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ON A CLASS OF FREE GALOIS EXTENSIONS

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Abstract: Let RG_f be a projective group algebra over a commutative ring R, where G is a finite group and f is a factor set. If RG_f is a central Galois R-algebra with inner Galois group G' induced by the basis of RG_f, then there exists a one-to-one correspondence between the set of subgroups H' of G' such that RH_f is Galois with a free basis induced by H' and the set of Azumaya subalgebras B over R such that $RG = B(G(B))_f$, where $G(B) = \{\alpha \in G \mid \alpha(b) = b\}$ for any b in B and $B(G(B))_f$ is a projective group ring over B.

1 – Introduction

Let RG_f $(= \sum RU_\alpha,\ \alpha \text{ in } G)$ be a projective group algebra with a free basis $\{U_\alpha \mid \alpha \text{ in } G\}$ over a commutative ring R where G is a finite group and f is a factor set: $G \times G \to U(R)$ which is the set of units of R. In [1] and [2], F.R. DeMeyer proved that RG_f is a central Galois R-algebra if and only if it is an Azumaya R-algebra, where the Galois group G' is inner induced by $\{U_\alpha\}$. If the coefficient ring R is noncommutative, then we call RG_f a projective group ring over R. Let RG_f be a central Galois R-algebra with inner Galois group G'. The purpose of the present paper is to show a fundamental theorem for RG_f. It will be shown that there exists a one-to-one correspondence between the set of subgroups H' of G' such that RG_f is a free Galois extension with basis $\{U_\alpha \mid \alpha \text{ in } H\}$ over $(RG_f)^{H'}$ and the set of Azumaya subalgebras B such that $RG_f = B(G(B))_f$ where $G(B) = \{\alpha \text{ in } G' \mid \alpha(b) = b \text{ for each } b \text{ in } B\}$ and $B(G(B))_f$ is a projective group ring over B where f on a subgroup is the restriction of f on G. In this case, $B = (RG_f)^{G(B)'}$ and $(RG_f)^{H'}$ is also an Azumaya subalgebra.

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2 – Definitions and notations

Throughout, we assume that all rings have an identity 1. A ring A is called a ring extension over a subring B if A and B have the same identity. For the definitions of separable extensions and algebras and Azumaya algebras, we refer to [1], [3] and [8]. Let A be a ring extension over B, G a finite automorphism group of A of order n for some integer n. Then A is called a Galois extension over B with Galois group G if $B = A^G$ $(= \{a \text{ in } A \mid \alpha(a) = a \text{ for each } \alpha \text{ in } G\})$ and there exist $\{a_i, b_i \text{ in } A \mid i = 1, ..., m \text{ for some integer } m\}$ such that $\sum_i a_i \alpha(b_i) = \delta_{1\alpha}$ (the Kronecker δ) for α in G. Such a set $\{a_i, b_i\}$ is called a Galois coordinate system for A. Let R be a ring with center C, $U(C)$ the set of units of C, and G a finite group. Then $f\colon\ G \to U(C)$ is called a factor set if $f(\alpha\beta,v)\, f(\alpha,\beta) = f(\alpha,\beta v)\, f(\beta, v)$ for all α, β, v in G, and RG_f $(= \sum RU_\alpha$, α in $G)$ is called a projective group ring over R if $\{U_\alpha\}$ are free over R, $rU_\alpha = U_\alpha r$ for each r in R and α in G, and $U_\alpha U_\beta = U_{\alpha\beta}\, f(\alpha,\beta)$ for all α, β in G. RG_f is called a projective group algebra when R is commutative (see [1] and [2]).

3 – Main results

In the following, we assume that A $(= RG_f)$ is a central Galois algebra over a commutative ring R with inner Galois group G', where G is a finite group of order n for some integer n, f is a factor set and G' is the inner automorphism group of A induced by $\{U_\alpha\}$; that is, $\alpha'(a) = U_\alpha\, a U_\alpha^{-1}$ for each a in A and α in G. We recall that RG_f is Galois over R with inner Galois group G' if and only if RG_f is an Azumaya R-algebra ([1], Theorems 1 and 2).

Theorem 1. *Let A $(= RG_f)$ be a central Galois R-algebra with inner Galois group G' and H' a subgroup of G'. Then, A is a free Galois extension with a free basis $\{U_\alpha \mid \alpha \text{ in } H\}$ over $A^{H'}$ with Galois group H' if and only if RH_f $(= \sum RU_\alpha$, α in $H)$ is an Azumaya R-algebra.*

Proof: Let RH_f be an Azumaya R-algebra. Then it is a central Galois R-algebra ([1], Theorem 3). Let $\{a_i, b_i \text{ in } RH_f \mid i = 1, ..., m \text{ for some integer } m\}$ be a Galois coordinate system. Then, for each b in $A^{H'}$ and α in H, $\alpha'(b) = b$, so $U_\alpha\, bU_\alpha^{-1} = b$. Hence $U_\alpha b = bU_\alpha$ for each α in H. Thus $bx = xb$ for each x in RH_f. Now we claim that $\{U_\alpha \mid \alpha \text{ in } H\}$ are free over $A^{H'}$. Let $\sum t_\alpha U_\alpha = 0$ for α in H and t_α in $A^{H'}$. Then for any α in H,

$$0 = \sum_i a_i \Big(\sum t_\alpha U_\alpha\Big)\, \beta'^{-1}(b_i) = \sum_\alpha t_\alpha \Big(\sum_i a_i\, \alpha'\, \beta'^{-1}(b_i)\Big) U_\alpha = t_\beta\, U_\beta\ .$$

