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ON A CERTAIN CONSTRUCTION OF MS-ALGEBRAS

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1 – Introduction

The first construction of MS-algebras from Kleene algebras and distributive lattices was presented by T.S. Blyth and J.C. Varlet in [3]. This was a construction by means of so-called "triples" which were successfully used in constructions of Stone algebras (see [6], [7]), distributive p-algebras (see [9]), pseudocomplemented semilattices (see [10]), etc. In [4], T.S. Blyth and J.C. Varlet improved their construction from [3] by means of "quadruples" and they showed that each MS-algebra from the subvariety $\mathbb{K}_2$ ($\mathbb{K}_2$-algebra) can be constructed in this way. This was independently done by T. Katriňák and K. Mikula (in an unpublished paper), who compared then both approaches in [11].

In this paper we establish in a particular case an essential simplification of the above mentioned constructions, which is based on the observation that a $\mathbb{K}_2$-algebra L in which $L^\vee$ is a principal filter is completely determined by the quadruple $(L^{00}, L^\vee, \varphi(L), \gamma(L))$, where φ is — in contrast to the constructions mentioned above — a certain mapping from L^{00} into $L^\vee$ (Section 3). Many complications involved in the previous constructions can be removed in this way. We also show that there exists a one-to-one correspondence between the mentioned class of MS-algebras and the class of so-called decomposable $\mathbb{K}_2$-quadruples (Section 4). In Section 5 we establish similar results for MS-algebras from the subvariety $\mathbf{S} \vee \mathbb{K}$. Two examples illustrate the results.

2 – Preliminaries

An *MS-algebra* is an algebra $(L; \vee, \wedge, {}^0, 0, 1)$ of type $(2,2,1,0,0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and ^{0} is a unary operation such that for all $x, y \in L$

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(1) $x \leq x^{00}$;

(2) $(x \wedge y)^0 = x^0 \vee y^0$;

(3) $1^0 = 0$.

The class of all MS-algebras is equational. Algebras from the subvariety $\mathbb{K}_2$ (we call them briefly $\mathbb{K}_2$-*algebras*) are described by the additional two identities:

(4) $x \wedge x^0 = x^{00} \wedge x^0$;

(5) $(x \wedge x^0) \vee y \vee y^0 = y \vee y^0$.

A $\mathbb{K}_2$-algebra satisfying the identity

(6) $x = x^{00}$

is called a *Kleene* algebra.

Let L be a $\mathbb{K}_2$-algebra. Then

i) $L^{00} = \{x \in L;\ x = x^{00}\}$ is a Kleene algebra;

ii) $L^\vee = \{x \vee x^0;\ x \in L\}$ is a filter of L;

iii) $L^\wedge = \{x \wedge x^0;\ x \in L\}$ is an ideal of L.

Further, for any MS-algebra L,

iv) The relation Φ defined by

$$x \equiv y(\Phi) \quad \text{iff} \ \ x^0 = y^0$$

is a congruence of L such that every Φ-class $[x]\Phi$ containing x contains also the element x^{00} which is the largest element of $[x]\Phi$ and $[x]\Phi \cap L^{00} = \{x^{00}\}$.

For these and other properties of MS-algebras we refer the reader to [1], [2] and [5].

3 – The quadruple construction

The quadruple constructions mentioned above provide a complete representation of any $\mathbb{K}_2$-algebra L by its quadruple $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ where $\varphi(L)$ is a certain mapping from L^{00} into $F(L^\vee)$, the lattice of all filters of $L^\vee$ and $\gamma(L)$ is the restriction of the congruence Φ to the filter $L^\vee$. These constructions can be essentially simplified for those algebras whose filter $L^\vee$ has a smallest element (e.g. finite MS-algebras) — we shall call them *locally bounded*.

First we shall present a simple method of how to construct some $\mathbb{K}_2$-algebras.

Definition 1. An (abstract) *triple* is (K, D, φ), where

i) K is a Kleene algebra;

ii) D is a bounded distributive lattice;

iii) φ is $(0,1)$-lattice homomorphism from K into D.

