $\bigoplus_{i=1}^{k} A/r_iA$ can be generated by n elements as A-module.

Consequently, $\bigoplus_{i=1}^{k} A/r_iA[x,\sigma,\delta]$ can be generated by n elements as R-module. Hence, I/J can be generated by n elements as a homomorphic image of $\bigoplus_{i=1}^{k} A/r_iA[x,\sigma,\delta]$. Then I can be generated by $n+1$ elements. ∎

Proposition 4. *Let A be a σ-prime right noetherian ring that satisfies $r - T(A) = n$. Then A is σ-simple.*

Proof: Suppose that A is not σ-simple, then it contains a proper σ-ideal P, take to be the direct sum of m copies of A/P where $m > n$. Since, A is a σ-prime right noetherian ring and P is a nonzero σ-ideal, then by Remark 6 $P \cap S \neq \varphi$. The regular elements that belong to P annihilate all components of $M = (A/P)^m$. Thus, $M = (A/P)^m$ is a finitely generated torsion right A-module which can't be generated by less than $m > n$ elements which contradicts our assumption. Thus, A is a σ-simple ring. ∎

Lemma 5. *Let A be a σ-prime right noetherian ring such that $A[x,\sigma]$ satisfies $r - Bnd(n)$, then A satisfies $r - T(A) = n$ and A is σ-simple.*

Proof: Consider a finitely generated torsion right A-module $M = \sum_{i=1}^{m} a_i A$, $m > n$. So, for each a_i there exists $r_i \in S$ such that $a_i r_i = 0$. Let α_i be an A-homomorphism: $A \to a_iA$, since $a_ir_i = 0$, then $r_iA \subseteq \ker\alpha_i$ for each $i = 1, ..., m$ and we have an onto A-homomorphism: $A \to A/\ker\alpha_i \cong a_iA$. Now, consider the A-homomorphism

$$\phi\colon \sum_{i=1}^{m \oplus} A/\ker\alpha_i \to \sum_{i=1}^{m} a_iA = M$$

defined by

$$(b_1 + \ker\alpha_1,\ ...,\ b_m + \ker\alpha_m) \to \sum_{i=1}^{m} a_ib_i\ .$$

ϕ is well defined, since if $(b_1 + \ker \alpha_1, ..., b_m + \ker \alpha_m) = 0$, then each $b_1 \in \ker \alpha_i$ (i.e., $a_i b_i = 0$) hence, $\sum_{i=1}^{m\oplus} a_i b_i = 0$ and it is clear that ϕ is onto. Let

$$\Phi \colon \sum_{i=1}^{m}{}^{\oplus} A/r_i A \to \sum_{i=1}^{m}{}^{\oplus} A/\ker \alpha_i$$

be an A-homomorphism defined by

$$(b_1 + r_1 A,\ ...,\ b_m + r_m A) \to (b_1 + \ker \alpha_1,\ ...,\ b_m + \ker \alpha_m)\ ;$$

it is clear that ϕ is a well defined and onto A-homomorphism. So, M is a homomorphic image of

$$N = \sum_{i=1}^{m}{}^{\oplus} A/r_i A = \sum_{i=1}^{m}{}^{\oplus} n_i A$$

give N an $A[x,\sigma]$-module structure by defining $Nx = 0$. Let I, J be a nonzero right ideals of $R = A[x,\sigma]$ given by

$$I = x^m R + x^m r_1^{\sigma^{-1}} R + x^{m-1} r_1^{\sigma^{-2}} r_2^{\sigma^{-1}} R + ... + x r_1^{\sigma^{-1}} \cdots r_{m-1}^{\sigma^{-1}} R$$

and

$$J = x^m r_1 R + x^{m-1} r_1^{\sigma^{-1}} r_2 R + x^{m-2} r_1^{\sigma^{-1}} r_3 R + ... + x r_1^{\sigma^{-m-1}} \cdots r_{m-1}^{\sigma^{-1}} r_m R\ .$$

Hence

$$\begin{aligned} I/J = {} & x^m R/x^m r_1 R \oplus x^{m-1} r_1^{\sigma^{-1}} R/x^{m-1} r_1^{\sigma^{-1}} r_2 R \oplus ... \oplus \\ & \oplus x r_{m-1}^{\sigma^{-m-1}} \cdots r_{m-1}^{\sigma^{-1}} R/x r_1^{\sigma^{-m-1}} \cdots r_{m-1}^{\sigma^{-1}} r_m R\ . \end{aligned}$$

