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ON THE IDEMPOTENT RANKS OF CERTAIN SEMIGROUPS OF ORDER-PRESERVING TRANSFORMATIONS

G.U. Garba

Abstract: The ranks of the semigroups O_n, PO_n and SPO_n (the semigroups of order-preserving singular selfmaps, partial and strictly partial transformations on $X_n = \{1, ..., n\}$ respectively), and the idempotent ranks of O_n and PO_n were studied by Gomes and Howie [2]. In this paper we generalize their results in line with Howie and McFadden [7], by considering the semigroups $L(n,r)$, $M(n,r)$ and $N(n,r)$, where, for $2 \leq r \leq n-2$, $L(n,r) = \{\alpha \in O_n \colon |\mathrm{Im}\,\alpha| \leq r\}$, $M(n,r) = \{\alpha \in PO_n \colon |\mathrm{Im}\,\alpha| \leq r\}$ and $N(n,r) = \{\alpha \in SPO_n \colon |\mathrm{Im}\,\alpha| \leq r\}$.

1 – Introduction

By the *rank* of a semigroup S we shall mean the cardinality of any subset A of minimal order in S such that $\langle A \rangle = S$. The cardinality of the smallest subset A *consisting of idempotents* for which $\langle A \rangle = S$ is called the *idempotent rank* of S.

Let $X_n = \{1, ..., n\}$, let T_n be the full transformation semigroup on X_n, and let $\mathrm{Sing}_n = \{\alpha \in T_n \colon |\mathrm{Im}\,\alpha| \leq n-1\}$ be the semigroup of all singular selfmaps on X_n. In [4], Sing_n was shown to be idempotent-generated; its rank and idempotent rank were shown by Gomes and Howie [1] to be equal to $n(n-1)/2$. This was generalized by Howie and McFadden [7], who considered the semigroup

$$K(n,r) = \{\alpha \in T_n \colon |\mathrm{Im}\,\alpha| \leq r\} ,$$

where $2 \leq r \leq n-1$, and showed that both the rank and the idempotent rank are equal to $S(n,r)$, the Stirling number of the second kind, defined by

$$S(n,1) = S(n,n) = 1 , \quad S(n,r) = S(n-1,r-1) + rS(n-1),r \quad (n \geq r \geq 1) .$$

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The semigroup $O_n = \{\alpha \in \mathrm{Sing}_n\colon (\forall\, x, y \in X_n)\ x \le y \Rightarrow x\alpha \le y\alpha\}$ of all order-preserving singular selfmaps of X_n, was shown to be idempotent-generated by Howie [5]; its rank and idempotent rank were shown to be n and $2n-2$ respectively, by Gomes and Howie [1]. In section 2 of this paper, we show that both the rank and the idempotent rank of

$$L(n,r) = \Big\{\alpha \in O_n\colon\ |\mathrm{Im}\,\alpha| \le r\Big\}\ ,$$

where $2 \le r \le n-2$, are equal to $\binom{n}{r}$.

Gomes and Howie [2] also considered the semigroup $PO_n = O_n \cup \{\alpha\colon \mathrm{dom}\,\alpha \subset X_n,\ (\forall\, x, y \in \mathrm{dom}\,\alpha)\ x \le y \Rightarrow x\alpha \le y\alpha\}$ of all partial order-preserving transformations of X_n (excluding the identity map). They showed that PO_n is idempotent-generated, its rank is equal to $2n-1$, and its idempotent rank is $3n-2$. In section 3 we show that the rank and the idempotent rank of

$$M(n,r) = \Big\{\alpha \in PO_n\colon\ |\mathrm{Im}\,\alpha| \le r\Big\}\ ,$$

where $2 \le r \le n-2$, are both equal to $\sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$.

In the final section we turn our attention to the semigroup $SPO_n = PO_n \backslash O_n$ of strictly partial order-preserving maps of X_n. This semigroup is not idempotent-generated, and as pointed out by Gomes and Howie [2], the question of its idempotent rank does not arise. However, they showed that its rank is $2n-2$. In this paper we show that the semigroup

$$N(n,r) = \Big\{\alpha \in SPO_n\colon\ |\mathrm{Im}\,\alpha| \le r\Big\}\ ,$$

where $2 \le r \le n-2$, is idempotent-generated, and that its rank and idempotent rank are both equal to $\sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$.

2 – Order-preserving singular selfmaps

By [6, Proposition 2.4.5 and Exercise 2.10] we have that in O_n

$$\alpha\,\mathcal{L}\,\beta \quad \text{if and only if} \quad \mathrm{Im}\,\alpha = \mathrm{Im}\,\beta\ ,$$

$$\alpha\,\mathcal{R}\,\beta \quad \text{if and only if} \quad \ker\alpha = \ker\beta\ ,$$

$$\alpha\,\mathcal{J}\,\beta \quad \text{if and only if} \quad |\mathrm{Im}\,\alpha| = |\mathrm{Im}\,\beta|\ .$$

Thus O_n is the union of $\mathcal{J}$-classes $J_1, J_2, \ldots, J_{n-1}$, where

$$J_r = \Big\{\alpha \in O_n\colon\ |\mathrm{Im}\,\alpha| = r\Big\}\ .$$

The $(\ker\alpha)$-classes are *convex* subsets C of X_n, in the sense that

$$x, y \in C \quad \text{and} \quad x \le z \le y \;\Rightarrow\; z \in C\ .$$

We shall refer to an equivalence ρ on the set X_n as *convex* if its classes are convex subsets of X_n, and we shall say that ρ is of weight r if $|X_n/\rho| = r$. Thus J_r has $\binom{n-1}{r-1}$ $\mathcal{R}$-classes corresponding to the $\binom{n-1}{r-1}$ convex equivalences of weight r on X_n, and $\binom{n}{r}$ $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ subsets of X_n of cardinality r.

Lemma 2.1. *Every element α in J_r $(r \le n-2)$ is expressible as a product of elements in J_{r+1}.*

Proof: Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}\ .$$

Then at least one block, say A_i, contains more than one element. Let $c = \min\{a_i : a_i \in A_i\}$. Suppose that $\{b_1, b_2, ..., b_r\}$ has a gap in position j, and let y be such that $b_{j-1} < y < b_j$. We distinguish four cases.

