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A STUDY OF K_W-SPACES AND K_W^*-SPACES

Carlos R. Borges

Abstract: Further study of K_W-spaces leads to the introduction of K_W^*-spaces. We obtain a characterization of K_W^*-spaces in terms of continuous real-valued functions which is dual to a characterization of K_0-spaces. We also get two characterizations of K_W-spaces, one of which exhibits their remarkable similarities with K_1-spaces; a consequence of the latter characterization is that K_W-spaces are collectionwise normal.

Throughout, we will use the terminology of [1].

We introduced the concept of K_W-spaces in [1; Definition 10], as follows: A space (X,τ) is a K_W-space provided that, for each closed $F \subset X$, there exists a function $k\colon \tau|F \to \tau$ (k is called a K_W-function) which satisfies the following:

(1) $F \cap k(U) = U$, for each $U \in \tau|F$, $k(F) = X$ and $k(\emptyset) = \emptyset$;

(2) $k(U) \subset k(V)$ whenever $U \subset V$;

(3) $k(U) \cup k(V) = X$ whenever $U \cup V = F$;

(4) $\overline{k(U)} \cap F = \overline{U}$.

Condition (3) naturally leads to one question if it can be replaced by the stronger condition below, without affecting the concept of a K_W-space:

(3*) $k(U) \cup k(V) = k(U \cup V)$.

We do not yet know the answer to this question. However, replacing (3) by (3*) in the definition of K_W-spaces leads to a (possibly new) class of spaces which we will call *K_W^*-spaces*, with remarkable properties which are dual to those of K_0-spaces (see Theorem 2 of [1] and compare it with Theorem 2 ahead). It is *noteworthy* that a K_0-function is a K_W-function if and only if it is a K_W^*-function (this follows from Theorem 12 of [1], and Theorems 2 c) and 3 b) v) ahead).

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Remark. Note that, for each closed subspace F of *any* space (X,τ) there exists $k:\ \tau|F \to \tau$ which satisfies (1), (2) and (3) above: Simply, let $k(U) = U \cup (X-F)$, for $U \neq \emptyset$, and $k(\emptyset)=\emptyset$.

Proposition 1. *Every K_W-space is completely normal.*

Proof: Let A, B be subsets of a K_W-space (X,τ) such that $\overline{A}\cap B = \emptyset = A\cap\overline{B}$. Let $F=\overline{A}\cup\overline{B}$ and let $k:\ \tau|F\to\tau$ be a K_W-function. Then $\overline{B}-\overline{A} = F-\overline{A} = U\in\tau|F$, $B\subset U$ and $\overline{U}\cap A=\emptyset$ (note that $a\in A$ implies $a\notin\overline{B}$ which implies that $a\in X-\overline{B}\in\tau$, since $(X-\overline{B})\cap(\overline{B}-\overline{A})=\emptyset$, $a\notin\overline{U}$). Since $\overline{k(U)}\cap F=\overline{U}$, by (4), we get that $\overline{k(U)}\cap A=\emptyset$; therefore, $k(U)$ and $X-\overline{k(U)}$ are disjoint τ-open subsets of X such that $B\subset k(U)$ and $A\subset X-\overline{k(U)}$. This completes the proof. ∎

Theorem 2. *For any space X, the following are equivalent:*

a) *X is a K^*_W-space;*

b) *X is completely normal and, for each nonempty closed subspace F of X, there exist extenders $\phi\colon\ C^*_{\text{usc}}(F)\to C^*_{\text{usc}}(X)$ and $\psi\colon\ C^*_{\text{lsc}}(F)\to C^*_{\text{lsc}}(X)$ such that*

- **i**) $\phi(f)\le\phi(g)$, *whenever* $f\le g$,
- **ii**) $\phi(f+g)\ge\phi(f)+\phi(g)$,
- **iii**) $\psi(f)\le\psi(g)$, *whenever* $f\le g$,
- **iv**) $\psi(f+g)\le\psi(f)+\psi(g)$,
- **v**) $\phi(f)\le\psi(f)$, *whenever* $f\in C^*(F)$,
- **vi**) $\phi(a_F)=a_X=\psi(a_F)$, *for* $a\in\mathbb{R}$,
- **vii**) $\psi(\sup(f,g))=\sup(\psi(f),\psi(g))$,
- **viii**) $\phi(\inf(f,g))=\inf(\phi(f),\phi(g))$,
- **ix**) $\phi(f)=-\psi(-f)$, *for each* $f\in C^*(F)$,
- **x**) *for any* $\{f_\alpha|\,\alpha\in\Lambda\}\subset C^*(F)$, $\overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[)}\cap F=\overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[)}$;

c) *X is completely normal and, for any nonempty closed $F\subset X$ there exists an extender $\phi\colon\ C^*(F)\to C^*_{\text{usc}}(X)$ which satisfies i), vi), viii) and x) of b) for functions in $C^*(F)$.*

