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A CONSTRUCTION OF A PARAMETRIX FOR AN ELLIPTIC BOUNDARY VALUE PROBLEM

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0 – Introduction

In this paper we present a method of construction of a parametrix for a general class of elliptic boundary value problems. Our method uses the fact that the given pseudodifferential operator $P(x, t, D_x, D_t)$ can be factorized as

$$P = M^- M^+ + R \ ,$$

where R is a regularizing operator, and M^+ and M^- are suitable pseudodifferential operators. Such a factorization is a consequence of the classical *ellipticity* condition for P stated at the beginning of Section 1. The principal symbols of M^+ and M^- are exactly the two factors of P^0, the principal symbol of P, containing respectively by the roots of P^0 with positive and imaginary parts. (See formulas (1.3) and (1.4).)

The components of a parametrix satisfying (1.10) are defined as pseudodifferential operators. Their formal symbols, constructed by successive approximations, belong to an appropriate symbol class described in Section 2.

What makes our method interesting is the fact that the problem discussed in this paper is a prototype of a more general boundary value problems for the class of *formally hypoelliptic* operators. A differential operator $P(x, D)$ with smooth coefficients is an open subset Ω of $\mathbb{R}^n$ is said to be formally hypoelliptic if:

i) $P(a, D)$ is hypoelliptic for some fixed $a \in \Omega$;

ii) All the operators $P(y, D)$ with $y \in \Omega$ are equally strong.

Formally hypoelliptic operators were studied by [6] and [8]. In a forthcoming paper [2] we show that it is possible to define the *type* and *index* of hypoellip-

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ticity for a formally hypoelliptic operator $P(x, D)$, provided that x remains in a compact subset of Ω. For such operators we can define in a natural way a general class of boundary value problems. We also show that for such operators, a factorization $P = M^- M^+ + R$, similar to that of the elliptic case, holds true. We can then apply the results of the present paper to construct a parametrix and study regularity up to the boundary for a general class of *hypoelliptic* boundary value problems.

We should mention that some of the techniques employed throughout this paper are very close to the ones used by Bergamasco and Petronilho in [4], where they studied the simpler case of a Dirichlet boundary value problem. However, our approach is different. We rely strongly on a representation formula for the solution of a suitable ordinary differential equation (see Sections 2 and 3) and follow closely the ideas used in our papers [1] and [3] on the constant coefficient case. In [4], to estimate the corresponding symbols, Bergamasco and Petronilho use a decomposition lemma (Lemma 2 of [4]) that splits the open set Ω into subsets where the roots of the characteristic polynomial have constant multiplicities, a technique that we were unable to apply to our case.

Section 1 contains generalities about the setting up of our problem. In Section 2, we obtain, by successive approximations, the formal symbols of the parametrix. In Section 3 and 4, we estimate all the terms appearing in the formal symbol and show that they belong to the class symbol $B_t^m(\Omega)$. Finally we show that the formal symbols define true symbols and obtain the desired regularity results.

1 – Preliminaries

In this paper we shall restrict our study to the region $\omega \times [0, T)$, where ω is an open set in $\mathbb{R}^n$. We assume that in this region $P = P(x, t, D_x, D_t)$ is a pseudodifferential operator of the form

$$P = D_t^\sigma + \sum_{j=1}^{\sigma} P_j(x, t, D_x)\, D_t^{\sigma-j} \ , \tag{1.1}$$

where, for each $1 \leq j \leq \sigma$, $P_j(x, t, D_x)$ is *classical pseudodifferential operator* of order j in ω whose coefficients are $\mathcal{C}^\infty$ functions in $\omega \times [0, T)$. Let

$$P^0 = \sigma(P) = \tau^\sigma + \sum_{j=1}^{\sigma} P_j^0(x, t, \xi)\, \tau^{\sigma-j} \tag{1.2}$$

be the *principal symbol* of P, where P^0 is the homogeneous term of degree j, with respect to ξ, in $P_j(x, t, \xi)$. We assume the following *ellipticity* condition:

There are two integers m^+ and m^-, such that

$$m^+ + m^- = \sigma\,, \qquad m^+ \geq 1\ ,$$

and for all $(x,\xi) \in T^*\omega\backslash\{0\}$, $t \in [0,T)$ the polynomial $\sigma(P)$ with respect to τ has exactly m^+ roots with positive imaginary part and m^- roots with negative imaginary parts.

We remark that when $\dim\omega > 1$ and P is a differential operator or, more generally, an *antipodal* pseudodifferential operator, [9, pg. 158], the above ellipticity condition implies that $m^+ = m^- = \sigma/2$.

The principal symbol P^0 factorizes as follows

$$P^0(x,t,\xi,\tau) = M^{+0}(x,t,\xi,\tau)\, M^{-0}(x,t,\xi,\tau)\ , \tag{1.3}$$

$$M^{\pm 0}(x,t,\xi,\tau) = \prod_{j=1}^{m^\pm}\Big(\tau - \tau_j^\pm(x,t,\xi)\Big)\ , \tag{1.4}$$

where τ_j^+ (resp. τ_j^-) are the roots with positive (resp. negative) imaginary part. Note that the roots $\tau_j^\pm(x,t,\xi)$ are positive homogeneous of degree 1 with respect to ξ.

Using the results of [9], it follows that the operator $P(x,t,D_x,D_t)$ can be written as

$$P(x,t,D_x,D_t) = M^-\, M^+ + R\ , \tag{1.5}$$

with

$$M^\pm(x,t,D_x,D_t) = D_t^{m^\pm} + \sum_{j=1}^{m^\pm} M_j^\pm(x,t,D_x)\, D_t^{m^\pm - j}\ ,$$

where $M_j^\pm(x,t,D_x)$ is, for $1 \leq j \leq m^\pm$ is a classical pseudodifferential operator of order j on ω and where R is a regularizing operator in ω, all depending smoothly on t. Also the principal symbols of the operators $M^\pm$ are given by equation (1.4), that is

$$\sigma(M^\pm) = M^{\pm 0}(x,t,\xi,\tau) = \tau^{m^\pm} + \sum_{j=1}^{m^\pm} M_j^{\pm 0}(x,t,\xi)\, \tau^{m^\pm - j}\ ,$$

where for each j the coefficients $M_j^{\pm 0}(x,t,\xi)$ are $\mathcal{C}^\infty$ functions in (x,t,ξ), positive-homogeneous of degree j with respect to ξ. This is so because $M_j^{+0}(x,t,\xi)$ (resp. $M_j^{-0}(x,t,\xi)$) is a symmetric function of the roots τ_j^+ (resp. τ_j^-) and that the two sets of roots stay apart as (x,t,ξ) varies.