Case 1. $i = j-1$. Let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i\backslash\{c\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+3 & \dots & r+2 \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1 & \dots & i-1 & \{i, i+1\} & i+2 & i+3 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & b_i & y & b_{i+1} & \dots & b_{r-1} & b_r \end{pmatrix}$$

where $A' = X_n\backslash\{1, 2, ..., r+1\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 2. $i < j-1$. Suppose here that β and δ are given by

$$\begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i\backslash\{c\} & A_{i+1} & \dots & A_{j-1} & A_j & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & j & j+2 & \dots & r+2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & \dots & i-1 & Y & i+2 & \dots & j & j+1 & j+2 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{j-1} & y & b_j & \dots & b_{r-1} & b_r \end{pmatrix}$$

respectively, where $Y = \{i, i+1\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 3. $i = j$. Let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i\backslash\{c\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i+1 & i+2 & i+3 & \dots & r+2 \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1 & \dots & i-1 & i & \{i+1, i+2\} & i+3 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & y_i & b_i & b_{i+1} & \dots & b_{r-1} & b_r \end{pmatrix}\ .$$

Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 4. $i > j-1$. Suppose β and δ are given by

$$\begin{pmatrix} A_1 & ... & A_{j-1} & A_j & ... & A_{i-1} & c & A_i\backslash\{c\} & A_{i+1} & ... & A_r \\ 1 & ... & j-1 & j+1 & ... & i & i+1 & i+2 & i+3 & ... & r+2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & ... & j-1 & j & j+1 & ... & i & \{i+1, i+2\} & i+3 & ... & r+1 & A' \\ b_1 & ... & b_{j-1} & y & b_j & ... & b_{i-1} & b_i & b_{i+1} & ... & b_{r-1} & b_r \end{pmatrix}$$

respectively. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$. Hence the proof. ∎

It follows from this Lemma that $\langle J_r \rangle = L(n,r)$. If we let E_r be the set of all idempotents in J_r, then by Lemma 1 in [3], and Theorem 1.1 in [5] we have $J_r \subseteq \langle E_r \rangle$. Thus

$$L(n,r) = \langle E_r \rangle \ .$$

From Lemma 3 in [7] we deduce that the rank of $L(n,r)$ must be at least the number of $\mathcal{L}$-classes in J_r. Thus we have

$$\text{rank}(L(n,r)) \geq \binom{n}{r} .$$

We now show

Theorem 2.2. *For $2 \leq r \leq n-2$, we have*

$$\text{rank}(L(n,r)) = \text{idrank}(L(n,r)) = \binom{n}{r} .$$

Proof: The proof depends on a Lemma very similar to Lemma 6 in [7]. By a *transversal* A of an equivalence relation π on a set X we mean a subset of X with the property that each a in A belongs to precisely one π-class. ∎

Lemma 2.3. *Let $\pi_1, \pi_2, ..., \pi_m$ (where $m = \binom{n-1}{r-1}$, $r \geq 3$) be a list of the convex equivalences of weight r on X_n. Suppose that there exist distinct subsets $A_1, A_2, ..., A_m$ of cardinality r of X_n with the property that A_i is a transversal of π_{i-1}, π_i ($i = 2, ..., m$) and A_1 is a transversal of π_1, π_m. Then each $\mathcal{H}$-class (π_i, A_i) consists of an idempotent ϵ_i, and there exist idempotents $\epsilon_{m+1}, ..., \epsilon_p$ (where $p = \binom{n}{r}$) such that $\{\epsilon_1, \epsilon_2, ..., \epsilon_p\}$ is a set of generators for $L(n,r)$.* ∎

Assuming the listing of convex equivalences and subsets as in Lemma 2.3 above, we now show that every idempotent in E_r is expressible as a product of the p idempotents $\epsilon_1, \epsilon_2, ..., \epsilon_p$. Notice first that the product $\epsilon_i\epsilon_{i-1}$ $(i = 2, ..., m)$ is an element of height r, since we have a configuration

$$\begin{array}{cc} \epsilon_{i-1} & \circ \\ * & \epsilon_i \end{array}$$

in which the $\mathcal{H}$-class labeled $\circ$ consists of an idempotent. Moreover, the element $\epsilon_i\epsilon_{i-1}$ is in the position $*$ by Lemma 1 in [7]. By the same token the product $\epsilon_1\epsilon_m$ is of height r, and $\epsilon_m \,\mathcal{L}\, \epsilon_1\epsilon_m \,\mathcal{R}\, \epsilon_1$.

Choose the idempotents $\epsilon_{m+1}, ..., \epsilon_p$ so that $\epsilon_1, \epsilon_2, ..., \epsilon_p$ covers all the $\mathcal{L}$-classes in J_r. Then if η is an arbitrary idempotent in J_r there exists a unique $i \in \{1, ..., p\}$ such that $\eta\mathcal{L}\epsilon_i$, and a unique $j \in \{1, ..., m\}$ such that $\eta\mathcal{R}\epsilon_j$,

$$\begin{array}{ccccc} \epsilon_k & \cdots & & \cdots & \epsilon_i \\ & & \vdots & & \vdots \\ & \epsilon_{j-1} & \circ & \cdots & \\ & & \epsilon_j & \cdots & \eta \end{array}$$

Moreover, there is a unique $k \in \{1, ..., m\}$ such that $\epsilon_i\mathcal{R}\epsilon_k$. (If $i \in \{1, ..., m\}$ then of course $k = i$.) If $k = j$ then $\eta = \epsilon_i$ and there is nothing to prove. If $k < j$ then

$$\eta = \epsilon_j\, \epsilon_{j-1} \cdots \epsilon_{k+1}\, \epsilon_i \ .$$

If $k > j$ then

$$\eta = \epsilon_j \cdots \epsilon_1\, \epsilon_m \cdots \epsilon_{k+1}\, \epsilon_i \ .$$

We have shown that every idempotent in J_r can be expressed as a product of the $p = \binom{n}{r}$ idempotents, $\epsilon_1, ..., \epsilon_p$. Hence

$$L(n, r) = \langle \epsilon_1, \epsilon_2, ..., \epsilon_p \rangle \ . \ \blacksquare$$

It remains to prove that the listing of convex equivalences and images postulated in the statement of Lemma 2.3 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n-2$, and consider the Proposition:

$P(n, r)$. *There is a way of listing the convex equivalences of weight r as $\pi_1, \pi_2, ..., \pi_m$ (with $m = \binom{n-1}{r-1}$) and π_1 having $\{r, r+1, ..., n\}$ as the only non-singleton class, π_2 having $\{r-1, r\}$ and $\{r+1, ..., n\}$ as the only non-singleton classes, π_m having $\{r-1, ..., n-1\}$ as the only non-singleton class) so that there exist subsets $A_1, ..., A_m$ of X_n of cardinality r with the property that A_i is a transversal of π_{i-1}, π_i $(i = 2, ..., m)$ and A_1 is a transversal of π_1, π_m.*

We shall prove this by a double induction on n and r, the key step being a kind of Pascal's Triangle implication

$$P(n-1, r-1) \ \text{ and } \ P(n-1, r) \quad \Rightarrow \quad P(n, r) \ .$$

First, however, we anchor the induction with two Lemmas.