Proof: a) implies b). By Proposition 1, X is completely normal. Let $k\colon\ \tau|F\to\tau$ be a K^*_W-function. For each $x\in X$, let

$$\phi(f)(x)=\inf\Big\{t\in\mathbb{R}\mid x\in k(f^{-1}(]-\infty,t[))\Big\}\ ,$$

$$\psi(g)(x)=\sup\Big\{t\in\mathbb{R}\mid x\in k(g^{-1}(]t,\infty[))\Big\}\ ,$$

where $f \in C^*_{\text{usc}}(F)$ and $g \in C^*_{\text{lsc}}(F)$. Since k is monotone, we immediately get that ϕ and ψ satisfy i) and iii), respectively. Since we also get that

$$\phi(f)^{-1}(]-\infty,t[) = \bigcup\Big\{k(f^{-1}(]-\infty,s[)) \mid s<t\Big\}\ ,$$
$$\psi(g)^{-1}(]t,\infty[) = \bigcup\Big\{k(f^{-1}(]s,\infty[)) \mid s>t\Big\}\ ,$$

we immediately get that ϕ is a usc-extender and ψ is an lsc-extender. (It is clear that, for $x \in F$, $\phi(f)(x) = f(x)$ and $\psi(g)(x) = g(x)$.)

Next, we show that ϕ satisfies ii): Pick $x \in X$ and say $\phi(f)(x) = t_1$, $\phi(g)(x) = t_2$, with $t_1 \le t_2$. Let $t = t_1 + t_2$ and note that, for any $\varepsilon > 0$,

$$(f+g)^{-1}(]-\infty,t-\varepsilon[) \subset f^{-1}(]-\infty,t_1-\tfrac{\varepsilon}{2}[) \cup g^{-1}(]-\infty,t_2-\tfrac{\varepsilon}{2}) \ .$$

(Pick any $z \in F$ such that $f(z)+g(z) < t-\varepsilon$. Note that if $f(z) < t_1 - \frac{\varepsilon}{2}$ then $z \in f^{-1}(]-\infty,t_1-\frac{\varepsilon}{2}[)$; if $f(z) \ge t_1 - \frac{\varepsilon}{2}$ then $g(z) < t_2 - \frac{\varepsilon}{2}$ which implies that $z \in g^{-1}(]-\infty,t_2-\frac{\varepsilon}{2}[)$.) Since $\phi(f)(x) = t_1$ and $\phi(g)(x) = t_2$, we get that $x \notin k(f^{-1}(]-\infty,t_1-\frac{\varepsilon}{2}[))$ and $x \notin k(g^{-1}(]-\infty,t_2-\frac{\varepsilon}{2}[))$; hence $x \notin k(f^{-1}(]-\infty, t_1-\frac{\varepsilon}{2}[)) \cup k(g^{-1}(]-\infty,t_2-\frac{\varepsilon}{2}[)) \supset k((f+g)^{-1}(]-\infty,t-\varepsilon[))$, by (2) and (3*), which implies that $\phi(f+g)(x) \ge t = \phi(f)(x)+\phi(g)(x)$, as required.