From now on we set $m = m^+$ and let $Q_\nu(x, D_x, D_t)$, $1 \leq \nu \leq m$, be given boundary operators of the form

$$Q_\nu(x, D_x, D_t) = \sum_{j=0}^{n_\nu} Q_{\nu j}(x, D_x)\, D_t^{n_\nu - j} \ , \tag{1.6}$$

where, for each $1 \leq j \leq n_\nu$, $Q_{\nu j}(x, D_x)$ is a differential operator of order j in ω whose coefficients are $\mathcal{C}^\infty$ functions in $\omega \times [0, T)$.

We wish to solve modulo regularizing operators the boundary value problem

$$\begin{cases} P(x, t, D_x, D_t)\, u(x, t) = f(x, t), \\ Q_\nu(x, D_x, D_t)\, u(x, t)_{|t=0} = g_\nu(x)\,, \quad 1 \leq \nu \leq m \ , \end{cases} \tag{1.7}$$

and to study the regularity of the solution. It suffices to study the homogeneous problem

$$\begin{cases} P(x, t, D_x, D_t)\, u(x, t) = 0, \\ Q_\nu(x, D_x, D_t)\, u(x, t)_{|t=0} = g_\nu(x)\,, \quad 1 \leq \nu \leq m \ , \end{cases} \tag{1.8}$$

for if u_1 is any solution of $Pu = f$ and u_2 is a solution of the homogeneous problem (1.8) with g_ν substituted for $g_\nu - Q_\nu u_1$, then $u = u_1 + u_2$ satisfies (1.7) In view of (1.5) this is equivalent modulo a regularizing operator to solve the problem

$$\begin{cases} M^+(x, t, D_x, D_t)\, u(x, t) = 0, \\ Q_\nu(x, D_x, D_t)\, u(x, t)_{|t=0} = g_\nu(x)\,, \quad 1 \leq \nu \leq m \ . \end{cases} \tag{1.9}$$

In order to do this, we are going to construct a *parametrix* for the problem (1.9).

Definition 1.1. By a parametrix to (1.9) we mean pseudodifferential operators $H_j(t)$, $1 \leq j \leq m$, on ω varying smoothly in $t \in [0, T)$, such that

$$\begin{cases} M^+(x, t, D_x, D_t)\, H_j(t) \sim 0 \\ Q_\nu(x, D_x, D_t)\, H_j(t)_{|t=0} \sim \delta_{\nu j}\, I\,, \quad 1 \leq \nu \leq m \ . \end{cases} \tag{1.10}$$

Once the parametrix is obtained, the solution of (1.9) is given by

$$u(x, t) \sim \sum_{j=1}^{m} H_j(t)\, g_j(x) \ .$$

Incidentally, if E is a parametrix of P (which exists because P is elliptic), then

$$u(x,t) = Ef + \sum_{j=1}^{m} H_j(t) \Big[g_j(x) - Q_j\, E\, f\Big]$$

satisfies (1.7).

Following [1] and [5] we define by

$$\mathcal{C}^0(x,\xi) = \frac{\det\Big(Q^0_\nu(x,\xi,\tau^+_\ell(x,0,\xi))\Big)}{\prod_{k<j}\Big(\tau^+_j(x,0,\xi) - \tau^+_k(x,0,\xi)\Big)} \tag{1.11}$$

the *characteristic function* of the boundary value problem (1.8) and assume

$$\mathcal{C}^0(x,\xi) \neq 0\,, \qquad \forall\, x \in \omega \times \mathbb{R}^n \backslash \{0\}\ . \tag{1.12}$$

This is equivalent to the classical Lopatinski–Shapiro conditions on Q_ν.

Note that $\mathcal{C}^0(x,\xi)$ is homogeneous of degree

$$A = n_1 + n_2 + ... + n_m - 1 - 2 - ... - (m-1)\ , \tag{1.13}$$

with respect to ξ.

2 – Construction of the parametrix

We try to construct H_j as a pseudodifferential operator

$$H_j(t)\, g(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\, h_j(x,t,\xi)\, \widehat{g}(\xi)\, d\xi\ , \tag{2.1}$$

with $g \in \mathcal{C}^\infty_c(\omega)$ and

$$h_j(x,t,\xi) \sim \sum_{\ell=0}^{\infty} h_{j\ell}(x,t,\xi)\ , \tag{2.2}$$

where $h_{j\ell}(x,t,\xi)$ will be determined by recurrence as follows.

Applying $M^+(x,t,D_x,D_t)$ to (2.1) and using the composition of pseudodifferential operators, we have

$$\begin{aligned} &M^+(x,t,D_x,D_t)\, H_j(t)\, g(x) = \\ &= (2\pi)^{-n} \int e^{ix\cdot\xi} \Big\{D^m_t + \sum_{q=1}^{m} M^+_q(x,t,D_x+\xi) D^{m-q}_t\Big\}\, h_j(x,t,\xi)\, \widehat{g}(\xi)\, d\xi\ . \end{aligned} \tag{2.3}$$