Theorem 1. *Let (K, D, φ) be a triple. Then*

$$L = \Big\{(x,y);\ x \in K,\ y \in D,\ y \leq \varphi(x)\Big\}$$

is an MS-algebra, if we define

$$\begin{aligned}(x_1,y_1) \vee (x_2,y_2) &= (x_1 \vee x_2, y_1 \vee y_2)\\ (x_1,y_1) \wedge (x_2,y_2) &= (x_1 \wedge x_2, y_1 \wedge y_2)\\ (x,y)^0 &= (x^0, \varphi(x^0))\\ 1_L &= (1,1)\\ 0_L &= (0,0)\ .\end{aligned}$$

Moreover, $L^{00} \cong K$.

Proof: It is easy to prove that L is a sublattice of $K \times D$. Obviously $(0,0), (1,1) \in L$. Thus L is a bounded distributive lattice. Clearly,

$$(x,y) \wedge (x,y)^{00} = (x \wedge x^{00}, y \wedge \varphi(x^{00})) = (x,y)\ ,$$

hence (1) is satisfied in L. The identities (2) and (3) can be verified in the similar way. Now

$$\begin{aligned}L^{00} &= \Big\{(x,y)^{00};\ (x,y) \in L\Big\} = \Big\{(x^{00}, \varphi(x^{00}));\ x \in K\Big\}\\ &= \Big\{(x, \varphi(x));\ x \in K\Big\} \quad \text{(by (6))}\\ &\cong K \quad \text{under the isomorphism } (x,\varphi(x)) \mapsto x\ . \ \blacksquare\end{aligned}$$

By a $\mathbb{K}_2$-*triple* we shall mean a triple (K, D, φ) in which $\varphi(K^\wedge) = \{0_D\}$.

Corollary 1. *Let (K, D, φ) be a $\mathbb{K}_2$-triple. Then the MS-algebra L from Theorem 1 is a $\mathbb{K}_2$-algebra.*

Proof: We shall prove that the identities (4), (5) hold in L. We have

$$\begin{aligned}(x,y) \wedge (x,y)^0 &= (x \wedge x^0, y \wedge \varphi(x^0)) = (x^{00} \wedge x^0, 0)\\ &= (x^{00} \wedge x^0, \varphi(x^{00} \wedge x^0)) = (x,y)^{00} \wedge (x,y)^0\end{aligned} \tag{4}$$

using the fact that

$$y \wedge \varphi(x^0) \le \varphi(x \wedge x^0) = 0_D \ .$$

The identity (5) can be verified in the similar way using the facts that $y = y \wedge \varphi(x)$ and (5) holds in K. ∎

Definition 2. An (abstract) $\mathbb{K}_2$-*quadruple* is (K, D, φ, γ), where (K, D, φ) is a $\mathbb{K}_2$-triple and γ is a monomial congruence on D, i.e. every γ-class $[y]\gamma$ has a largest element — we shall denote it by $\max[y]\gamma$.

Corollary 2. *Let (K, D, φ, γ) be a $\mathbb{K}_2$-quadruple. Then*

$$L = \Big\{(x, y);\ x \in K,\ y \in D,\ y \le \varphi(x) \le \max[y]\gamma\Big\}$$

is a $\mathbb{K}_2$-algebra, if the operations are defined in the same way as in Theorem 1. Moreover, $L^{00} \cong K$.

Proof: It suffices to verify that for any $(x, y), (z, w) \in L$

$$\varphi(x \vee z) \le \max[y \vee w]\gamma \quad \text{and} \quad \varphi(x \wedge z) \le \max[y \wedge w]\gamma$$

hold in L, but this follows from the facts that

$$\varphi(x) \le \max[y]\gamma \quad \text{and} \quad \varphi(z) \le \max[w]\gamma \ . ∎$$

We shall say that the MS-algebra L from Corollary 2 is *associated* with the $\mathbb{K}_2$-quadruple (K, D, φ, γ) and the construction of L described in Corollary 2 will be called a $\mathbb{K}_2$-*construction*.

Let L be a locally bounded $\mathbb{K}_2$-algebra and let b be the smallest element of $L^\vee$. Define a mapping $\varphi(L):\ L^{00} \to L^\vee$ by $\varphi(L)(x) = x \vee b$. Let $\gamma(L)$ be the restriction of the congruence Φ to $L^\vee$. Obviously, $\varphi(L)$ is a $(0,1)$-homomorphism and $\gamma(L)$ is a monomial congruence on $L^\vee$.