Let Ω be an $A[x,\sigma]$-module homomorphism: $I/J \to N$ defined by $\Omega\overline{(x^m)} = n_1$ and $\Omega\overline{(x_1^{\sigma^{-m-1}} \cdots r_{m-i}^{\sigma-i})} = n_{i+1}$, then it can be easily shown that Ω is an isomorphism of $A[x,\sigma]$-module. Since, $A[x,\sigma]$ satisfies $r - Bnd(n)$, then i can be generated by $n < m$ elements. Hence, as a homomorphic image I/J can be generated by $n<m$ elements. Consequently, N can be generated by $n<m$ elements as an $A[x,\sigma]$-module. Since, N has a trivial structure as $A[x,\sigma]$-module, the same n elements will generate N as an A-module. Consequently A satisfies $r - T(A) = n$ and by Lemma 4 A is σ-simple. ∎

Now, if we put $\delta = 0$ in Lemma 2, then using the above proposition it follows that

Theorem 6. *If A is a prime right noetherian ring and σ is an automorphism of A then*

1) *If A satisfies $r - T(A) = n$, then $A[x,\sigma]$ satisfies $r - Bnd(n+1)$.*

2) *If $A[x,\sigma]$ satisfies $r - Bnd(n)$, then A satisfies $r - T(A) = n$.* ∎

Proposition 7. *Let A be a ring such that $A[x,\sigma]$ satisfies $r - Bnd(n)$, then A/P satisfies $r - T(A/P) = n$ for each σ-prime ideal P of A.*

Proof: Since $A[x,\sigma]$ satisfies $r - Bnd(n)$, then $A[x,\sigma]$ is right noetherian. So, by Remark 7 A is right noetherian. Since P is a σ-prime ideal, then A/P is a σ'-prime ring, where σ' is an automorphism of A/P induced by σ and $A/P[x,\sigma'] \cong A[x,\sigma]/P[x,\sigma]$. Since $A[x,\sigma]$ satisfies $r - Bnd(n)$, then $A[x,\sigma]/P[x,\sigma]$ satisfies $r - Bnd(n)$. Hence, $A/P[x,\sigma']$ satisfies $r - Bnd(n)$. Hence, $A/P[x,\sigma']$ satisfies $r - Bnd(n)$. Using Lemma 5 A/P satisfies $r - T(A/P) = n < \infty$. ∎

Corollary 8. *Let A be a ring such that $A[x,\sigma]$ satisfies $r - Bnd(n)$, then all σ-prime ideals of A are σ-maximal.*

Proof: This follows directly from Propositions 4 and 7. ∎

ACKNOWLEDGEMENTS – I would like to express my gratitude to Professors J.M. Goursaud and M.H. Fahmy, for their helpful suggestions which has improved the results of this paper.

REFERENCES

[1] BONNEFOND, G. – *Thesis University of Poitiers*, France, 1979.

[2] GAUCHON, G. – *Thesis University of Orsay*, France, 1977.

[3] EL-AHMER, A. – *Thesis University of Poitiers*, France, 1979.

[4] MCCONNELL, J.C. and ROBSON, J.C. – *Non commutative noetherian rings*, A. Wiely Interscience Publication, 1987.

[5] RENTSCHLER, R. and GABRIEL, P. – Sur la dimension des anneaux et ensemble ordonnes, *C.R. Acad. Sci. Paris*, 265 (1967), 712–715.

[6] ROBSON, J.C. – Cyclic and Faithful objects in quotient categories with application to noetherian simple or Asano rings, in *Noncommutative ring theory*, Lecture Notes in Math., No. 545, Springer-Verlag, 1975.

[7] STAFFORD, J.T. – Completely faithful modules and ideals of simple noetherian rings, *Bull. Lond. Math. Soc.*, 8 (1976), 168–173.

[8] STAFFORD, J.T. – Bounded number of generators of right ideals in polynomial rings, *Comm. in Alg.*, 8(16) (1980), 1513–1518.

Refaat M. Salem,
Department of Mathematics, Faculty of Science,
Al Azhar University, Nasr City, Cairo – EGYPT