Since U_β is a unit, $t_\beta = 0$ for each β in H. Thus $(A^{H'})H_f$ is a projective group ring over $A^{H'}$ in A. Moreover, $(A^{H'})H_f \supset RH_f$, so $(A^{H'})H_f$ is Galois over $A^{H'}$ with inner Galois group H' (for RH_f is Galois over R with inner Galois group H'). But A is also Galois over $A^{H'}$ with inner Galois group H', so $A = (A^{H'})H_f$. This proves the sufficiency. For the necessity, let A be a free Galois extension over $A^{H'}$ with a basis $\{U_\alpha \mid \alpha \text{ in } H\}$. Then $A = (A^{H'})H_f$. Denote $A^{H'}$ by B and let C be the center of B. Then $A = BH_f$, a projective group ring over B. Clearly, $A = BH_f = B \otimes CH_f$. Since A has center R, it is easy to see that $R = C$. Moreover, since A is Azumaya C-algebra, both B and CH_f are Azumaya C-algebras ([3], Chapter 4, Theorem 4.4). Thus RH_f is an Azumaya R-algebra. ∎

By the proof of the necessity of the above theorem, we have

Corollary 2. *By keeping the notations and hypotheses of Theorem 1, if A is a free Galois extension with a basis $\{U_\alpha \mid \alpha \text{ in } H\}$ over $A^{H'}$ with Galois group H', then $A^{H'}$ is an Azumaya R-algebra such that $A = (A^{H'})H_f$.* ∎

Next, we want to show the converse. Let B be an Azumaya subalgebra of A, $G(B) = \{\alpha \text{ in } G \mid \alpha'(b) = b \text{ for each } b \text{ in } B\}$. Then $B(G(B))$ $(= \sum BU_\alpha$, α in $G(B))$ is a subalgebra of A such that $bU_\alpha = U_\alpha b$ for each α in $G(B)$ and b in B.

Theorem 3. *Let B be an Azumaya subalgebra of A. Then, $B(G(B))$ is an Azumaya R-algebra if and only if $R(G(B))_f$ is Galois over R with inner Galois group $(G(B))'$.*

Proof: Let $R(G(B))_f$ be Galois over R with inner Galois group $(G(B))'$. Then it is an Azumaya R-algebra. Since $bU_\alpha = U_\alpha b$ for each b in B and α in $G(B)$, $\{U_\alpha \mid \alpha \text{ in } G(B)\}$ are free over B as proved in Theorem 1. Hence $B(G(B)) = B(G(B))_f$, a projective group ring over B. Since $B(G(B))_f = B \otimes R(G(B))_f$, $B(G(B))_f$ is an Azumaya R-algebra (for B and $R(G(B))_f$ are so) ([3], Chapter 2, Proposition 3.3). Conversely, let $B(G(B))$ be an Azumaya R-algebra. Since $bU_\alpha = U_\alpha b$ for each b in B and α in $G(B)$, the center of $R(G(B))_f$ is contained in R. Hence the center of $R(G(B))_f$ is R. On the other hand, A $(= RG_f)$ is a separable R-algebra if and only if the order of G is a unit ([1], Lemma 1). Hence the order of $G(B)$ is a unit. Thus $R(G(B))_f$ is a separable R-algebra. But then $R(G(B))_f$ is Galois over R with inner Galois group $(G(B))'$. ∎

We note that the Azumaya subalgebra $B(G(B))_f$ in the above theorem may not be A. We are going to show that there exists a bigger Azumaya subalgebra $D \supset B$ such that $A = D(G(B))_f$ and $G(D) = G(B)$.

Theorem 4. *Let A and B be given as in Theorem 3 such that $B(G(B))$ is an Azumaya subalgebra of A. Then there exists a unique Azumaya subalgebra D containing B such that $A = D(G(B))_f$ and $G(D) = G(B)$.*

Proof: By Theorem 3, $B(G(B)) = B(G(B))_f$, a projective group ring over B. Since B and $B(G(B))_f$ are Azumaya subalgebras of A, there exists an Azumaya subalgebra D' of A such that $A \cong B(G(B))_f \otimes D' \cong B \otimes R(G(B))_f \otimes D'$ as Azumaya R-algebras ([3], Chapter 2, Theorem 4.3). Let $D = B \otimes D'$. Then $A \cong D \otimes R(G(B))_f$ where D and $R(G(B))_f$ are commutant Azumaya subalgebras in A. Hence, by $A(G(B))'$ Theorem 3, $A \cong D \otimes R(G(D))_f$; and so $R(G(D))_f = R(G(B))_f$ ([3], Chapter 2, Theorem 4.3). Thus $G(D) = G(B)$. Also, by Theorem 1, $A \cong A^{(G(D))'} \otimes R(G(D))_f$, so $D = A^{(G(D))'} = A^{(G(B))'}$ by the commutant theorem again. Thus $A = D(G(B))_f$. Moreover, let D'' be another subalgebra of A such that $A = D''(G(B))_f$; then clearly $D'' = D$. ∎

By Theorems 2 and 4, we have a one-to-one correspondence theorem for central Galois algebras with an inner Galois group.