We say that $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ is a quadruple *associated* with L.

Since $L^\vee = [b) = [c \vee c^0)$ for some $c \in L$ and (5) holds in L, we have $\varphi(a \wedge a^0) = (a \wedge a^0) \vee c \vee c^0 = c \vee c^0 = b$ for every $a \in L$. Hence the quadruple $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ associated with L is a $\mathbb{K}_2$-quadruple.

The next theorem states that every locally bounded $\mathbb{K}_2$-algebra can be obtained by the $\mathbb{K}_2$-construction.

Theorem 2. *Let L be a locally bounded $\mathbb{K}_2$-algebra. Let $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ be the quadruple associated with L. Then the MS-algebra L_1 associated with $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ is isomorphic to L.*

Proof: Let $L=[b)$. We shall prove that the mapping $f\colon\, L\to L_1$ defined by

$$f(a)=(a^{00},a\vee b)$$

is the desired isomorphism. Obviously $f(a)\in L_1$, since

$$a\vee b\le a^{00}\vee b=\varphi(a^{00})\le a^{00}\vee b^{00}=\max[a\vee b]\,\gamma(L)\ .$$

Evidently, f is a lattice homomorphism and $f(1)=(1,1)$, $f(0)=(0,b)$. Further, we get

$$(f(a))^0=(a^{00},a\vee b)^0=(a^0,\varphi(a^0))=(a^0,a^0\vee b)=f(a^0)\ ,$$

hence f is a homomorphism of MS-algebras. Now assume $f(a_1)=f(a_2)$. Then $a_1^{00}=a_2^{00}$ and $a_1\vee b=a_2\vee b$. Thus $a_1^{00}\wedge(a_1\vee b)=a_2^{00}\wedge(a_2\vee b)$, hence $a_1\vee(a_1^{00}\wedge b)=a_2\vee(a_2^{00}\wedge b)$. Further, for $i\in\{1,2\}$, we have

$$\begin{aligned}
(a_i^{00}\wedge b)\wedge(a_i^0\wedge b)&=a_i^{00}\wedge a_i^0\wedge b\\
&=a_i\wedge a_i^0\wedge b\quad\text{(by (4))}\\
&=(a_i\wedge b)\wedge(a_i^0\wedge b)\ .\\
(a_i^{00}\wedge b)\vee(a_i^0\wedge b)&=(a_i^{00}\vee a_i^0)\wedge b=b\\
&=(a_i\vee a_i^0)\wedge b\\
&=(a_i\wedge b)\vee(a_i^0\wedge b)\ .
\end{aligned}$$

Since L is distributive, we obtain $a_i^{00}\wedge b=a_i\wedge b$, thus $a_i^{00}\wedge b\le a_i$. Hence, $a_1=a_2$ and f is injective. It remains to prove that f is an onto map. Let $(x,y)\in L_1$. Put $a=x\wedge y$. Then we have

$$\begin{aligned}
f(a)&=\Big((x\wedge y)^{00},(x\wedge y)\vee b\Big)=\Big(x^{00}\wedge y^{00},(x\vee b)\wedge(y\vee b)\Big)\\
&=\Big(x\wedge y^{00},(x\vee b)\wedge y\Big)=\Big(x,\varphi(x)\wedge y\Big)=(x,y)
\end{aligned}$$

using the facts that $x=x^{00}$ as $x\in L^{00}$, $b\le y$ as $y\in L^\vee$ and $x\le x\vee b=\varphi(x)\le\max[y]\,\gamma(L)=y^{00}$, $y\le\varphi(x)$ follow from rules of the $\mathbb{K}_2$-construction of L_1. The proof of Theorem 2 is complete. ∎

3 – MS-algebras from $\mathbb{K}_2$ and decomposable $\mathbb{K}_2$-quadruples

In the previous section we presented a simple triple construction of some $\mathbb{K}_2$-algebras, then its modification by quadruples ($\mathbb{K}_2$-construction) and we proved that every locally bounded $\mathbb{K}_2$-algebra is obtained in this way. In this section we shall investigate a relation between the $\mathbb{K}_2$-quadruples which give rise to the same (up to isomorphism) MS-algebra by $\mathbb{K}_2$-construction.