Next, we show that ψ satisfies iv): Pick $x \in X$ and say $\psi(f)(x) = t_1$, $\psi(g)(x) = t_2$, with $t_1 \le t_2$. Let $t = t_1+t_2$ and note that, for any $\varepsilon > 0$,

$$(f+g)^{-1}(]t+\varepsilon,\infty[) \subset f^{-1}(]t_1+\tfrac{\varepsilon}{2},\infty[) \cup g^{-1}(]t_2+\tfrac{\varepsilon}{2},\infty[) \ .$$

(Pick any $z \in F$ such that $f(z)+g(z) > t+\varepsilon$. Note that if $f(z) > t_1 + \frac{\varepsilon}{2}$ then $z \in f^{-1}(]t_1+\frac{\varepsilon}{2},\infty[)$; if $f(z) \le t_1+\frac{\varepsilon}{2}$ then $g(z) > t_2+\frac{\varepsilon}{2}$ which implies that $z \in g^{-1}(]t_2+\frac{\varepsilon}{2},\infty[)$.) Since $\psi(f)(x) = t_1$ and $\psi(g)(x) = t_2$, we get that $x \notin k(f^{-1}(]t_1+\frac{\varepsilon}{2},\infty[))$ and $x \notin k(g^{-1}(]t_2+\frac{\varepsilon}{2},\infty[))$; hence $x \notin k(f^{-1}(]t_1+\frac{\varepsilon}{2}, \infty[)) \cup k(g^{-1}(]t_2+\frac{\varepsilon}{2},\infty[)) \supset k((f+g)^{-1}(]t+\varepsilon,\infty[))$, by (2) and (3*), which implies that $\psi(f+g)(x) \le t = \psi(f)(x)+\psi(g)(x)$, as required.

In order to show that v) is satisfied, let $f \in C^*(F)$ and say $\phi(f)(x) = t_0$. Then $x \notin k(f^{-1}(]-\infty,t[))$ for $t < t_0$. Therefore, by conditions (3) for a K_W-function, $x \in k(f^{-1}(]s,\infty[))$ for $s < t < t_0$ (because $F = f^{-1}(]s,\infty[) \cup f^{-1}(]-\infty,t[)$); therefore, $\psi(f)(x) \ge t_0 = \phi(f)(x)$.

It is easily seen from the definitions of ϕ and ψ that they satisfy vi).

Next, we show that ψ satisfies vii): Note that, for $f,g \in C^*_{\text{lsc}}(F)$ and $t \in \mathbb{R}$,

$$\sup(f,g)^{-1}(]t,\infty[) = f^{-1}(]t,\infty[) \cup g^{-1}(]t,\infty[) \ .$$

Pick $x \in X$ and let $\psi(f)(x) = t_1$, $\psi(g)(x) = t_2$; say $t_1 \le t_2$. Then $x \notin k(f^{-1}(]t,\infty[))$ for $t > t_1$, and $x \notin k(g^{-1}(]t,\infty[))$ for $t > t_2$; therefore, by (3*),

$$x \notin k(f^{-1}(]t,\infty[)) \cup k(g^{-1}(]t,\infty[)) \quad \text{for} \ \ t > t_2 \ .$$

Therefore, $x \notin k(\sup(f,g)^{-1}(]t,\infty[))$ for $t > t_2$, which implies that $\psi(\sup(f,g))(x) \leq t_2 = \sup(\psi(f)(x), \psi(g)(x))$. Since $\psi(\sup(f,g)) \geq \sup(\psi(f), \psi(g))$, because of iii), we get that ψ satisfies vii).

Similarly, one can prove that ϕ satisfies viii); also, ix) follows immediately from the definitions of ϕ and ψ.

Finally, we show that x) is satisfied: Note that

$$\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[) = \bigcup_\alpha \Big(\bigcup \Big\{ k(f_\alpha^{-1}(]-\infty,r[)) \mid r < 0 \Big\}\Big) \subset \bigcup_\alpha k(f_\alpha^{-1}(]-\infty,0[)) \subset k(\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[)).$$

Hence,

$$\overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[)} \cap F \subset \overline{k(\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[))} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[)}\ ,$$

by (4). Since, for $A \subset X$, $\overline{A} \cap F \supset \overline{A \cap F}$, letting $A = \bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[)$, we then get that

$$\overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[)} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[)}\ .$$

This completes the proof that a) implies b).

Since it is obvious that b) implies c), let us prove that c) implies a). Define $k\colon \tau|F \to \tau$ by

$$k(U) = \bigcup \Big\{ \phi(f)^{-1}(]-\infty,0[) \mid f \in C^*(F,]-\infty,1]),\ f(F-U) \subset \{1\} \Big\}\ .$$

Since ϕ is a usc-extender and F is a Tychonoff space, one easily gets that $k(U) \in \tau$ and $k(U) \cap F = U$, for each $U \in \tau|F$; also, $k(\emptyset) = \emptyset$ and $k(F) = X$, because of vi).