Lemma 2.4. *$P(n,2)$ holds for every $n \geq 4$.*

Proof: Consider the list $\pi_1, ..., \pi_{n-1}$ of convex equivalences of weight 2 on X_n, where

$$\pi_i = 1\ 2\ 3 \cdots i/i+1 \cdots n \ .$$

Let

$$A_1 = \{1, n\}\,, \qquad A_2 = \{1, 3\} \qquad \text{and} \qquad A_i = \{i-1, n\}$$

for $i=3, ..., n-1$. Then it is easy to verify that $\pi_1, \pi_2, ..., \pi_{n-1}$ and $A_1, A_2, ..., A_{n-1}$ have the required property. ∎

Lemma 2.5. *$P(n, n-2)$ holds for every $n \geq 4$.*

Proof: The proof is by induction. We shall show that for $k \geq 4$,

$$P(k, k-2) \quad \Rightarrow \quad P(k+2, k) \ .$$

For $n = 4$ the result follows from Lemma 2.4, and for $n = 5$ we have the list of the six convex equivalences and the six subsets as follows:

$$\begin{array}{ll}
1/2/3\ 4\ 5 & \{1,2,5\}\ , \\
1/2\ 3/4\ 5 & \{1,2,4\}\ , \\
1\ 2/3/4\ 5 & \{1,3,4\}\ , \\
1\ 2/3\ 4/5 & \{1,3,5\}\ , \\
1\ 2\ 3/4/5 & \{2,4,5\}\ , \\
1/2\ 3\ 4/5 & \{1,4,5\}\ .
\end{array}$$

Suppose inductively that $P(k, k-2)$ holds $(k \geq 4)$. Thus we have a list $\pi_1, \pi_2, ..., \pi_m$ (with $m = \binom{k-1}{k-3}$) of convex equivalences of weight $k-2$ on X_k, and a list $A_1, A_2, ..., A_m$ of subsets of X_k of cardinality $k-2$ such that A_i is a transversal of π_{i-1}, π_i $(i = 2, ..., m)$ and A_1 is a transversal of π_1, π_m. We may also assume that

π_1 has $\{k-2, k-1, k\}$ as the only non-singleton class ,

π_2 has $\{k-3, k-2\}$ and $\{k-1, k\}$ as the only non-singleton class ,

π_m has $\{k-3, k-2, k-1\}$ as the only non-singleton class ,

$A_2 = X_k \backslash \{k-2, k\}$.

Let $\sigma_1, ..., \sigma_k$ be the list of convex equivalences of weight k on X_{k+1}, where

$$\sigma_i \text{ has } \{k-i+1, k-i+2\} \text{ as the only non-singleton class .}$$

(Thus in particular σ_1, σ_2 and σ_k have $\{k, k+1\}$, $\{k-1, k\}$ and $\{1, 2\}$ as the only non-singleton classes respectively.) Let $\tau_1, \tau_2, ..., \tau_{k-1}$ be the list of convex equivalences of weight $k-1$ on X_k, where

$$\tau_i \text{ has } \{k-i, k-i+1\} \text{ as the only non-singleton class.}$$

(In particular each of τ_1, τ_2 and τ_{k-1} has $\{k-1, k\}$, $\{k-2, k-1\}$ and $\{1, 2\}$ as the only non-singleton class respectively.) Define the convex equivalences

$\pi_i' = \pi_i \cup \{(k+1, k+1)\} \cup \{(k+2, k+2)\}$, for $i = 1, ..., m$,

$\sigma_i' = \sigma_i$ with $k+2$ adjoined to the class containing $k+1$, $i = 1, ..., k$,

$\tau_i' = \tau_i$ with $k+1$ adjoined to the class containing k,
and $k+2$ as a singleton class, for $i = 1, ..., k-1$.

Then arrange them as follows:

$$\sigma_1', ..., \sigma_k', \tau_{k-1}', ..., \tau_2', \pi_2', ..., \pi_m', \pi_1', \tau_1' \ . \qquad (2.6)$$

Notice that these convex equivalences are all distinct, and (2.6) is a complete list of the convex equivalences of weight k on X_{k+2}, since $m + k + k - 1 = \binom{k+1}{k-1}$.

We now define the subsets

$$\begin{aligned} A_i' &= A_i \cup \{k+1, k+2\} && \text{for } i = 1, ..., m \ , \\ B_i &= X_{k+2} \backslash \{k-i+2, k+2\} && \text{for } i = 2, ..., k \ , \\ C_i &= X_{k+2} \backslash \{k-i+1, k+1\} && \text{for } i = 1, 3, ..., k \ , \\ D &= X_{k+2} \backslash \{k-1, k\} \ . \end{aligned}$$

It follows from the hypothesis that A_i' is a transversal of π_{i-1}', π_i' for $i = 3, ..., m$ and that A_1' is a transversal of π_m', π_1'. It is also not difficult to verify that, for $i = 2, ..., k$, B_i is a transversal of σ_{i-1}', σ_i'; for $i = 3, ..., k-1$, C_i is a transversal of τ'_{i-1}, τ'_i; C_1 is a transversal of σ'_1, τ'_1; C_k is a transversal of σ'_k, τ'_{k-1}; A'_2 is a transversal of τ'_2, π'_2; and finally D is a transversal of π'_1, τ'_1. It therefore remains to show that the subsets

$$C_1, B_2, ..., B_k, C_k, C_{k-1}, ..., C_3, A'_2, A'_3, ..., A'_m, A'_1, D \tag{2.7}$$

are all distinct. It is clear that the A''s, B's and C's are all distinct. (The A''s contain $k+1$ and $k+2$, the B's contain $k+1$ but not $k+2$, while the C's contain $k+2$ but not $k+1$.) Also D is distinct from the B's and the C's (since the latter must not contain $k+1$ or $k+2$). Note that for $i = 1, 2, ..., m$ the $\mathcal{L}$-class characterized by A_i contains at least two idempotents (since A_i is a transversal of π_{i-1} and π_i for $i = 2, ..., m$ and A_1 is a transversal of π_1 and π_m). But the $\mathcal{L}$-class characterized by $D \backslash \{k+1, k+2\}$ contains only one idempotent, namely

$$\begin{pmatrix} 1 & 2 & ... & k-3 & \{k-2, k-1, k\} \\ 1 & 2 & ... & k-3 & k-2 \end{pmatrix}.$$