Definition 3. An *isomorphism* of the $\mathbb{K}_2$-quadruples (K, D, φ, γ) and $(K_1, D_1, \varphi_1, \gamma_1)$ is a pair (f, g), where f is an isomorphism of K and K_1, g is an isomorphism of D and D_1 such that $x \equiv y(\gamma)$ iff $g(x) \equiv g(y)(\gamma_1)$ and the diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & D \\ f\downarrow & & \downarrow g \\ K_1 & \xrightarrow{\varphi_1} & D_1 \end{array}$$

is commutative.

Lemma 1. *If two $\mathbb{K}_2$-algebras are isomorphic then their associated quadruples are isomorphic, too.*

The proof is straightforward.

Theorem 3. *Assume that the $\mathbb{K}_2$-quadruples (K,D,φ,γ) and $(K_1, D_1, \varphi_1, \gamma_1)$ are isomorphic under an isomorphism (f, g) and let L and L_1 be their associated $\mathbb{K}_2$-algebras, respectively. Then*

$$L \cong L_1 \ ,$$

where the isomorphism is defined by the rule

$$h((x,y)) = (f(x), g(y)) \ .$$

Proof: Obviously, h is a lattice homomorphism. Further, we have

$$\begin{aligned} h((x,y)^0) &= h(x^0, \varphi(x^0)) = (f(x^0), g(\varphi(x^0))) \\ &= (f(x^0), \varphi_1(f(x^0))) = (f(x)^0, \varphi_1((f(x))^0)) = (f(x), g(y))^0 = h((x,y))^0 \ . \end{aligned}$$

Obviously, h is bijective, thus h is an isomorphism. ∎

We get immediately from Lemma 1 and Theorems 2, 3:

Corollary 3. *Two locally bounded $\mathbb{K}_2$-algebras are isomorphic if and only if their associated quadruples are isomorphic.*

Let us now observe that the converse statement to Theorem 3 is not true, i.e. a $\mathbb{K}_2$-algebra can be obtained from non-isomorphic $\mathbb{K}_2$-quadruples as well. Hence, it is not true, that every $\mathbb{K}_2$-quadruple is isomorphic to a quadruple associated with some $\mathbb{K}_2$-algebra. We illustrate this observation on the next example.

Example 1. Let K be a subdirectly irreducible Kleene algebra, let D be a two-element distributive lattice and let $\varphi\colon\ K \to D$ be the mapping defined by the rule

$$\varphi(0) = \varphi(a) = 0_D\,, \qquad \varphi(1) = 1_D$$

(see Figure 1a).

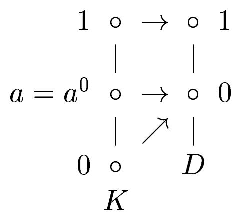

Fig. 1a

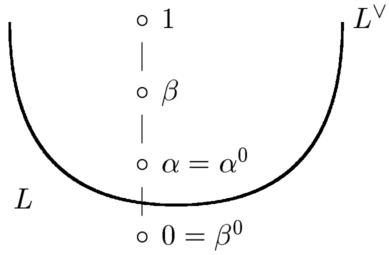

Fig. 1b

Let $\gamma = D \times D$. Then (K, D, φ, γ) is a $\mathbb{K}_2$-quadruple and by the $\mathbb{K}_2$-construction we obtain a (subdirectly irreducible) MS-algebra L, where

$$L = \Big\{(0,0), (a,0), (1,0), (1,1)\Big\}$$

and

$$(a,0)^0 = (a,0)\,, \qquad (1,0)^0 = (0,0)$$

(see Figure 1b – we renamed the elements of L). Obviously,

$$(K, D, \varphi, \gamma) \ncong (L^{00}, L^{\vee}, \varphi(L), \gamma(L))\ ,$$

since $L^{\vee} = \{(a,0), (1,0), (1,1)\}$ is a three element chain. Hence the subdirectly irreducible $\mathbb{K}_2$-algebra L is obtained from two non-isomorphic $\mathbb{K}_2$-quadruples by the $\mathbb{K}_2$-construction, and the $\mathbb{K}_2$-quadruple (K, D, φ, γ) is not isomorphic to any associated quadruple.