Next, note that k is monotone: Let $U, V \in \tau|F$ such that $U \subset V$. Note that $f(F-U) \subset \{1\}$ implies that $f(F-V) \subset \{1\}$, by i), which shows that $k(U) \subset k(V)$.

Next, we prove that, for each $U, V \in \tau|F$, $k(U \cup V) = k(U) \cup k(V)$; i.e., k satisfies (3*): Since k is monotone, we need only prove that $k(U \cup V) \subset k(U) \cup k(V)$. Let $x \in k(U \cup V)$. Then there exists a function $f \in C^*(F,]-\infty,1])$ such that $f(F - U \cup V) \subset \{1\}$ and $\phi(f)(x) < 0$. By Lemma 1 in the Appendix, there exist functions $f_1, f_2, f_3 \in C^*(F,]-\infty,1])$ such that $f_1(F-U) \cup f_2(F-V) \cup f_3(F - U \cap V) \subset \{1\}$ and $\inf(f_1, f_2, f_3) \leq f$. Then

$$0 > \phi(f)(x) \geq \phi(\inf(f_1,f_2,f_3))(x) = \inf(\phi(f_1)(x), \phi(f_2)(x), \phi(f_3)(x))\ .$$

Note that if $\phi(f_1)(x) < 0$ then $x \in k(U)$; if $\phi(f_2)(x) < 0$ then $x \in k(V)$; if $\phi(f_3)(x) < 0$ then $x \in k(U \cap V) \subset k(U) \cup k(V)$. Hence, $x \in k(U) \cup k(V)$.

Finally, we prove that $\overline{k(U)} \cap F = \overline{U}$: Let us say that $k(U) = \bigcup\{\phi(f_\alpha)^{-1}(]-\infty, 0[) \mid \alpha \in \Lambda\}$. Then, since ϕ satisfies property x) of b), we get that

$$\overline{\mu(U)} \cap F = \overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,0[)} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty,0[)} = \overline{U}\ .$$

Hence, $\overline{k(U)} \cap F = \overline{U}$, which completes the proof that c) implies a). ∎

Theorem 3. *For any space X, the following are equivalent:*

a) *X is a K_W-space;*

b) *X is a normal space and, for each nonempty closed subspace F of X, there exist extenders $\phi\colon C^*_{\mathrm{usc}}(F) \to C^*_{\mathrm{usc}}(X)$ and $\psi\colon C^*_{\mathrm{lsc}}(F) \to C^*_{\mathrm{lsc}}(X)$ such that*

- **i**) *$\phi(f) \le \phi(g)$ whenever $f \le g$,*
- **ii**) *$\psi(f) \le \psi(g)$ whenever $f \le g$,*
- **iii**) *$\phi(a_F) = a_X = \psi(a_F)$, for each $a \in \mathbb{R}$,*
- **iv**) *$\phi(f) \le \psi(f)$ whenever $f \in C^*(F)$,*
- **v**) *For any subset $\{f_\alpha \mid \alpha \in \Lambda\}$ of $C^*(F)$ and $a \in \mathbb{R}$,*

$$\overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty,a[)} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty,a[)}\ ,$$

$$\overline{\bigcup_\alpha \psi(f_\alpha)^{-1}(]a,\infty[)} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]a,\infty[)}\ .$$

c) *X is normal and, for any nonempty closed $F \subset X$, there exist extenders $\phi\colon C^*(F) \to C^*_{\mathrm{usc}}(X)$ and $\psi\colon C^*(F) \to C^*_{\mathrm{lsc}}(X)$ which satisfy iii), iv) and v) of b) for functions in $C^*(F)$.*

Proof: a) implies b). This is essentially Proposition 11 of [1]. (The proof of condition v) in Proposition 11 of [1] can obviously be adapted to the more general condition v) of this result.)