Hence $D \backslash \{k+1, k+2\}$ is not one of the A's, and consequently D is distinct from the A''s. So all the subsets in (2.7) are distinct. ∎

Lemma 2.8. *Let $n \geq 6$ and $3 \leq r \leq n-3$. Then $P(n-1, r-1)$ and $P(n-1, r)$ together imply $P(n, r)$.*

Proof: From the assumption $P(n-1, r)$ we have a list $\sigma_1, \sigma_2, ..., \sigma_m$ (where $m = \binom{n-2}{r-1}$) of convex equivalences of weight r on X_{n-1} and a list $A_1, ..., A_m$ of distinct subsets of X_{n-1} of cardinality r such that A_i is a transversal of σ_{i-1}, σ_i $(i = 2, ..., m)$ and A_1 is a transversal of σ_m, σ_1. We may also assume that

σ_1 has $\{r, ..., n-1\}$ as the only non-singleton class ,

σ_2 has $\{r-1, r\}$ and $\{r+1, ..., n-1\}$ as the only non-singleton classes ,

σ_m has $\{r-1, ..., n-2\}$ as the only non-singleton class ,

$A_2 = \{1, 2, ..., r-1, r+1\}$.

From the assumption $P(n-1, r-1)$ we have a list $\tau_1, ..., \tau_t$ (where $t = \binom{n-2}{r-2}$) of convex equivalences of weight $r-1$ on X_{n-1} and a list $B_1, ..., B_t$ of distinct subsets of cardinality $r-1$ on X_{n-1} such that B_i is a transversal of τ_{i-1}, τ_i $(i = 2, ..., t)$ and B_1 is a transversal of τ_t, τ_1. We may also assume that

τ_1 has $\{r-1, ..., n-1\}$ as the only non-singleton class ,

τ_2 has $\{r-2, r-1\}$ and $\{r, ..., n-1\}$ as the only non-singleton classes ,

τ_t has $\{r-2, ..., n-2\}$ as the only non-singleton class ,

$B_2 = \{1, 2, ..., r-2, r\}$.

Now, for $i = 1, ..., m$ let

$$\sigma'_i = \sigma_i \text{ with } n \text{ adjoined to the class containing } n-1 \ ,$$

for $j = 1, ..., t$ let

$$\tau'_j = \tau_j \cup \{(n, n)\} \ .$$

Then arrange the convex equivalences as follows:

$$\sigma'_1, ..., \sigma'_m, \tau'_2, ..., \tau'_t, \tau'_1 \ . \tag{2.9}$$

Note that $m + t = \binom{n-1}{r-1}$. Hence above is a complete list of all the convex equivalences of weight r on X_n. Next we define

$$A = \{1, 2, ..., r-1, n\} \ ,$$

$$B'_i = B_i \cup \{n\} \quad \text{for} \quad i = 1, ..., t \ ,$$

and arrange the subsets as follows:

$$A, A_2, A_3, ..., A_m, B'_2, ..., B'_t, B'_1 \ . \tag{2.10}$$

Then A_i is a transversal of σ'_{i-1}, σ'_i $(i = 2, ..., m)$; B'_i is a transversal of τ'_{i-1}, τ'_i $(i = 3, ..., t)$; B'_1 is a transversal of τ'_t, τ'_1; A is a transversal of σ'_1, τ'_1 and B'_2 is a transversal of σ'_m, τ'_2.

It is clear that $A_2, ..., A_m, B'_1, ..., B'_t$ are all distinct subsets of X_n of cardinality r, and A is distinct from $A_2, ..., A_m$. If $A = B'_i$ for some $i = 1, ..., t$, then

$$A \backslash \{n\} = B_i = \{1, 2, ..., r-1\} \ .$$

But the $\mathcal{L}$-class characterized by $\{1, 2, ..., r-1\}$ has only one idempotent, namely

$$\begin{pmatrix} 1 & 2 & 3 & ... & r-2 & A' \\ 1 & 2 & 3 & ... & r-2 & r-1 \end{pmatrix} ,$$

where $A' = X_n \backslash \{1, 2, ..., r-2\}$. This is contrary to the hypothesis that the $\mathcal{L}$-class characterized by B_i must contain at least two idempotents. Hence all the subsets are distinct. Thus the induction step is complete, and we may deduce that $P(n, r)$ is true for all $n \geq 4$ and all r such that $2 \leq r \leq n-2$.

The pattern of deduction is

$$\begin{array}{ccccccc} & & & P(4,2) & & & \\ & & P(5,2) & & P(5,3) & & \\ & P(6,2) & & P(6,3) & & P(6,4) & \\ P(7,2) & & P(7,3) & & P(7,4) & & P(7,5) \end{array}$$

3 – Order-preserving partial transformation semigroups

As usual, we shall refer to an element α in PO_n, and indeed in the larger semigroup P_n of all partial transformations of X_n, as being of type (k, r), or belonging to the set $[k, r]$ if $|\operatorname{dom}\alpha| = k$ and $|\operatorname{Im}\alpha| = r$.

The $\mathcal{J}$-class $J_r = \{\alpha \in PO_n \colon |\operatorname{Im}\alpha| = r\}$ is the union of the sets $[k, r]$, where $r \le k \le n$. The number of $\mathcal{L}$-classes in J_r is the number of image sets in X_n of cardinality r, namely $\binom{n}{r}$. The number of $\mathcal{R}$-classes in J_r is the number of convex equivalences of weight r on all the subsets of X_n of cardinality k, where $r \le k \le n$. This number is $\sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$.

Lemma 3.1. *$J_r \subseteq (J_{r+1})^2$ for $1 \le r \le n-3$.*

Proof: Let α in J_r be in $[k, r]$, $2 \le r \le k \le n$. If $k = r$, the result follows from Lemma 3.4 in [2], that $[r, r] \subseteq ([r+1, r+1])^2$. If $k > r$, then the proof of Lemma 2.1 above applies equally to this case by adjusting A' to $\{r+2\}$. ∎

From Lemma 3 in [7] we also deduce that the rank of $M(n, r)$ must be at least as large as the number of $\mathcal{R}$-classes in J_r. Thus we have

$$\operatorname{rank}(M(n,r)) \ge \sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}.$$

Theorem 3.2. *For $1 \le r \le n-2$,*

$$\operatorname{rank}(M(n,r)) = \operatorname{idrank}(M(n,r)) = \sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}.$$

The proof follows the same basic strategy as that of Theorem 2.2. It depends on the following Lemma.