Now

$$
\begin{aligned}
M_q^+(x,t,D_x+\xi) &= \sum_\alpha \frac{1}{\alpha!} M_q^{+(\alpha)}(x,t,\xi)\, D_x^\alpha \\
&= \sum_{\ell=0}^\infty M_{q\ell}^+(x,t,\xi,D_x)\ ,
\end{aligned}
\tag{2.4}
$$

where $M_{q\ell}^+$ is a differential operator of order $\leq \ell$ in D_x, whose coefficients depend on (x,t,ξ). Also observing that $M_{q0}^+ = M_q^+(x,t,\xi)$ can be written as follows

$$
M_{q0}^+(x,t,\xi,D_x) = M_{q0}^+(x,0,\xi) + t\,\widetilde{M}_{q0}^+(x,t,\xi)\ , \tag{2.5}
$$

we set

$$
L(x,\xi,D_t) = D_t^m + \sum_{q=1}^m M_{q0}^+(x,0,\xi)\, D_t^{m-q}\ . \tag{2.6}
$$

In view of (1.10) we set

$$
D_t^m h_j(x,t,\xi) + \sum_{q=1}^m \sum_{\ell=0}^\infty M_{q\ell}^+(x,t,\xi,D_x)\, D_t^{m-q} h_j(x,t,\xi) = 0\ ,
$$

or equivalently

$$
\begin{aligned}
L(x,\xi,D_t)\, h_j(x,t,\xi) + t\sum_{q=1}^m \widetilde{M}_{q0}^+(x,t,\xi)\, D_t^{m-q} h_j(x,\xi,t) &+ \\
+ \sum_{q=1}^m \sum_{\ell=1}^\infty M_{q\ell}^+(x,t,\xi,D_x)\, D_t^{m-q} h_j(x,t,\xi) &= 0\ .
\end{aligned}
$$

If each h_j, $1 \leq j \leq m$, is given by an asymptotic expansion

$$
h_j(x,t,\xi) \sim \sum_{\ell=0}^\infty h_{j\ell}(x,t,\xi)\ ,
$$

then we may determine the terms $h_{j\ell}$ by recurrence as follows

$$
Lh_{j0} = 0\ ,
$$

and

$$
Lh_{j\ell} = -t\sum_{q=1}^m \widetilde{M}_{q0}^+\, D_t^{m-q} h_{j,\ell-1} - \sum_{q=1}^m \sum_{k=0}^{\ell-1} M_{q,\ell-k}^+\, D_t^{m-q} h_{jk}\ ,
$$

for $\ell = 1, 2, \dots$.

To these equations we have to adjoin the corresponding boundary conditions. As before, we have

$$(2.7)\qquad Q_\nu\, H_j(t)\, g(x) = (2\pi)^{-n} \int e^{ix\xi}\, Q_\nu(x, D_x + \xi, D_t)\, h_j(x, t, \xi)\, \widehat{g}(\xi)\, d\xi\ ,$$

with

$$Q_\nu(x, D_x + \xi, D_t) = Q_\nu^0(x, \xi, D_t) + \sum_{|\alpha| \geq 1} \widetilde{Q}_{\nu\alpha}(x, \xi, D_t)\, D_x^\alpha\ ,$$

where

$$Q_\nu^0(x, \xi, \tau) = \sum_{j=0}^{n_\nu} Q_{\nu j}^0(x, \xi)\, \tau^{n_\nu - j}$$

is homogeneous of degree n_ν and $\widetilde{Q}_{\nu\alpha}(x, \xi, \tau)$ has degree $< n_\nu - |\alpha|$. In view of (1.10) we set

$$Q_\nu(x, D_x + \xi, D_t)\, h_{j\,|t=0} = \delta_{\nu j}\ .$$

Since $h_j \sim \sum h_{j\ell}$, we require that

$$Q_\nu^0(x, \xi, D_t)\, h_{j0\,|t=0} = \delta_{\nu j}$$

and

$$Q_\nu^0(x, \xi, D_t)\, h_{j\ell\,|t=0} = -\sum_{k=0}^{\ell-1} \sum_{|\alpha| = \ell - k} \widetilde{Q}_{\nu\alpha}(x, \xi, D_t)\, D_x^\alpha h_{jk\,|t=0}\ .$$

Thus we have to solve by recurrence the systems of ordinary differential equations

$$(2.8)\qquad \begin{cases} L(x, \xi, D_t)\, h_{j0}(x, t, \xi) = 0, \\ Q_\nu^0(x, \xi, D_t)\, h_{j0}(x, t, \xi)_{|t=0} = \delta_{\nu j}\,, \qquad 1 \leq \nu \leq m\ , \end{cases}$$

and

$$(2.9)\qquad \begin{cases} L h_{j\ell} = -t \displaystyle\sum_{q=1}^{m} \widetilde{M}_{q0}^{+}\, D_t^{m-q} h_{j,\ell-1} - \sum_{q=1}^{m} \sum_{k=0}^{\ell-1} M_{q,\ell-k}^{+}\, D_t^{m-q} h_{jk}, \\ Q_\nu^0\, h_{j\ell\,|t=0} = -\displaystyle\sum_{k=0}^{\ell-1} \sum_{|\alpha| = \ell - k} \widetilde{Q}_{\nu\alpha}\, D_x^\alpha h_{jk\,|t=0}\,, \qquad 1 \leq \nu \leq m\ , \end{cases}$$

for $\ell = 1, 2, \ldots$.

3 – Estimating h_{j0}

In this section, by using a representation formula for h_{j0} the solution of equation (2.8) in section 2 we prove the following theorem.

Theorem 3.1. *For each j, $1 \le j \le m$ there is a positive constant B such that for any $\alpha, \beta \in \mathbf{Z}_+^n$, $k \in \mathbf{Z}_+$ there are constants $C = C(\alpha,\beta,k) > 0$ and $N = N(\alpha,\beta,k) > 0$ such that*

$$\left|D_x^\alpha D_\xi^\beta D_t^k h_{j0}(x,t,\xi)\right| \le C(1+t|\xi|)^N\, |\xi|^{-n_j-|\beta|+k}\, e^{-Bt|\xi|}\ , \tag{3.1}$$

for all $(x,t,\xi) \in \omega \times [0,T) \times (\mathbb{R}^n \backslash \{0\})$.