Thus the class of all $\mathbb{K}_2$-quadruples is "too large" for establishing a one-to-one correspondence between locally bounded $\mathbb{K}_2$-algebras and $\mathbb{K}_2$-quadruples by means of the $\mathbb{K}_2$-construction. The next theorem gives a characterization of the class of $\mathbb{K}_2$-quadruples for which such a correspondence exists.

Theorem 4. *A $\mathbb{K}_2$-quadruple (K, D, φ, γ) is isomorphic to a quadruple associated with some $\mathbb{K}_2$-algebra if and only if it satisfies the following two conditions:*

i) *For every $y \in D$ there exists a unique element $x_y \in K^\vee$ such that $y \leq \varphi(x_y) \leq \max[y]\gamma$;*

ii) $y_1 \equiv y_2(\gamma)$ *iff* $x^0_{y_1} = x^0_{y_2}$ *for any* $y_1, y_2 \in D$.

Definition 4. A $\mathbb{K}_2$-quadruple (K, D, φ, γ) satisfying the conditions i), ii) from Theorem 4 will be called a *decomposable* $\mathbb{K}_2$-quadruple.

Lemma 2. *Let L be a locally bounded $\mathbb{K}_2$-algebra. Then its associated quadruple $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ is a decomposable $\mathbb{K}_2$-quadruple.*

Proof: We have already observed that $(L^{00}, L^\vee, \varphi(L), \gamma(L))$ is a $\mathbb{K}_2$-quadruple. To prove that it satisfies the condition i), suppose $y \in L^\vee$, i.e., $y = a \vee a^0$ for some $a \in L$. Put $x_y = x = a^{00} \vee a^0 = y^{00}$. Obviously, $x \in (L^{00})^\vee$ and $y \leq \varphi(L)(x) = a^0 \vee a^{00} = \max[y]\,\gamma(L)$, i.e. $(x, y) \in L_1$ where L_1 is a $\mathbb{K}_2$-algebra associated with $(L^{00}, L^\vee, \varphi(L), \gamma(L))$. To prove the uniqueness, suppose that $(x', y) \in L_1$ for an element $x' \in (L^{00})^\vee$. Then $y \leq x' \vee b \leq y^{00}$, hence $x'^{00} \vee b^{00} = y^{00}$. Since $x' \in (L^{00})^\vee$, we have $b \leq x'$ and $x' = x' \vee b^{00} = x'^{00} \vee b^{00} = y^{00} = x$. Now we shall prove ii). Let $y_1, y_2 \in L^\vee$, $y_1 = c \vee c^0$, $y_2 = d \vee d^0$ for some $c, d \in L$. Then $y_1 \equiv y_2(\gamma(L))$ iff $c^0 \wedge c^{00} = d^0 \wedge d^{00}$ and this is equivalent to $x^0_{y_1} = x^0_{y_2}$. ∎

Theorem 5. *Let (K, D, φ, γ) be a decomposable $\mathbb{K}_2$-quadruple. Then there exists a $\mathbb{K}_2$-algebra L such that*

$$\Big(L^{00}, L^\vee, \varphi(L), \gamma(L)\Big) \cong (K, D, \varphi, \gamma)\ .$$

Proof: Let L be a $\mathbb{K}_2$-algebra associated with (K, D, φ, γ). By Theorem 1, the mapping $f\colon L^{00} \to K$ defined by the rule $f(x, \varphi(x)) = x$ is an isomorphism of Kleenean algebras. Now,

$$L^\vee = \Big\{(x, y) \vee (x, y^0);\ (x, y) \in L\Big\} = \Big\{(x \vee x^0, y \vee \varphi(x^0));\ (x, y) \in L\Big\}\ .$$

We shall prove that the mapping $g\colon L^\vee \to D$ defined by the rule

$$g(x \vee x^0, y \vee \varphi(x^0)) = y \vee \varphi(x^0)$$

is a lattice isomorphism. Obviously, g is a lattice homomorphism. Let $y \in D$. By i) of Definition 4 there exists a unique element $x \in K^\vee$ such that $(x, y) \in L$. We have $x = z \vee z^0$ for some $z \in K$, hence $x^0 = z^0 \wedge z^{00}$ and $x = x \vee x^0$. Further $\varphi(x^0) = \varphi(z^{00} \wedge z^0) = 0_D$ as $\varphi(K^\wedge) = \{0_D\}$. Hence $y = y \vee \varphi(x^0)$. Therefore for every $y \in D$ there exists an element $x \in K$ such that $y = y \vee \varphi(x^0)$ and $(x \vee x^0, y) \in L^\vee$. This proves the surjectivity of g. The injectivity of g immediately follows from the condition i) of Definition 4.