Clearly, b) implies c).

c) implies a). (The proof of Theorem 4.1 in [2] surely helped us in devising this argument.) Let F be a nonempty closed subspace of (X, τ). For each $U \in \tau|F$, let

$$\mu(U) = \bigcup \Big\{ \phi(f)^{-1}(]-\infty, 1[) \mid f \in C(F, [-2,2]),\ f(F-U) \subset \{2\} \Big\}\ ,$$
$$\nu(U) = \bigcup \Big\{ \psi(f)^{-1}(]-1, \infty[) \mid f \in C(F, [-2,2]),\ f(F-U) \subset \{-2\} \Big\}\ ,$$
$$k(U) = \mu(U) \cup \nu(U)\ .$$

If $U \in \tau|F$ and $z \in U$, then there exists $f \in C(F, [-2,2])$ such that $f(z) = -2$ and $f(F-U) \subset \{2\}$ (because X is Tychonoff). Since ϕ is an extender, we get that $U \cap \mu(U) = U$; similarly, $U \cap \nu(U) = U$. Hence, $F \cap k(U) = U$, for each $U \in \tau|F$. Clearly, $k(F) = X$ and $k(\emptyset) = \emptyset$, because of iii).

It is easily seen that $k(U) \subset k(V)$ whenever $U \subset V$ (indeed, $\mu(U) \subset \mu(V)$ and $\nu(U) \subset \nu(V)$).

Next, we prove that if $U \cup V = F$ then $k(U) \cup k(V) = X$ (Wlog, let us assume that $U \neq F \neq V$). Let $x \in X$ and suppose that $x \notin \mu(U)$. Then, for each $f \in C(F, [-2,2])$ such that $f(F-U) = 2$, we get that $\phi(f)(x) \geq 1$. Pick $h \in C(F, [-2,2])$ such that $h(F-V) = -2$ and $h(F-U) = 2$ (this can be done because F is normal). It follows that $\psi(h)(x) \geq \phi(h)(x) \geq 1$, which implies that $x \in \nu(V)$. Similarly, if $x \notin \mu(V)$ then $x \in \nu(U)$. Consequently, we get that $x \in k(U) \cup k(V)$, as required.

Finally, we prove that, for each $U \in \tau|F$, $\overline{k(U)} \cap F = \overline{U}$, by proving that $\overline{\mu(U)} \cap F = \overline{U} = \overline{\nu(U)} \cap F$ (we will prove the first equality and note that the second equality can be similarly proved): Let us assume that $\mu(U) = \bigcup \{\phi(f_\alpha)^{-1} (]-\infty, 1[) \mid \alpha \in \Lambda\}$. Since ϕ satisfies condition v) of b), we get that

$$\overline{\mu(U)} \cap F = \overline{\bigcup_\alpha \phi(f_\alpha)^{-1}(]-\infty, 1[)} \cap F = \overline{\bigcup_\alpha f_\alpha^{-1}(]-\infty, 1[)} = \overline{U}\ .$$

This completes the proof. ∎

Theorem 4. *For a space (X, τ), the following are equivalent:*

i) *X is a K_W-space;*

ii) *For each closed subspace F of X there exists a function $k\colon \tau|F \to \tau$ such that*

(1′) *$F \cap k(U) = U$, for each $U \in \tau|F$, $k(F) = X$, $k(\emptyset) = \emptyset$,*

(2′) *$k(U) \subset k(V)$ whenever $U \subset V$,*

(3′) *$U, V \in \tau|F$, $\overline{U} \cap \overline{V} = \emptyset$ implies $\overline{k(U)} \cap \overline{k(V)} = \emptyset$,*

(4′) *$\overline{k(U)} \cap F = \overline{U}$.*

Proof: i) implies ii). Let $\sigma\colon \tau|F \to \tau$ be a K_W-function and define $k\colon \tau|F \to \tau$ by $k(U) = U \cup (X - [F \cup \overline{\sigma(F - \overline{U})}])$. (Note that

$$\begin{aligned} k(U) &= U \cup \Big((X - F) \cap (X - \overline{\sigma(F - \overline{U})})\Big) \\ &= \Big(U \cup (X - F)\Big) \cap \Big(U \cup [X - \overline{\sigma(F - \overline{U})}]\Big) \end{aligned}$$

and $X - \overline{\sigma(F - \overline{U})} \supset U$ because, by (4),

$$\Big(X - \overline{\sigma(F - \overline{U})}\Big) \cap F = F - \Big(\overline{\sigma(F - \overline{U})} \cap F\Big) = F - \overline{F - \overline{U}} \supset U\ .$$

Hence, we do get that $k(U) \in \tau$.)