Lemma 3.3. *Let $A_1, ..., A_m$ (where $m = \binom{n}{r}$ and $r \ge 2$) be a list of subsets of X_n with cardinality r. Suppose that there exist distinct convex equivalences $\pi_1, ..., \pi_m$ of weight r on X_n with the property that A_{i-1}, A_i are both transversals of π_i $(i = 2, ..., m)$ and A_m, A_1 are both transversals of π_1. Then each $\mathcal{H}$-class (π_i, A_i) consists of an idempotent ϵ_i, and there exist idempotents $\epsilon_{m+1}, ..., \epsilon_p$ (where $p = \sum_{k=r}^{n}\binom{n}{k}\binom{k-1}{r-1}$) such that $\{\epsilon_1, ..., \epsilon_p\}$ is a set of generators for $M(n,r)$.*

Assuming the listing of convex equivalences and images as in Lemma 3.3 above, we now show that every idempotent in J_r is expressible as a product of the p idempotents $\epsilon_1, \epsilon_2, ..., \epsilon_p$.

Notice first that $\epsilon_{i-1}\,\epsilon_i$ $(i = 2, ..., m)$ is an element of height r, since we have a configuration

$$\begin{array}{cc} \epsilon_{i-1} & * \\ \circ & \epsilon_i \end{array}$$

in which the $\mathcal{H}$-class labeled $\circ$ contains an idempotent. Moreover, by Lemma 1 in [7], the element $\epsilon_{i-1}\epsilon_i$ is in position $*$. By the same token, the product $\epsilon_m\epsilon_1$ is also of height r, and $\epsilon_m \mathcal{R} \epsilon_m\epsilon_1 \mathcal{L} \epsilon_1$.

Choose the idempotents $\epsilon_{m+1}, ..., \epsilon_p$ so that $\epsilon_1, ..., \epsilon_p$ cover all the $\mathcal{R}$-classes in J_r. Then if η is an arbitrary idempotent in J_r there exists a unique i in $\{1, ..., p\}$ and a unique j in $\{1, ..., m\}$ such that $\eta \mathcal{R} \epsilon_i$ and $\eta \mathcal{L} \epsilon_j$.

$$\begin{array}{ccc} \epsilon_k & & \\ \circ & \epsilon_{k+1} & \\ \vdots & & \\ \vdots & & \epsilon_j \\ \vdots & & \vdots \\ \epsilon_i & & \eta \end{array}$$

Moreover, there is a unique K in $\{1, ..., m\}$ such that $\epsilon_i \mathcal{L} \epsilon_k$. (If $i \in \{1, ..., m\}$ then of course $k = i$.) If $k = j$ then $\eta = \epsilon_i$ and there is nothing to prove. If $k < j$ then

$$\eta = \epsilon_i\,\epsilon_{k+1}\,\epsilon_{k+2}\cdots\epsilon_j\ .$$

If $k > j$ then

$$\eta = \epsilon_i\,\epsilon_{k+1}\cdots\epsilon_m\,\epsilon_1\cdots\epsilon_j\ . \blacksquare$$

Note that in O_n, the number of $\mathcal{L}$-classes in any $\mathcal{J}$-class exceeds the number of $\mathcal{R}$-classes, in PO_n the number of $\mathcal{L}$-classes in a $\mathcal{J}$-class is smaller than the number of $\mathcal{R}$-classes. This accounts for the difference of the strategies in Lemmas 2.3 and 3.3.

It remains to prove that the listing of images and convex equivalences postulated in the statement of Lemma 3.3 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n - 2$, and consider the Proposition:

$P(n, r)$. *There is a way of listing the subsets of X_n of cardinality r as $A_1, ..., A_m$ (with $m = \binom{n}{r}$), $A_1 = \{1, 2, ..., r\}$, $A_2 = \{1, 2, ..., r-1, r+1\}$, $A_m = \{1, 2, ..., r-1, n\}$) so that there exist distinct convex equivalences $\pi_1, ..., \pi_m$ of weight r with the property that A_{i-1}, A_i are both transversals of π_i $(i = 2, ..., m)$ and A_m, A_1 are both transversals of π_1.*

The proof is by double induction on n and r, the key step being again a Pascal's triangular implication

$$P(n-1, r-1) \text{ and } P(n-1, r) \quad \Rightarrow \quad P(n, r) \ .$$

First, however, we anchor the induction with two Lemmas.

Lemma 3.4. *$P(n,2)$ holds for every $n \geq 4$.*

Proof: The proof is by induction. For $n = 4$ we have the list of 6 subsets and 6 equivalences as follows:

$$\begin{array}{ll} \{1,2\} & 1/2\ 4\ , \\ \{1,3\} & 1/2\ 3\ , \\ \{2,3\} & 1\ 2/3\ , \\ \{2,4\} & 2/3\ 4\ , \\ \{3,4\} & 2\ 3/4\ , \\ \{1,4\} & 1\ 3/4\ . \end{array}$$

Suppose inductively that $P(n-1,2)$ holds ($n \geq 5$). Thus we have a list $A_1, ..., A_t$ (where $t = \binom{n-1}{2}$) of subsets of X_{n-1} of cardinality 2, and a list $\pi_1, ..., \pi_t$ of distinct convex equivalences of weight 2 such that for $i = 2, ..., t$ the sets A_{i-1}, A_i are both transversals of π_i and A_t, A_1 are both transversals of π_1. Suppose moreover that $A_1 = \{1,2\}$, $A_2 = \{1,3\}$ and $A_t = \{1, n-1\}$. Let

$$B_i = \{i, n\}$$

for $i = 1, ..., n-1$, and define

$$\begin{aligned} \pi_1' &= \pi_1 \ \text{ with } n-1 \text{ being replaced by } n\ , \\ \sigma_1 &= 1\ 2/n-1\ n\ , \\ \sigma_i &= i\ i+1/n \ \text{ for } i = 2, ..., n-2\ , \\ \sigma_{n-1} &= 1\ n-1/n\ . \end{aligned}$$

Arrange the subsets and the convex equivalences as follows:

$$\begin{gathered} A_1, A_2, ..., A_t, B_2, B_3, ..., B_{n-1}, B_1\ , \\ \pi_1', \pi_2, ..., \pi_t, \sigma_1, \sigma_2, ..., \sigma_{n-2}, \sigma_{n-1}\ . \end{gathered}$$

Then, it is easy to verify that the subsets and the convex equivalences as arranged above satisfy $P(n,2)$. Notice that these subsets are all the subsets of X_n of cardinality 2, and the convex equivalences are all distinct. ∎

Lemma 3.5. *$P(n, n-2)$ holds for every $n \geq 4$.*

Proof: We shall show that for $k \geq 4$,

$$P(k, k-2) \quad \Rightarrow \quad P(k+2, k) \ .$$

But first we show that $P(4,2)$ and $P(5,3)$ are true.