Proof: Using (2.24) in [1] we can represent the solution of (2.8) as

$$h_{j0}(x,t,\xi) = \frac{R\Big(L, Q_1^0(x,\xi,\tau), ..., Q_{j-1}^0(x,\xi,\tau), e^{it\tau}, Q_{j+1}^0(x,\xi,\tau), ..., Q_m^0(x,\xi,\tau)\Big)}{\mathcal{C}^0(x,\xi)}\ ,$$

where

$$R\Big(L, Q_1^0(x,\xi,\tau), ..., Q_{j-1}^0(x,\xi,\tau), e^{it\tau}, Q_{j+1}^0(x,\xi,\tau), ..., Q_m^0(x,\xi,\tau)\Big) =$$

$$= \frac{\begin{vmatrix} Q_1^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_1^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ \vdots & & \vdots \\ Q_{j-1}^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_{j-1}^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ e^{it\tau_1^+(x,0,\xi)} & \dots & e^{it\tau_m^+(x,0,\xi)} \\ Q_{j+1}^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_{j+1}^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ \vdots & & \vdots \\ Q_m^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_m^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \end{vmatrix}}{\prod\limits_{k<j}\Big(\tau_j^+(x,0,\xi) - \tau_k^+(x,0,\xi)\Big)}\ .$$

Since each $h_{j0}(x,t,\xi)$ is a solution of an ordinary differential equation in t depending upon parameters x and ξ, it follows, by the theorem on smoothness of dependence on parameters, that h_{j0} is a $\mathcal{C}^\infty$ function of $(x,\xi) \in \omega \times \mathbb{R}^n \backslash \{0\}$.

From a classical theorem on algebraic functions (see, for example, [5]), the roots $\tau_j^+(x,0,\xi)$ are bounded and their imaginary parts $\operatorname{Im}\tau_j^+(x,0,\xi)$ are bounded away from zero for $(x,\xi) \in \omega \times S^{n-1}$ (by shrinking ω if necessary). Thus, there exist constants $C > 0$, $B > 0$ such that $\operatorname{Im}\tau_j^+(x,0,\xi) \ge B > 0$, $|\tau_j^+(x,0,\xi)| \le C$, for all $(x,\xi) \in S^{n-1}$, and by homogeneity

$$\operatorname{Im}\tau_j^+(x,0,\xi) \ge B|\xi|\ , \qquad |\tau_j^+(x,0,\xi)| \le C|\xi|\ , \tag{3.2}$$

for all $(x,\xi) \in \omega \times \mathbb{R}^n \backslash \{0\}$.

By hypothesis since $\mathcal{C}^0(x,\xi)$ is homogeneous of degree A given by (1.13), there is a constant $C_1 > 0$, such that

$$|\mathcal{C}^0(x,\xi)| \geq C_1 |\xi|^A\,, \qquad \forall\,(x,\xi) \in \omega \times \mathbb{R}^n \backslash \{0\}\,. \tag{3.3}$$

We now want to estimate the derivatives $D_t^k h_{j0}$, for $k \geq 0$. First we note that $D_t^k R$ is equal

$$R\Big(L, Q_1^0, ..., Q_{j-1}^0, (i\tau)^k e^{it\tau}, Q_{j+1}^0, ..., Q_m^0\Big) = \tag{3.4}$$

$$= \frac{\begin{vmatrix} Q_1^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_1^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ \vdots & & \vdots \\ Q_{j-1}^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_{j-1}^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ (i\,\tau_1^+)^k\, e^{it\tau_1^+(x,0,\xi)} & \dots & (i\,\tau_m^+)^k\, e^{it\tau_m^+(x,0,\xi)} \\ Q_{j+1}^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_{j+1}^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \\ \vdots & & \vdots \\ Q_m^0\Big(x,\xi,\tau_1^+(x,0,\xi)\Big) & \dots & Q_m^0\Big(x,\xi,\tau_m^+(x,0,\xi)\Big) \end{vmatrix}}{\displaystyle\prod_{k<j}\Big(\tau_j^+(x,0,\xi) - \tau_k^+(x,0,\xi)\Big)}\,.$$

If $z_1, ..., z_r$ are variable elements in the domain of definition of f, a single variable function, denote by

$$f(z_1,z_2) = \frac{f(z_1) - f(z_2)}{z_1 - z_2}$$

and

$$f(z_1,...,z_s) = \frac{f(z_1,...,z_{s-1}) - f(z_2,...,z_s)}{z_1 - z_s}\,, \qquad s \leq r\,,$$

the *iterated divided differences* of f. Now setting $g_j(\tau) = (i\tau)^k\, e^{it\tau}$ and using divided differences, it is easy to see that (3.4) can be written as the determinant

$$\begin{vmatrix} Q_1^0(x,\xi,\tau_1^+) & Q_1^0(x,\xi,\tau_1^+,\tau_2^+) & \dots & Q_1^0(x,\xi,\tau_1^+,...,\tau_m^+) \\ \vdots & \vdots & & \vdots \\ Q_{j-1}^0(x,\xi,\tau_1^+) & Q_{j-1}^0(x,\xi,\tau_1^+,\tau_2^+) & \dots & Q_{j-1}^0(x,\xi,\tau_1^+,...,\tau_m^+) \\ g_j(\tau_1^+) & g_j(\tau_1^+,\tau_2^+) & \dots & g_j(\tau_1^+,...,\tau_m^+) \\ Q_{j+1}^0(x,\xi,\tau_1^+) & Q_{j+1}^0(x,\xi,\tau_1^+,\tau_2^+) & \dots & Q_{j+1}^0(x,\xi,\tau_1^+,...,\tau_m^+) \\ \vdots & \vdots & & \vdots \\ Q_m^0(x,\xi,\tau_1^+) & Q_m^0(x,\xi,\tau_1^+,\tau_2^+) & \dots & Q_m^0(x,\xi,\tau_1^+,...,\tau_m^+) \end{vmatrix}\,.$$