From the definition of k we immediately get that k satisfies $(1')$.

k satisfies $(2')$: $U \subset V$ implies $\overline{U} \subset \overline{V}$ implies $F - \overline{V} \subset F - \overline{U}$ implies $\overline{\sigma(F - \overline{V})} \subset \overline{\sigma(F - \overline{U})}$ implies $k(U) \subset k(V)$.

k satisfies $(3')$: $\overline{U} \cap \overline{V} = \emptyset$ implies $(F - \overline{U}) \cup (F - \overline{V}) = F$ implies $\sigma(F - \overline{U}) \cup \sigma(F - \overline{V}) = X$ implies

$$\begin{aligned} &\overline{X - [F \cup \overline{\sigma(F - \overline{U})}]} \cap \overline{X - [F \cup \overline{\sigma(F - \overline{V})}]} = \\ &\quad = X - [F \cup \overline{\sigma(F - \overline{U})}]^0 \cap X - [F \cup \overline{\sigma(F - \overline{V})}]^0 \\ &\quad = X - \Big([F \cup \overline{\sigma(F - \overline{U})}]^0 \cup [F \cup \overline{\sigma(F - \overline{V})}]^0\Big) \subset \\ &\qquad \subset X - \Big(\overline{\sigma(F - \overline{U})}^0 \cup \overline{\sigma(F - \overline{V})}^0\Big) \subset X - \Big(\sigma(F - \overline{U}) \cup \sigma(F - \overline{V})\Big) = \emptyset\ . \end{aligned}$$

Also, $\overline{U} \cap \overline{V} = \emptyset$ implies $\overline{U} \subset F - \overline{V}$ implies $\overline{U} \subset \sigma(F - \overline{V})$ implies $\overline{U} \subset \overline{\sigma(F - \overline{V})}^0$ implies $\overline{U} \cap (X - [F \cup \overline{\sigma(F - \overline{V})}]^0) = \emptyset$; similarly, $\overline{V} \cap (X - [F \cup \overline{\sigma(F - \overline{U})}]^0) = \emptyset$. Consequently, $\overline{k(U)} \cap \overline{k(V)} = \emptyset$.

k satisfies $(4')$: $\overline{k(U)} \cap F = \overline{U} \cup (\overline{(X - [F \cup \overline{\sigma(F - \overline{U})}])} \cap F) \supset \overline{U}$; since

$$\begin{aligned} &\overline{X - [F \cup \overline{\sigma(F - \overline{U})}]} \cap F \subset \overline{X - \overline{\sigma(F - \overline{U})}} \cap F = \Big(X - \overline{[\sigma(F - \overline{U})]^0}\Big) \cap F = \\ &\quad = F - \Big(F \cap \overline{[\sigma(F - \overline{U})]^0}\Big) \subset F - \Big(F \cap \sigma(F - \overline{U})\Big) = F - (F - \overline{U}) = \overline{U}\ , \end{aligned}$$

we then get that $\overline{k(U)} \cap F = \overline{U}$.

ii) implies i). One need only check that the preceding arguments are essentially reversible; that is, starting with k, which satisfies $(1')$–$(4')$, define σ by $\sigma(U) = U \cup (X - [F \cup \overline{k(F - \overline{U})}])$; then σ is a K_W-function: It is easily seen that $F \cap \sigma(U) = U$, for each $U \in \tau|F$, $\sigma(F) = X$, $\sigma(\emptyset) = \emptyset$, and $\sigma(U) \subset \sigma(V)$

whenever $U \subset V$. Also, $U, V \in \tau|F$ and $U \cup V = F$ implies $\overline{U}^0 \cup \overline{V}^0 = F$ (here, interiors refer to $\tau|F$) implies $(F - \overline{U}^0) \cap (F - \overline{V}^0) = \emptyset$ if and only if $\overline{(F - \overline{U})} \cap \overline{(F - \overline{V})} = \emptyset$ implies $\overline{k(F - \overline{U})} \cap \overline{k(F - \overline{V})} = \emptyset$ implies

$$\begin{aligned} U \cup \Big(X - [F \cup \overline{k(F - \overline{U})}]\Big) \cup V \cup \Big(X - [F \cup \overline{k(F - \overline{V})}]\Big) = \\ &= (U \cup V) \cup \Big(X - [F \cup \overline{k(F - \overline{U})}] \cap [F \cup \overline{k(F - \overline{V})}]\Big) \\ &= F \cup \Big(X - [F \cup (\overline{k(F - \overline{U})} \cap \overline{k(F - \overline{V})})]\Big) = F \cup (X - F) = X\ . \end{aligned}$$