Now, let $u=(x_1,y_1)$, $v=(x_2,y_2)\in L^{\vee}$. We know that u, v can be expressed in the form

$$u=(w_1\vee w_1^0,y_1)\,,\qquad v=(w_2\vee w_2^0,y_2)\ ,$$

where $w_1,w_2\in K$ and $\varphi(w_1^0)=\varphi(w_2^0)=0_D$. Thus $u\equiv v(\gamma(L))$ iff $u^0=v^0$, that is equivalent to $w_1^0\wedge w_1^{00}=w_2^0\wedge w_2^{00}$. By ii) this holds iff $y_1\equiv y_2(\gamma)$, that is equivalent to $g(u)\equiv g(v)(\gamma)$.

It remains to prove that the following diagram

$$\begin{array}{ccc} L^{00} & \xrightarrow{\varphi(L)} & L^{\vee} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{\varphi} & D \end{array}$$

is commutative. Using the fact that $g\colon L^{\vee}\to D$ is an isomorphism we can assume that the smallest element of $L^{\vee}$ is of the form $v=(z,0_D)$ for some $z\in K^{\vee}$. Now, let $u\in L^{00}$. Then $u=(x,\varphi(x))$ for some $x\in K$ and we have

$$g(\varphi(L)(u))=g\Big((x,\varphi(x))\vee(z,0_D)\Big)=\varphi(x)=\varphi(f(x,\varphi(x)))=\varphi(f(u))\ .$$

This completes the proof of Theorem 5. ∎

Now we shall prove Theorem 4:

Proof of Theorem 4: Let L be a $\mathbb{K}_2$-algebra such that $(L^{00},L^{\vee},\varphi(L),\gamma(L))\cong(K,D,\varphi,\gamma)$ and (f,g) be the corresponding isomorphism. First we shall prove that (K,D,φ,γ) satisfies the condition i). Let $y\in D$. Using Lemma 2 for an element $y'=g^{-1}(y)\in L^{\vee}$ there exists a unique element $x_{y'}=x'\in(L^{00})^{\vee}$ such that $y'\le\varphi(L)(x')\le\max[y']\gamma(L)$. Put $x_y=x=f(x')$. Clearly, $x\in K^{\vee}$ and

$$y=g(y')\le g(\varphi(L)(x'))=\varphi(f(x'))=\varphi(x)\ ,$$

$$\begin{aligned}\varphi(x)=\varphi(f(x'))=g(\varphi(L)(x'))&\le g(\max[y']\gamma(L))\\ &=\max[g(y')]\gamma=\max[y]\gamma\ .\end{aligned}$$

Thus for every $y\in D$ there exists an element $x_y\in K^{\vee}$ such that $y\le\varphi(x_y)\le\max[y]\gamma$. From the uniqueness of the element x_y follows the uniqueness of the element x_y. The condition ii) can be verified in the similar way.

The converse statement follows from Theorem 5. ∎

Note that the $\mathbb{K}_2$-quadruple (K,D,φ,γ) from Example 1 is not decomposable. Now, we summarize the previous results:

Corollary 4. *There exists a one-to-one (up to isomorphism) correspondence between locally bounded $\mathbb{K}_2$-algebras and decomposable $\mathbb{K}_2$-quadruples by means of the $\mathbb{K}_2$-construction. More precisely:*

i) *Let (K, D, φ, γ) be a decomposable $\mathbb{K}_2$-quadruple. Then its associated MS-algebra L is a locally bounded $\mathbb{K}_2$-algebra and*

$$(L^{00}, L^{\vee}, \varphi(L), \gamma(L)) \cong (K, D, \varphi, \gamma) \ .$$

ii) *Let L be a locally bounded $\mathbb{K}_2$-algebra. Then its associated quadruple $(L^{00}, L^{\vee}, \varphi(L), \gamma(L))$ is a decomposable $\mathbb{K}_2$-quadruple and if L_1 is an MS-algebra associated with the quadruple $(L^{00}, L^{\vee}, \varphi(L), \gamma(L))$ then*

$$L \cong L_1 \ . \blacksquare$$

5 – A construction of MS-algebras from the subvariety $\mathbf{S} \vee \mathbb{K}$