This determinant is a sum of products of the form

$$\prod_{\substack{p=1\\p\neq j}}^{m} Q_p^0(\tau_1,...,\tau_{\ell_p})\, g_j(\tau_1,...,\tau_{\ell_j}) \;, \tag{3.5}$$

where $\ell_1, \ell_2, ..., \ell_m$ is a permutation of $1, 2, ..., m$. Note that for simplicity of notation we have dropped the variables x and ξ, and the superscript $+$. By using the inequality (2.79) of [1] we obtain the following estimates

$$|Q_p^0(\tau_1,...,\tau_{\ell_p})| \le \frac{1}{(\ell_p-1)!}\sup_{\mathcal{K}} |(Q_p^0)^{(\ell_p-1)}(\tau)| \le C|\xi|^{n_p-(\ell_p-1)} \tag{3.6}$$

and

$$|g_j(\tau_1,...,\tau_{\ell_j})| \le \frac{1}{(\ell_j-1)!}\sup_{\mathcal{K}} |g_j^{(\ell_j-1)}(\tau)| \;, \tag{3.7}$$

where $\mathcal{K}$ denotes the convex hull of the set of zeros $\{\tau_j^+(x,0,\xi)\}$. Now we set $\ell_j - 1 = \ell$ and estimate the derivative

$$g_j^\ell(\tau) = \frac{d^\ell}{d\tau^\ell}((i\tau)^k\, e^{it\tau}) \;.$$

We have

$$\begin{aligned}\frac{d^\ell}{d\tau^\ell}((i\tau)^k\, e^{it\tau}) &= \sum_{q=0}^{\ell}\binom{\ell}{q}\frac{d^q}{d\tau^q}((i\tau)^k)\,\frac{d^{\ell-q}}{d\tau^{\ell-q}}(e^{it\tau})\\ &= \sum_{q=0}^{\ell}\binom{\ell}{q} i^q\, k(k-1)\cdots(k-q-1)\,(i\tau)^{k-q}\,(it)^{\ell-q}\, e^{it\tau} \;.\end{aligned}$$

By using the estimates in (3.2) we get

$$\begin{aligned}\left|\frac{d^\ell}{d\tau^\ell}((i\tau)^k\, e^{it\tau})\right| &\le C\sum_{q=0}^{\ell}|\xi|^{k-q}\, t^{\ell-q}\, e^{-tB|\xi|}\\ &\le C\sum_{q=0}^{\ell}(t|\xi|)^{\ell-q}\,|\xi|^{k-\ell}\, e^{-tB|\xi|}\\ &\le C\Big(1+t|\xi|+...+(t|\xi|)^\ell\Big)|\xi|^{k-\ell}\, e^{-tB|\xi|} \;.\end{aligned} \tag{3.8}$$

By combining the inequalities (3.6), (3.7) and (3.8), we see that the product (3.5) can be estimated by

$$C\Big(1+t|\xi|+...+(t|\xi|)^{m-1}\Big)|\xi|^{\sum_{\ell=1}^m n_\ell - n_j - \widetilde{N}+k}\, e^{-tB|\xi|} \;,$$

where $\widetilde{N} = 1+2+...+(m-1) = m(m-1)/2$. This combined with the inequality (3.3)

$$\frac{1}{|\mathcal{C}^0(x,\xi)|} \leq C|\xi|^{-A} = C|\xi|^{-\sum_{\ell=1}^m n_\ell + \widetilde{N}} \ , \tag{3.9}$$

yields the following inequality

$$|D_t^k h_{j0}(x,\xi,t)| \leq C(1+t|\xi|)^N \, |\xi|^{-n_j+k} \, e^{-tB|\xi|} \tag{3.10}$$

for all $(x,t,\xi) \in \omega \times [0,T) \times \mathbb{R}^n \backslash \{0\}$, which proves Theorem 3.1 for $\alpha, \beta = 0$.

Assuming the theorem is true for all k and for all α and β such that $|\alpha|+|\beta|<r$, we want to prove it for all k and for $|\alpha|+|\beta| = r$. If we set

$$h_{j0}^{\#}(x,t,\xi) = D_x^\alpha D_\xi^\beta h_{j0}(x,t,\xi)$$

(no differentiation with respect to t) and differentiate the equations in (2.8), we get

$$\begin{cases} D_x^\alpha D_\xi^\beta [L h_{j0}] = 0, \\ D_x^\alpha D_\xi^\beta [Q_\nu^0(x,\xi,D_t)\, h_{j0}]_{|t=0} = 0\,, \quad 1 \leq \nu \leq m \ , \end{cases}$$

Next, by using Leibnitz formula and rearranging the terms, we obtain that $h_{j0}^{\#}(x,t,\xi)$ satisfies the system

$$\begin{cases} L(x,\xi,D_t)\, h_{j0}^{\#}(x,t,\xi) = \phi(x,t,\xi), \\ Q_\nu^0(x,\xi,D_t)\, h_{j0}^{\#}(x,t,\xi)_{|t=0} = \psi_{\nu j}(x,\xi)\,, \quad 1 \leq \nu \leq m \ , \end{cases} \tag{3.11}$$

where

$$\phi(x,t,\xi) = -\sum_{q=1}^m \Bigg(\sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta \\ |\alpha'|+|\beta'|<|\alpha|+|\beta|}} C_{\alpha\beta} \Big(D_x^{\alpha-\alpha'} D_\xi^{\beta-\beta'} M_{q0}^+ \, D_x^{\alpha'} D_\xi^{\beta'} D_t^{m-q} h_{j0}(x,t,\xi) \Big) \Bigg) \tag{3.12}$$

and

$$\psi_{\nu j} = - \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta \\ |\alpha'|+|\beta'|<|\alpha|+|\beta|}} C_{\alpha\beta} \Big(D_x^{\alpha-\alpha'} D_\xi^{\beta-\beta'} Q_\nu^0(x,\xi,D_t) \Big) D_x^{\alpha'} D_\xi^{\beta'} h_{j0}{}_{|t=0} \ . \tag{3.13}$$