Therefore, $\sigma(U) \cup \sigma(V) = X$ whenever $U \cup V = F$. Finally, $\sigma(\overline{U}) \cap F = \overline{\overline{U} \cup (X - [F \cup \overline{k(F - \overline{U})}]} \cap F) \supset \overline{U}$; since $\overline{X - [F \cup \overline{k(F - \overline{U})}]} \cap F \subset \overline{U}$, we then get that $\sigma(\overline{U}) \cap F = \overline{U}$. We have thus shown that σ is a K_W-function, which completes the proof. ∎

Corollary 5. *K_W-spaces are collectionwise normal.*

Proof: Let (X, τ) be a K_W-space and $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda\}$ be a discrete collection of closed subsets of X. Letting $F = \bigcup \mathcal{A}$, we get that each $A_\alpha \in \tau|F$. Letting $k : \tau|F \to \tau$ be a function which satisfies conditions $(1')$ and $(3')$ of Theorem 4, we then get that $\{\overline{k(A_\alpha)} \mid \alpha \in \Lambda\}$ is a pairwise-disjoint collection of closed subsets of X with each $A_\alpha \subset k(A_\alpha)$. This shows that X is collectionwise normal. ∎

Appendix

The following result is crucial to our work. It probably is folklore.

Lemma 1. *Let F be a completely normal space, U and V open subsets of F and $f\colon F \to]-\infty, 1]$ be a continuous function such that $f(F - U \cup V) \subset \{1\}$. Then there exist continuous functions $f_1, f_2, f_3\colon F \to]-\infty, 1]$ such that*

- **i)** $f_1(F - U) \cup f_2(F - V) \cup f_3(F - U \cap V) \subset \{1\}$;
- **ii)** $\inf(f_1, f_2, f_3) \leq f$.

Proof: Let us first consider the case $U \cup V \neq F$. Since F is completely normal and $\overline{U - V} \cap (V - U) = \emptyset = (U - V) \cap \overline{V - U}$, pick disjoint open U', V' such that $U - V \subset U'$ and $V - U \subset V'$. Let $f_1 = f$ on $U - V'$ and $f_1 = 1$ on $F - U$ and extend f_1 to $f_1 : F \to]-\infty, 1]$. Let $f_2 = f$ on $V - U'$ and $f_2 = 1$ on $F - V$ and extend f_2 to $f_2\colon F \to]-\infty, 1]$. Let $f_3 = f$ on $U \cap V - (U' \cup V')$ and $f_3 = 1$ on $F - U \cap V$ and extend f_3 to $f_3\colon F \to]-\infty, 1]$. Since $U \cup V = (U - V') \cup (V - U') \cup [(U \cap V) - (U' \cup V')]$, we immediately get that $\inf(f_1, f_2, f_3) \leq f$.

It is now clear that the result is also valid if $U \cap V = \emptyset$. Finally, let us show that it also remains valid if $U \cup V = F$: (Wlog, assume $U \neq F \neq V$). Simply pick open U', V' such that $\overline{U}' \subset U$, $\overline{V}' \subset V$ and $\overline{U}' \cup \overline{V}' = F$. Let $f_1 = f$ on U' and $f_1 = 1$ on $F - U$ and extend f_1 to $f_1\colon F \to]-\infty, 1[$. Let $f_2 = f$ on V' and $f_2 = 1$ on $F - V$ and extend f_2 to $f_2\colon F \to]-\infty, 1[$. Let $f_3 = 1_F$. One immediately gets that $\inf(f_1, f_2, f_3) \leq f$. ∎

Remark. Clearly, the preceding result remains valid for $f\colon F \to [-1, \infty[$, $f(F - U \cup V) \subset \{-1\}$ and $\sup(f_1, f_2, f_3) \geq f$.

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Carlos R. Borges,
Dep. of Mathematics, University of California,
Davis, California 95616-8633 – U.S.A.