In this section we give an analogue construction of locally bounded MS-algebras from the subvariety $\mathbf{S} \vee \mathbb{K}$ ($\mathbf{S} \vee \mathbb{K}$-*algebras*). The subvariety $\mathbf{S} \vee \mathbb{K}$ is the join of the variety $\mathbf{S}$ of Stonean algebras and the variety $\mathbb{K}$ of Kleenean algebras and is defined by the identities (4), (5) and

$$x \vee y^0 \vee y^{00} = x^{00} \vee y^0 \vee y^{00} \ . \tag{6}$$

Lemma 3. *Let L be a locally bounded $\mathbf{S}\vee\mathbb{K}$-algebra and let $(L^{00}, L^{\vee}, \varphi(L), \gamma(L))$ be its associated quadruple. Let $c \le \varphi(L)(a) \le \max[c]\gamma(L)$ for any $a \in L^{00}$, $c \in L^{\vee}$. Then*

$$c \vee \varphi(L)(d) = \varphi(L)(a) \vee \varphi(L)(d) \quad \text{for any } \ d \in (L^{00})^{\vee} \ .$$

Proof: Let b be the smallest element of $L^{\vee}$ and let $d = e^0 \vee e^{00}$, where $e \in L$. By the hypothesis $c \le a \vee b \le c^{00}$, which implies $c^{00} = a^{00} \vee b^{00} = a \vee b^{00}$. Thus

$$\begin{aligned} c \vee \varphi(L)(d) &= c \vee e^0 \vee e^{00} = c^{00} \vee e^0 \vee e^{00} = a \vee b^{00} \vee e^0 \vee e^{00} \\ &= a \vee e^0 \vee e^{00} = \varphi(L)(a) \vee \varphi(L)(d) \end{aligned}$$

using the fact that (6) holds in L. $\blacksquare$

Definition 5. By a *decomposable* $\mathbf{S}\vee\mathbb{K}$-quadruple we mean a decomposable $\mathbb{K}_2$-quadruple (K, D, φ, γ) satisfying the following condition:

$$\begin{array}{lll} \text{if} & y \le \varphi(x) \le \max[y]\gamma & \text{for any } \ x \in K, \ \ y \in D \ , \\ \text{then} & y \vee \varphi(z) = \varphi(x) \vee \varphi(z) & \text{for any } \ z \in K^{\vee} \ . \end{array}$$

Theorem 6. *There exists a one-to-one correspondence between locally bounded* $\mathbf{S} \vee \mathbb{K}$*-algebras and decomposable* $\mathbf{S} \vee \mathbb{K}$*-quadruples by means of the* $\mathbb{K}_2$*-construction. More precisely:*

i) *Let* (K, D, φ, γ) *be a decomposable* $\mathbf{S} \vee \mathbb{K}$*-quadruple. Then its associated MS-algebra* L *is a locally bounded* $\mathbf{S} \vee \mathbb{K}$*-algebra and*

$$(L^{00}, L^{\vee}, \varphi(L), \gamma(L)) \cong (K, D, \varphi, \gamma) \ .$$

ii) *Let* L *be a locally bounded* $\mathbf{S} \vee \mathbb{K}$*-algebra. Then its associated quadruple* $(L^{00}, L^{\vee}, \varphi(L), \gamma(L))$ *is a decomposable* $\mathbf{S} \vee \mathbb{K}$*-quadruple. If* L_1 *is an MS-algebra associated with* $(L^{00}, L^{\vee}, \varphi(L), \gamma(L))$ *then*

$$L \cong L_1 \ .$$

Proof:

i) It suffices to prove that (6) holds in L. Let $(x,y),(z,w) \in L$. Then by Definition 5

$$y \vee \varphi(z^0 \vee z^{00}) = \varphi(x) \vee \varphi(z^0 \vee z^{00}) \ ,$$