For $n = 4$, the result follows from Lemma 3.4. For $n = 5$, we have the list of 10 subsets and 10 equivalences as follows:

$$\begin{array}{ll}
\{1,2,3\} & 1/2/3\ 5\ , \\
\{1,2,4\} & 1/2/3\ 4\ , \\
\{1,3,4\} & 1/2\ 3/4\ , \\
\{2,3,4\} & 1\ 2/3/4\ , \\
\{2,3,5\} & 2/3/4\ 5\ , \\
\{2,4,5\} & 2/3\ 4/5\ , \\
\{3,4,5\} & 2\ 3/4/5\ , \\
\{1,4,5\} & 1\ 3/4/5\ , \\
\{1,3,5\} & 1/3\ 4/5\ , \\
\{1,2,5\} & 1/2\ 3/5\ .
\end{array}$$

Suppose inductively that $P(k, k-2)$ holds ($k \geq 4$). Thus we have a list $A_1, ..., A_m$ (where $m = \binom{k}{k-2}$) of subsets of X_k of cardinality $k-2$, and a list $\pi_1, ..., \pi_m$ of distinct convex equivalences of weight $k-2$ such that for $i = 2, ..., m$ the sets A_{i-1}, A_i are both transversals of π_i and A_m, A_1 are both transversals of π_1. We may also assume that

$$A_1 = \{1, 2, ..., k-2\}\,, \qquad A_2 = \{1, 2, ..., k-3, k-1\}$$

and

$$A_m = \{1, 2, ..., k-3, k\} \ .$$

Let $B_1, ..., B_{k+1}$ be the list of subsets of X_{k+1} of cardinality k, where $B_i = X_{k+1} \backslash \{k+2-i\}$. (Thus in particular $B_1 = \{1, 2, ..., k\}$ and $B_{k+1} = \{2, 3, ..., k+1\}$.) Let $C_1, ..., C_k$ be the list of subsets of X_k of cardinality $k-1$, where $C_i = X_k \backslash \{k+1-i\}$. (In particular $C_1 = \{1, 2, ..., k-1\}$ and $C_k = \{2, 3, ..., k\}$.) Define

$$\begin{aligned}
A'_i &= A_i \cup \{k+1, k+2\} && \text{for } i = 1..., m\ , \\
C'_i &= C_i \cup \{k+2\} && \text{for } i = 1, ..., k\ .
\end{aligned}$$

Notice that the subsets $A'_1, ..., A'_m, B_1, ..., B_{k+1}, C'_1, ..., C'_k$ are all distinct, and form a complete list of subsets of X_{k+2}, of cardinality k, since $m + k + (k+1) = \binom{k+2}{k}$.

Denote by $|i,j|$ the convex equivalence of weight $n-1$ on a set T of n elements, where $\{i,j\}$ is the only non-singleton class. Then define

$$
\begin{array}{lll}
\sigma_i = |k+2-i, k+3-i| & \text{on } X_{k+1} & \text{for } i = 2, ..., k+1 \ , \\
\tau_i = |k+1-i, k+2-i| & \text{on } X_k \cup \{k+2\} & \text{for } i = 2, 4, ..., k-1 \ , \\
\pi'_i = \pi_i \cup \{(k+1, k+1)\} \cup \{(k+2, k+2)\} & & \text{for } i = 1, 3, ..., m \ , \\
\delta_1 = |k, k+2| & \text{on } X_k \cup \{k+2\} \ , & \\
\delta_2 = |k+1, k+2| & \text{on } X_{k+2} \backslash \{1\} \ , & \\
\delta_3 = |k, k+1| & \text{on } X_{k+2} \backslash \{k-2\} \ , & \\
\delta_4 = |k, k+1| & \text{on } X_{k+2} \backslash \{k-1\} \ . &
\end{array}
$$

Now, arrange the subsets and the convex equivalences as follows:

$$
B_1, B_2, ..., B_{k+1}, C'_k, C'_{k-1}, ..., C'_3, A'_2, A'_3, ..., A'_m, A'_1, C'_2, C'_1 \ ,
$$
$$
\delta_1, \sigma_2, ..., \sigma_{k+1}, \delta_2, \tau_k, ..., \tau_4, \delta_3, \pi'_3, ..., \pi'_m, \pi'_1, \delta_4, \tau_2 \ .
$$

With this arrangement it is easy to verify that the subsets and the convex equivalences satisfy $P(k+2, k)$.

Since an $\mathcal{R}$-class characterized by a convex equivalence of weight $n-1$ on a set of n elements contains only two idempotents, the convex equivalences above are unique, and therefore distinct. ∎

Lemma 3.6. *Let $n \geq 5$ and $3 \leq r \leq n-3$. Then $P(n-1, r-1)$ and $P(n-1, r)$ together imply $P(n, r)$.*

Proof: From the assumption $P(n-1, r)$ we have a list $A_1, ..., A_m$ (where $m = \binom{n-1}{r}$) of the subsets of X_{n-1} with cardinality r and a list $\sigma_1, ..., \sigma_m$ of distinct convex equivalences of weight r such that A_{i-1}, A_i $(i = 2, ..., m)$ are transversals of σ_i, and A_1, A_m are transversals of σ_1. We may also assume that

$$
A_1 = \{1, 2, ..., r\} \ , \quad A_2 = \{1, ..., r-1, r+1\} \ , \quad A_m = \{1, ..., r-1, n-1\}
$$

and σ_2 has $\{r, r+1\}$ as the only non-singleton class.

From the assumption $P(n-1, r-1)$ we have a list $B_1, ..., B_t$ (where $t = \binom{n-1}{r-1}$) of subsets of X_{n-1} of cardinality $r-1$, and a list $\tau_1, ..., \tau_t$ of distinct convex equivalences of weight $r-1$ such that B_{j-1}, B_j $(j = 2, ..., t)$ are transversals of τ_j, and B_1, B_t are transversals of τ_1. We may also assume that

$$
B_1 = \{1, 2, ..., r-1\} \ , \quad B_2 = \{1, ..., r-2, r\} \ , \quad B_t = \{1, .., r-2, n-1\}
$$

and that τ_2 has $\{r-1, r\}$ as the only non-singleton class.