Following the same argument we used to obtain formula (2.31) of [1] we can represent the solution of (3.11) explicitly as follows

$$h_{j0}^{\#}(x,t,\xi) = \int_0^\infty G(x,\xi,t,s)\, \phi(x,s,\xi)\, ds + \sum_{\nu=1}^m \psi_{\nu j}(x,\xi)\, h_{\nu 0}(x,t,\xi) \ , \tag{3.14}$$

where

$$G(x,\xi,t,s) = G_0(x,\xi,t-s) - \sum_{\nu=1}^{m} \Big(Q_\nu^0(x,\xi,D_t)\, G_0\Big)(x,\xi,-s)\, h_{\nu 0}(x,t,\xi) \tag{3.15}$$

and

$$G_0(x,\xi,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{L(x,0,\xi,\tau)}\, d\tau \tag{3.16}$$

is a fundamental solution of L. Now we set $\widetilde{h}_{j0} = D_t^k h_{j0}^{\#} = D_x^\alpha D_\xi^\beta D_t^k h_{j0}$, differentiate (3.14) k times with respect to t, and get

$$\begin{aligned} \widetilde{h}_{j0}(x,t,\xi) = \int_0^\infty & D_t^k G(x,\xi,t,s)\, \phi(x,s,\xi)\, ds \\ & + \sum_{\nu=1}^{m} \psi_{\nu j}(x,\xi)\, D_t^k h_{\nu 0}(x,t,\xi) \ , \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} D_t^k G(x,\xi,t,s) &= D_t^k G_0(x,\xi,t-s) \\ & \quad - \sum_{\nu=1}^{m} \Big(Q_\nu^0(x,\xi,D_t)\, G_0\Big)(x,\xi,-s)\, D_t^k h_{\nu 0}(x,t,\xi) \ . \end{aligned} \tag{3.18}$$

We estimate each term in (3.17) using the induction hypothesis and (3.10). To estimate (3.12) we use the fact that $M_{q0}^+(x,0,\xi)$ is homogeneous of degree q with respect to ξ, so that

$$\Big| D_x^{\alpha-\alpha'} D_\xi^{\beta-\beta'} M_{q0}^+(x,0,\xi) \Big| \le C|\xi|^{q-|\beta-\beta'|} \ . \tag{3.19}$$

By the induction hypothesis

$$\Big| D_x^{\alpha'} D_\xi^{\beta'} D_t^{m-q} h_{j0}(x,t,\xi) \Big| \le C(1+t|\xi|)^N\, |\xi|^{m-n_j-q-|\beta'|}\, e^{-Bt|\xi|} \ . \tag{3.20}$$

Combining (3.19) and (3.20) we obtain from (3.12)

$$|\phi(x,t,\xi)| \le C(1+t|\xi|)^N\, |\xi|^{m-n_j-|\beta|}\, e^{-tB|\xi|} \ . \tag{3.21}$$

To estimate $\psi_{\nu j}(x,\xi)$ note that

$$\begin{aligned} & \Big| (D_x^{\alpha-\alpha'} D_\xi^{\beta-\beta'} Q_\nu^0(x,\xi,D_t)) D_x^{\alpha'} D_\xi^{\beta'} h_{j0} \Big| \le \\ & \qquad \le C|\xi|^{n_\nu-|\beta-\beta'|}\, (1+t|\xi|)^N\, |\xi|^{-n_j-|\beta'|}\, e^{-Bt|\xi|} \ . \end{aligned}$$

Thus

$$|\psi_{\nu j}(x,\xi)| \le C|\xi|^{n_\nu-|\beta|-n_j} \ . \tag{3.22}$$

In order to estimate $G(x,\xi,t,s)$ we need the following lemma.

Lemma 3.1. *Suppose that $L(\xi,\tau)$ is a homogeneous polynomial of degree m with respect to (ξ,τ) and assume that the roots $\tau_j(\xi)$ of $L(\xi,\tau)$ satisfy*

$$|\tau_j(\xi)| \le C_1|\xi|, \quad |\operatorname{Im}\tau_j(\xi)| \ge B_1|\xi|, \qquad \forall\,\xi \ne 0 \; ; \tag{3.23}$$

then there exists constants $C, B > 0$, such that

$$G(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it\tau}}{L(\xi,\tau)}\,d\tau$$

satisfies

$$|D_t^j G(\xi,t)| \le C|\xi|^{-m+1+j}\, e^{-B|\xi|\,|t|} \; . \tag{3.24}$$

Proof: We first prove the case $j=0$ and $m \ge 2$. Let $0 < B < B_1$ and move the contour of integration so that

$$G(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(\tau \pm iB|\xi|)}}{L(\xi,\tau \pm iB|\xi|)}\,d\tau \; .$$

Using (3.23), we get

$$|G(\xi,t)| \le C\, e^{\mp tB|\xi|} \int_{-\infty}^{\infty} \frac{d\tau}{|L(\xi,\tau \pm iB|\xi|)|} \; .$$

The last integral is absolutely convergent and homogeneous of degree $-m+1$ with respect to ξ. Consequently

$$|G(\xi,t)| \le C|\xi|^{-m+1}\, e^{-B|\xi|\,|t|} \; .$$

Now assume that the result is true for $s < j$ for operators of order $< m$. If $\tau(\xi)$ is a root of $L(\xi,\tau)$, then $[D_t - \tau(\xi)]\,G(\xi,t)$ is a fundamental solution of the operator (of order $m-1$) $L(\xi,D_t)/(D_t - \tau(\xi))$. For $s < j$

$$\Big|(D_t^{s+1} - \tau(\xi)\,D_t^s)\,G(\xi,t)\Big| \le C|\xi|^{s-m+2}\, e^{-B|\xi|\,|t|} \; .$$

Multiplying this inequality by $\tau(\xi)^{j-s-1}$ and adding for $s = 0,1,...,j-1$, we obtain

$$\Big|D_t^j G(\xi,t) - \tau^j(\xi)\,G(x,\xi)\Big| \le C\sum_{s=0}^{j-1} |\xi|^{s-m+2}\,|\tau(\xi)|^{j-s-1}\, e^{-B|\xi|\,|t|} \; .$$