thus

$$\begin{aligned}(x,y) \vee (z,w)^0 \vee (z,w)^{00} &= \Big(x \vee z^0 \vee z^{00}, y \vee \varphi(z^0 \vee z^{00})\Big) \\ &= \Big(x \vee z^0 \vee z^{00}, \varphi(x) \vee \varphi(z^0 \vee z^{00})\Big) \\ &= (x,y)^{00} \vee (z,w)^0 \vee (z,w)^{00} \ .\end{aligned}$$

ii) The statement follows immediately from Lemma 3 and Corollary 4. ∎

Example 2. Let K and D be the Kleenean algebra and the distributive lattice depicted respectively on Figure 2a.

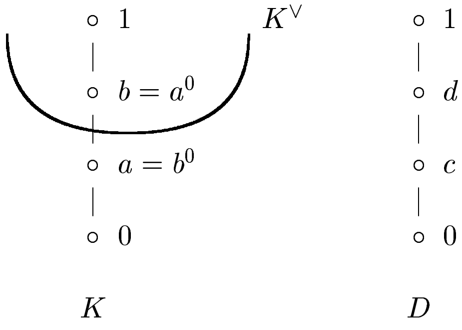

Fig. 2a

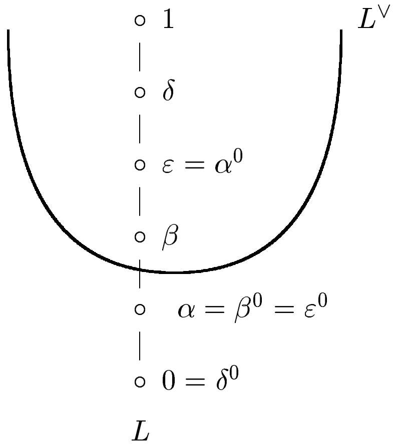

Fig. 2b

Define a homomorphism $\varphi\colon\ K \to D$ by the rule

$$\begin{aligned} &\varphi(0) = \varphi(a) = 0_D \ , \\ &\varphi(b) = c\,, \qquad \varphi(1) = 1_D \ , \end{aligned}$$

and a congruence γ on D having two classes $\{0,c\}\gamma$ and $\{d,1\}\gamma$. Clearly, $\varphi(K^\wedge) = \{0_D\}$, thus (K,D,φ,γ) is a $\mathbb{K}_2$-quadruple. It is easy to verify that it satisfies the conditions i), ii) from Theorem 4, where 1 and b are elements of $K^\vee$ corresponding to the elements $1,d$ and $c,0$ of D in the required correspondence between $K^\vee$ and D. Hence (K,D,φ,γ) is a decomposable $\mathbb{K}_2$-quadruple. But it is not a decomposable $\mathbf{S}\vee\mathbb{K}$-quadruple, since for $x=1$, $y=d$, $z=b$ we have $y \le \varphi(x) \le \max[y]\gamma$, but $y \vee \varphi(z) \ne \varphi(x) \vee \varphi(z)$.

By means of the $\mathbb{K}_2$-construction we get an MS-algebra L such that

$$L = \Big\{(0,0), (a,0), (b,0), (b,c), (1,d), (1,1)\Big\}$$

and

$$\begin{aligned} &(0,0)^0 = (1,1) \ , \\ &(a,0)^0 = (b,c) \ , \\ &(b,0)^0 = (b,c)^0 = (a,0) \ , \\ &(1,d)^0 = (1,1)^0 = (0,0) \ . \end{aligned}$$

The algebra L is represented on Figure 2b (we again renamed its elements). The homomorphism $\varphi(L):\ L^{00} \to L^\vee$ is defined by $\varphi(L)(x) = x \vee b$. One can verify that the algebra L is a $\mathbb{K}_2$-algebra, but it is not an $\mathbf{S}\vee\mathbb{K}$-algebra since $\delta = \delta\vee\beta^0\vee\beta^{00} < \delta^{00}\vee\beta^0\vee\beta^{00} = 1$. Moreover,

$$\Big(L^{00}, L^\vee, \varphi(L), \gamma(L)\Big) \cong (K,D,\varphi,\gamma) \ .$$

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