Let

$$B_i' = B_i \cup \{n\} \quad (i = 1, ..., t) \ .$$

Then $A_1, ..., A_m, B_1', ..., B_t'$ is a complete list of the subsets of X_n of cardinality r. (Notice that $m + t = \binom{n}{r}$.) Define

$$\sigma_1' = \sigma_1 \quad \text{with} \;\; n-1 \text{ replaced by } n \ ,$$

$$\tau_i' = \tau_i \cup \{(n,n)\} \quad \text{for} \;\; i = 1, 3, ..., t \ ,$$

while τ_2' is an equivalence of weight r whose only non-singleton class is $\{n-1, n\}$. Then $\sigma_2, ..., \sigma_m, \tau_1', ..., \tau_t'$ are all distinct (since the σ's do not contain n, while the τ''s contain n). Also σ_1' is distinct from all of them, since σ_1' contains r and n in the same equivalence class; and τ_2' is distinct from all the others, since it has $n-1$ and n in the same equivalence class.

Arrange the subsets and the convex equivalences as follows:

$$A_1, A_2, ..., A_m, B_2', ..., B_t', B_1' \ ,$$

$$\sigma_1', \sigma_2, ..., \sigma_m, \tau_2', ..., \tau_t', \tau_1' \ .$$

With this arrangement it is easy to verify that the convex equivalences and the subsets satisfy $P(n, r)$. ∎

The pattern of deduction here is

$$\begin{array}{ccccccc} & & & P(4,2) & & & \\ & & P(5,2) & & P(5,3) & & \\ & P(6,2) & & P(6,3) & & P(6,4) & \\ P(7,2) & & P(7,3) & & P(7,4) & & P(7,5) \end{array}$$

Remark 3.7. Observe that in Lemmas 3.4, 3.5 and 3.6 (proof) all the convex equivalences used have only one non-singleton class, except for τ_2' in Lemma 3.6 which has two. In all cases the non-singleton class (or classes) contained only two elements, and since $n \geq 4$, $r = 2$ in 3.4, $n \geq 4$, $r = n-2$ in 3.5 and $n \geq 5$, $r \leq n-3$ in 3.6 the convex equivalences are all partial. Thus in the generating set $\{\epsilon_1, ..., \epsilon_p\}$ of Lemma 3.3, $\epsilon_1, ..., \epsilon_m$ need not be full idempotents.

We shall find this useful in the next section.

4 – Strictly partial order-preserving transformations

It is clear that the number of $\mathcal{L}$-classes in a $\mathcal{J}$-class J_r of SPO_n is $\binom{n}{r}$, and the number of $\mathcal{R}$-classes is $\sum_{k=r}^{n-1}\binom{n}{k}\binom{k-1}{r-1}$.

Similar to Lemmas 2.1 and 3.1 we have:

Lemma 4.1. *For $1 \le r \le n-3$, we have $J_r \subseteq (J_{r+1})^2$.*

Proof: The proof of Lemma 2.1 applies to this case also by adjusting A' to $\{r+2\}$. ∎

The next result proves that $N(n,r)$ is idempotent-generated.

Proposition 4.2. *Let E_{n-2} be the set of all idempotents in J_{n-2}. Then $J_{n-2} \subseteq \langle E_{n-2} \rangle$.*

Proof: Notice that $J_{n-2} = [n-1, n-2] \cup [n-2, n-2]$. We shall first consider an element $\alpha \in [n-2, n-2]$. Let $\operatorname{dom}\alpha = X_n \backslash \{i, j\}$ and assume that $i < j$, and $\operatorname{Im}\alpha = X_n \backslash \{k, l\}$ with $k < l$. Let ϵ be the partial identity on $\operatorname{dom}\alpha$. We now distinguish several cases.

Case 1. $i = k$.

a) $j < l$. Let $A = \operatorname{dom}\alpha \cup \{j\}$. For $s = 1, ..., l-j$ define the idempotents ϵ_s on A by

$$\{j+s-1, j+s\}\,\epsilon_s = j+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A \backslash \{j+s-1, j+s\}$. Then

$$\alpha = \epsilon\,\epsilon_1\,\epsilon_2 \cdots \epsilon_{l-j}\ .$$

b) $j > l$. Let $A = \operatorname{dom}\alpha \cup \{j\}$. For $s = 1, ..., j-l$ define the idempotents ϵ_s on A by

$$\{j-s, j-s+1\}\,\epsilon_s = j-s+1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all x in $A \backslash \{j-s, j-s+1\}$. Then $\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{j-l}$.

c) $j = l$. Here α is an idempotent.

Case 2. $j = l$.

a) $i < k$. Let $A = \operatorname{dom}\alpha \cup \{i\}$. For $s = 1, ..., k-i$, define ϵ_s by

$$\{i+s-1, i+s\}\,\epsilon_s = i+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$. Then

$$\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{k-i}\ .$$

b) $i > k$. Let $A = \operatorname{dom}\alpha \cup \{i\}$. For $s = 1, ..., i-k$ define ϵ_s by

$$\{i-s, i-s+1\}\,\epsilon_s = i-s+1\ ,$$

and $x\,\epsilon_s = x$ for all x in $A\backslash\{i-s, i-s+1\}$. Then $\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{i-k}$.

Case 3.

a) $i < k < j < l$. Let $A = \operatorname{dom}\alpha \cup \{i\}$, $B = X_n\backslash\{k\}$. For $s = 1, ..., k-i$ and $t = 1, ..., l-j$, define ϵ_s and η_t as follows:

$$\{i+s-1, i+s\}\,\epsilon_s = i+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A\backslash\{i+s-1, i+s\}$;

$$\{j+t-1, j+t\}\,\eta_t = j+t-1 \quad \text{and} \quad x\,\eta_t = x$$

for all $x \in B\backslash\{j+t-1, j+t\}$. Then

$$\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{k-i}\,\eta_1 \cdots \eta_{l-j}\ .$$

b) $k < i < l < j$. Let $A = \operatorname{dom}\alpha \cup \{i\}$, $B = X_n\backslash\{k\}$. For $s = 1, ..., i-k$ and $t = 1, ..., j-l$ define ϵ_s and η_t as follows:

$$\{i-s, i-s+1\}\,\epsilon_s = i-s+1\,, \qquad x\,\epsilon_s = x \quad (x \in A\backslash\{i-s, i-s+1\})\ ,$$

$$\{j-t, j-t+1\}\,\eta_t = j-t+1\,, \qquad x\,\eta_t = x \quad (x \in B\backslash\{j-t, j-t+1\})\ .$$

Then $\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{i-k}\,\eta_1 \cdots \eta_{j-l}$.