Then

$$|D_t^j G(\xi,t)| \le C|\xi|^{j-m+1}\, e^{-B|\xi|\,|t|} \ .$$

The case $m=1$ is easy (see [1]). ∎

Returning to the proof of Theorem 3.1 and applying Lemma 3.1 to $G_0(x,\xi,t-s)$, we obtain

$$|D_t^k G_0(x,\xi,t-s)| \le C|\xi|^{-m+1+k}\, e^{-B|\xi|\,|t-s|}$$

and

$$\begin{aligned}\Big|(Q_\nu^0(x,\xi,D_t)\, G_0)(x,\xi,-s)\, D_t^k h_{\nu 0}(x,\xi,t)\Big| &\le \\ &\le C|\xi|^{-m+1+n_\nu}\, e^{-B|\xi|\,|s|}(1+t|\xi|)^N\, |\xi|^{-n_\nu+k}\, e^{-B|\xi|\,t} \\ &\le C(1+t|\xi|)^N\, |\xi|^{-m+1+k}\, e^{-B|\xi|\,|t-s|} \ .\end{aligned}$$

Therefore, it follows from (3.18) that

$$|D_t^k G(x,\xi,t,s)| \le C(1+t|\xi|)^N\, |\xi|^{-m+1+k}\, e^{-B|\xi|\,|t-s|} \ . \tag{3.25}$$

We can now estimate the first term in (3.17). Write

$$\int_0^\infty D_t^k G(x,\xi,t,s)\,\phi(x,s,\xi)\,ds = I + II \ ,$$

where

$$I = \int_0^t D_t^k G(x,\xi,t,s)\,\phi(x,s,\xi)\,ds$$

and

$$II = \int_t^\infty D_t^k G(x,\xi,t,s)\,\phi(x,s,\xi)\,ds \ .$$

Now using the inequalities (3.21), (3.25), and noticing that $s<t$ in I, we obtain the estimate

$$\begin{aligned}|I| &\le C(1+t|\xi|)^N\, |\xi|^{k+1-|\beta|-n_j}\, e^{-B|\xi|t}\int_0^t (1+s|\xi|)^N\,ds \\ &\le C(1+t|\xi|)^N\, |\xi|^{k-|\beta|-n_j}\, e^{-(B/2)|\xi|t} \ ,\end{aligned}$$

since t is bounded. As for the second integral, using again the inequalities (3.21), (3.25), and noticing that $s>t$, we obtain

$$|II| \le C(1+t|\xi|)^N\, |\xi|^{k+1-|\beta|-n_j}\, e^{B|\xi|t}\int_t^\infty (1+s|\xi|)^N\, e^{-2Bs|\xi|}\,ds \ .$$

Now

$$e^{B|\xi|t} \int_t^\infty (1+s|\xi|)^N \, e^{-2Bs|\xi|} \, ds \le \frac{C}{B|\xi|} \, e^{-(B/2)|\xi|t}$$

so that

$$|II| \le C(1+t|\xi|)^N \, |\xi|^{k-|\beta|-n_j} \, e^{-\frac{B}{2}|\xi|t} \,.$$

Consequently

$$\left| \int_0^\infty D_t^k G(x,\xi,t,s) \, \phi(x,s,\xi) \, ds \right| \le C(1+t|\xi|)^N \, |\xi|^{k-|\beta|-n_j} \, e^{-(B/2)|\xi|t} \,. \tag{3.26}$$

Finally, to estimate the last term in (3.17) we use inequality (3.22) together with the estimate for $D_t^k h_{\nu 0}$ to get

$$\left| \psi_{\nu j}(x,\xi) \, D_t^k h_{\nu 0}(x,\xi,t) \right| \le C(1+t|\xi|)^N \, |\xi|^{k-|\beta|-n_j} \, e^{-B|\xi|t} \,. \tag{3.27}$$

Estimates (3.26) and (3.27) prove Theorem 3.1. ∎

4 – The symbol class $B_t^m(\Omega)$

The aim of this section is to first estimate the terms $h_{j\ell}(x,t,\xi)$ for $\ell = 1,2,...$, and then introduce our symbol class.

Theorem 4.1. *The symbols $h_{j\ell}(x,t,\xi)$ defined by equations (2.8) and (2.9) satisfy an inequality of the type (3.1) with $-n_j$ replaced by $-n_j-\ell$, for all $j=1,2,...,m$ and $\ell=0,1,...$.*

Proof: We reason by induction on ℓ and we prove the desired inequality when $\alpha=\beta=k=0$. The general case follows the same argument used in Section 3. Recall that $h_{j\ell}(x,t,\xi)$ satisfy the system of equations

$$\begin{cases} L h_{j\ell} = F(x,t,\xi), \\ Q_\nu^0 h_{j\ell}{}_{|t=0} = \Phi_\nu(x,\xi) \,, \qquad 1 \le \nu \le m \,, \end{cases} \tag{4.1}$$

with

$$F(x,t,\xi) = -t \sum_{q=1}^m \widetilde{M}_{q0}^+ \, D_t^{m-q} h_{j,\ell-1} - \sum_{q=1}^m \sum_{k=0}^{\ell-1} M_{q,\ell-k}^+ \, D_t^{m-q} h_{jk}$$

and

$$\Phi_\nu(x,\xi) = -\sum_{k=0}^{\ell-1} \sum_{|\alpha|=\ell-k} \widetilde{Q}_{\nu\alpha} \, D_x^\alpha h_{jk}{}_{|t=0} \,.$$

As we did in Section 3 formula (3.14), we can write an explicit expression for $h_{j\ell}(x,t,\xi)$, namely

$$h_{j\ell}(x,t,\xi) = \int_0^\infty G(x,\xi,t,s)\,F(x,s,\xi)\,ds + \sum_{\nu=1}^{m} \Phi_\nu(x,\xi)\,h_{\nu 0}(x,t,\xi)\ .$$

By the induction assumption, we have

$$|D_t^{m-q} h_{j,\ell-1}| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j+m-q-(\ell-1)}\,e^{-B|\xi|t}$$

and since $\widetilde{M}_{q0}^+$ is homogeneous of degree q with respect to ξ we get

$$|t\widetilde{M}_{q0}^+\,D_t^{m-q} h_{j,\ell-1}| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j-\ell+m}\,e^{-(B/2)|\xi|t}\ . \tag{4.2}$$