Theorem 5. *Let A $(= RG_f)$ be a central Galois R-algebra with inner Galois group G'. Then there exists a one-to-one correspondence between the set S of subgroups H' of G' such that A is a free Galois extension of $A^{H'}$ with a basis $\{U_\alpha \mid \alpha \text{ in } H\}$ and the set T of Azumaya subalgebras B of A such that $A = B(G(B))_f$.*

Proof: For a subgroup H' of G' in S, A is a free Galois extension of $A^{H'}$ with a basis $\{U_\alpha \mid \alpha \text{ in } H\}$, so Theorem 1 implies that $A^{H'}$ is an Azumaya subalgebra of A such that $A = A^{H'}H_f$. Then $A^{H'}$ is in T. Then the map $\phi\colon H' \to A^{H'}$ is defined. On the other hand, let B be an Azumaya subalgebra of A in T. Then $A = B(G(B))_f$. Hence, by Theorem 1, $B = A^{G(B)'}$. Thus $(G(B))'$ is a subgroup of G' such that A is a free Galois extension over B $(= A^{G(B)'})$ with a basis $\{U_\alpha \mid \alpha \text{ in } G(B)\}$. This implies that $(G(B))'$ is in S; and so the map $\psi\colon B \to (G(B))'$ is defined. Moreover, since $A = A^{H'}H_f = A^{H'}(G(A^{H'}))_f$, $H = G(A^{H'})$. Thus $H' = (G(A^{H'}))' = \psi\phi(H')$; and so $\psi\phi = 1$. Also, $A = B(G(B))_f = A^{(G(B))'}(G(B))_f$, so $B = A^{(G(B))'} = \phi\psi(B)$. Thus $\phi\psi = 1$. Therefore the correspondence is one-to-one. ∎

We note that there exist subgroups H' of G' not belonging to the correspondence as given in Theorem 5. For example, the ordinary real quaternion algebra RG_f where $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group of order 8 and R is the real field. Then RG_f is an Azumaya R-algebra but $R\langle i\rangle_f$, $R\langle j\rangle_f$ and $R\langle k\rangle_f$ are not Azumaya algebras over R. Thus the subgroups $\langle i\rangle'$, $\langle j\rangle'$ and $\langle k\rangle'$ do not belong to the correspondence. We conclude the paper with a theorem suggested by the referee. The theorem identifies which subgroups of G' belong to the correspondence as given in Theorem 5.

Theorem 6. *Let G be a finite Abelian group of exponent m, and assume R contains no more than m distinct m^{th} roots of 1. If RG_f and RH_f are R-Azumaya algebras for a subgroup H of G, then there exists a subgroup K of G such that $G = H \times K$ where RK_f is an Azumaya R-algebra and the commutator subalgebra of RH_f in RG_f.*

Proof: From the proof of Theorem 4 in [1] (p. 292), the map $\psi\colon G \times G \to U(R)$ by $\psi(\alpha,\beta) = f(\alpha,\beta)(f(\beta,\alpha))^{-1}$ is a nonsingular skew pairing because RG_f is an Azumaya R-algebra. Also, since R contains no more than m distinct m^{th} roots of 1 by hypothesis, the map $\alpha \to \psi(\alpha,\_)$ for α in G is an isomorphism from G to $\mathrm{Hom}(G, U(R))$. Similarly, since RH_f is an Azumaya R-algebra, the map $\alpha \to \psi(\alpha,\_)$ for α in H is an isomorphism from H to $\mathrm{Hom}(H, U(R))$. Moreover, the map $\alpha \to \psi(\alpha,\_)$ for any α in G defines a homomorphism from G onto $\mathrm{Hom}(H, U(R))$ with kernel K. We then have that $G/K \cong \mathrm{Hom}(H, U(R))$ where $K = \{\alpha \text{ in } G \mid \psi(\alpha,\beta) = 1 \text{ for any } \beta \text{ in } H\}$. Next, we claim that $K \cap H = \{e\}$, the identity of G, and $G = HK$. In fact, let α be an element in $H \cap K$. Then $\psi(\alpha,\beta) = 1$ for any β in H. Hence $f(\alpha,\beta) = f(\beta,\alpha)$ for any β in H; and so U_α is in the center of RH_f. Thus $\alpha = e$ (for RH_f has center R). Noting that $\mathrm{Hom}(H, U(R)) \cong H$, we conclude that $G = KH$. But then $G = H \times K$. Therefore, $RG_f \cong RH_f \otimes RK_f$ as Azumaya R-algebras where RK_f is an Azumaya R-algebra and the commutator subalgebra of RH_f in RG_f as noted in [1] (p. 203). ∎

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