Case 4.

a) $i < k < l < j$. Let $A = \operatorname{dom}\alpha \cup \{i\}$, $B = X_n\backslash\{k\}$. For $s = 1, ..., k-i$ and $t = 1, ..., j-l$ define ϵ_s and η_t as follows:

$$\{i+s-1, i+s\}\,\epsilon_s = i+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A\backslash\{i+s-1, i+s\}$,

$$\{l+t-1, l+t\}\,\eta_t = l+t \quad \text{and} \quad x\,\eta_t = x$$

for all $x \in B\backslash\{l+t-1, l+t\}$. Then

$$\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{k-i}\,\eta_{j-l} \cdots \eta_1\ .$$

b) $k < i < j < l$. Let $A = \operatorname{dom}\alpha \cup \{i\}$, $B = X_n\backslash\{k\}$. For $s = 1, ..., i-k$ and $t = 1, ..., l-j$ define ϵ_s and η_t as follows:

$$\{i-s, i-s+1\}\,\epsilon_s = i-s+1\,, \qquad x\,\epsilon_s = x \quad (x \in A\backslash\{i-s, i-s+1\})\ ,$$

$$\{l-t, l-t+1\}\,\eta_t = l-t\,, \qquad x\,\eta_t = x \quad (x \in B\backslash\{l-t, l-t+1\})\ .$$

Then $\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{i-k}\,\eta_1 \cdots \eta_{l-j}$.

Case 5.

a) $i < j < k < l$. Let $A = \operatorname{dom}\alpha \cup \{i\}$. For $s = 1,..,j-i-1$, $t = 1,...,k-j$ and $u = 1,...,l-k-1$ define ϵ_s, η_t and δ_u as follows:

$$\{i+s-1, i+s\}\,\epsilon_s = i+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$;

$$\{j-1, j+1\}\,\epsilon_{j-1} = j-1 \quad \text{and} \quad x\,\epsilon_{j-i} = x$$

for all $x \in A \backslash \{j-1, j+1\}$;

$$\{j+t-1, j+t+1\}\,\eta_t = j+t-1 \quad \text{and} \quad x\,\eta_t = x$$

for all $x \in B_t = X_n \backslash \{j+t-1, j+t, j+t+1\}$;

$$\{k+u, k+u+1\}\,\delta_u = k+u \quad \text{and} \quad x\,\delta_u = x$$

for all $x \in X_n \backslash \{k, k+u, k+u+1\}$. Then

$$\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{j-i}\,\eta_1 \cdots \eta_{k-j}\,\delta_1 \cdots \delta_{l-k-1}\ .$$

b) $k < l < i < j$. Let $A = \operatorname{dom}\alpha \cup \{j\}$. For $s = 1,...,j-i-1$, $t = 1,...,i-l$ and $u = 1,...,l-k-1$ define η_s, η_t and δ_u as follows:

$$\{j-s, j-s+1\}\,\epsilon_s = j-s+1\,, \quad x\,\epsilon_s = x \ \ (x \in A \backslash \{j-s, j-s+1\})\ ,$$

$$\{i-t, i-t+1\}\,\eta_t = i-t+1\,, \quad x\,\eta_t = x \ \ (x \in B_t = X_n \backslash \{i-t-1, i-t, i-t+1\})\ ,$$

$$\{l-u-1, l-u\}\,\delta_u = l-u\,, \quad x\,\delta_u = x \ \ (x \in X_n \backslash \{l-u-1, l-u, l-u+1\})\ .$$

Case 6. $i < j = k < l$. Let $A = \operatorname{dom}\alpha \cup \{i\}$. For $s = 1,...,j-i-1$ and $t = 2,...,l-j$ define ϵ_s, η_1 and η_t as follows:

$$\{i+s-1, i+s\}\,\epsilon_s = i+s-1 \quad \text{and} \quad x\,\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$;

$$\{j-1, j+1\}\,\eta_1 = j-1 \quad \text{and} \quad x\,\eta_1 = x$$

for all $x \in A \backslash \{j-1, j+1\}$;

$$\{j+t-1, j+t\}\,\eta_t = j+t-1 \quad \text{and} \quad x\,\eta_t = x$$

for all $x \in A\backslash\{j+t-1, j+t\}$. Then

$$\alpha = \epsilon\,\epsilon_1 \cdots \epsilon_{j-i-1}\,\eta_1 \cdots \eta_{l-j} \ .$$

Now, if $\alpha \in [n-1, n-2]$ then it can be expressed as follows:

$$\begin{pmatrix} a_1 & ... & a_{i-1} & \{a_i, a_{i+1}\} & a_{i+2} & ... & a_{n-1} \\ b_1 & ... & b_{i-1} & b_i & b_{i+1} & ... & b_{n-2} \end{pmatrix} .$$

But then

$$\alpha = \epsilon\,\beta \ ,$$

where

$$\epsilon = \begin{pmatrix} a_1 & ... & a_{i-1} & \{a_i, a_{i+1}\} & a_{i+2} & ... & a_{n-1} \\ a_1 & ... & a_{i-1} & a_i & a_{i+2} & ... & a_{n-1} \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} a_1 & ... & a_{i-1} & a_i & a_{i+2} & ... & a_{n-1} \\ b_1 & ... & b_{i-1} & b_i & b_{i+1} & ... & b_{n-2} \end{pmatrix} .$$

Note that ϵ is an idempotent, and that $\beta \in [n-2, n-2]$. Hence α is expressible in terms of idempotents in E_{n-2}. ∎

Theorem 4.3. *For $1 \le r \le n-2$ we have*

$$\operatorname{rank}(N(n,r)) = \operatorname{idrank}(N(n,r)) = \sum_{k=r}^{n-1} \binom{n}{k} \binom{k-1}{r-1} \ .$$

Proof: The reason for choosing $\epsilon_1, ..., \epsilon_m$ in the generating set $\{\epsilon_1, ..., \epsilon_p\}$ for $M(n,r)$ to be non-full idempotents (see Remark 3.7) is to make the corresponding result for $N(n,r)$ much easier to deduce; since we may choose the same idempotents $\epsilon_1, ..., \epsilon_m$ and $\epsilon_{m+1}, ..., \epsilon_q$ (where $q = \sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$) from the remaining $\mathcal{R}$-classes to obtain the generating set $\{\epsilon_1, ..., \epsilon_q\}$ of $N(n,r)$. And the result follows from Lemma 3 in [7]. ∎

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G.U. Garba, Department of Mathematical and Computational Sciences,
University of St Andrews, Scotland – U.K.
and
Department of Mathematics,
Ahmadu Bello University Zaria – NIGERIA