On the other hand, since $M_{q,\ell-k}^+$ is homogeneous of degree $q-\ell+k$ we have

$$|M_{q,\ell-k}^+\,D_t^{m-q} h_{jk}| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j+m-\ell}\,e^{-B|\xi|t}\ . \tag{4.3}$$

From inequalities (4.2) and (4.3) it follows that

$$|F(x,t,\xi)| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j+m-\ell}\,e^{-(B/2)|\xi|t}\ .$$

Using the same argument we used to establish (3.26) one shows that

$$\left|\int_0^\infty G(x,t,\xi,s)\,F(x,s,\xi)\,ds\right| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j-\ell}\,e^{-(B/2)|\xi|t}\ . \tag{4.4}$$

Next, using again the induction hypothesis and the fact that the differential operator $\overline{Q}_{\nu\alpha}(x,\xi,D_t)$ has degree $\leq n_\nu - |\alpha|$, we obtain

$$|\overline{Q}_{\nu\alpha}(x,\xi,D_t)\,D_x^\alpha h_{jk}| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j-k+n_\nu-|\alpha|}\,e^{-B|\xi|t}\ .$$

Now $|\alpha| = \ell - k$ so that

$$|\Phi(x,\xi)| \leq C\,|\xi|^{-n_j-\ell+n_\nu}\ ,$$

and therefore

$$|\Phi_\nu h_{\nu 0}| \leq C(1+t|\xi|)^N\,|\xi|^{-n_j-\ell}\,e^{-B|\xi|t}\ . \tag{4.5}$$

Inequalities (4.4) and (4.5) yield the desired estimate for $h_{j\ell}$. ∎

We now introduce our symbol class. Let $\Omega = \omega \times [0,T)$ where ω is an open set in $\mathbb{R}^n$.

Definition 4.1. A function $b(x,t,\xi)$ is said to belong to the class $B_t^m(\Omega)$ if $b(x,t,\xi) \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n \backslash \{0\})$ and if there exists a constant $B > 0$ such that given $\alpha, \beta \in \mathbf{Z}_+^n$, $k \in \mathbf{Z}^+$ and a compact set $K \subset \omega \times [0,T)$, there exists a constant $C = C(\alpha,\beta,k,K)$ and a nonnegative integer $N = N(\alpha,\beta,k,K)$ such that

$$|D_x^\alpha D_\xi^\beta D_t^k b(x,t,\xi)| \leq C(1+t|\xi|)^N \, (1+|\xi|)^{m+k-|\beta|} \, e^{-B|\xi|t} \tag{4.6}$$

for all $(x,t) \in K$, $\xi \in \mathbb{R}^n$.

If $b \in B_t^m(\Omega)$ and $a \in B_t^\ell(\Omega)$, then the following properties are easily verifiable:

i) $b(x,t,\xi)\, a(x,t,\xi) \in B_t^{m+\ell}(\Omega)$;

ii) $D_t^k b(x,t,\xi) \in B_t^{m+k}(\Omega)$;

iii) $tb(x,t,\xi) \in B_t^{m-1}(\Omega)$;

iv) $D_x^\alpha b(x,t,\xi) \in B_t^m(\Omega)$;

v) If $b(x,t,\xi) \in B_t^m(\Omega)$, then $D_t^k b(x,t,\xi)_{|t=0} \in S^{m+k}(\omega)$, where $S^j(\omega)$ is the standard symbol class of pseudodifferential operators studied in [9].

vi) If $b(x,\xi) \in S^{m_1}(\Omega)$ and $c(x,t,\xi) \in B_t^{m_2}(\Omega)$, then $b(x,\xi)\, c(x,t,\xi) \in B_t^{m_1+m_2}(\Omega)$.

It is now a simple matter to verify that after multiplication by a suitable cutoff function each symbol $h_{j\ell}(x,t,\xi)$ belongs to the class $B_t^{-n_j-\ell}(\Omega)$.

Finally, each formal series

$$\sum_{\ell=0}^{\infty} h_{j\ell}(x,t,\xi)\,, \qquad 1 \leq j \leq m\ ,$$

can be transformed into a symbol of a pseudodifferential operator in a standard way. Let $\chi\colon \mathbb{R} \to \mathbb{R}$, $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}^\infty(\mathbb{R})$, be such that $\chi(s) = 0$ if $|s| \leq 1/2$ and $\chi(s) = 1$ if $|s| \geq 1$. Then it is possible to find a sequence of real numbers, $\rho_\ell \to +\infty$, such that

$$\widetilde{h}_j(x,t,\xi) = \sum_{\ell=0}^{\infty} h_{j\ell}(x,t,\xi)\, \chi(|\xi|/\rho_\ell)$$

is a smooth function. Furthermore, given $\alpha, \beta \in \mathbf{Z}_+^n$ and $k, N \in \mathbf{Z}_+$, there exists a constant $C = C(\alpha,\beta,k,N)$ such that

$$\left| D_x^\alpha D_\xi^\beta D_t^k \Big[\widetilde{h}_j(x,t,\xi) - \sum_{\ell=0}^{N-1} h_{j\ell}(x,t,\xi) \Big] \right| \leq C(1+|\xi|)^{-N-n_j+k-|\beta|} \, e^{-Bt|\xi|/2}$$

for all $(x,t,\xi) \in \omega \times [0,T) \times \mathbb{R}^n$.

It follows from our estimates that each of the operators $\widetilde{H}_j(t)$, $1 \le j \le m$, defined by

$$\widetilde{H}_j(t)\, g(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\, \widetilde{h}_j(x,t,\xi)\, \widehat{g}(\xi)\, d\xi \ ,$$

is a pseudodifferential operator of order $\le -n_j$ over ω, depending smoothly on t. Moreover, $\widetilde{H}_j(t)$ is regularizing for $t > 0$ and satisfy (1.10). By a known argument (see [9]), one can show regularity up to the boundary for the solution of the problem (1.7).

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